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Seshadri fibrations on algebraic surfaces

Dedicated to Professor Andrzej Zajtz, on occasion of his 70th birthday

Abstract. We show that small Seshadri constants in a general point of a surface have strong geometrical implications, the surface is fibered by curves computing the Seshadri constant. We give a sharp bound in terms of the selfintersection of the given ample line bundle and discuss some examples.

Introduction

Seshadri constants were introduced by Demailly [2] in the late 80's in connection with attempts to tackle the Fujita Conjecture. They express, roughly speaking, how ample a line bundle is locally. Seshadri constants quickly gained considerable interest on their own.

Recently Nakamaye [5] showed that these local invariants when studied at a general point of a variety carry interesting global geometric information. In particular he was interested in to which extend Seshadri constants capture existence of morphisms to lower dimensional varieties. This problem was considered also by Hwang and Keum [3]. In both papers it is shown that a small Seshadri constant in a generic point of a variety forces a fibration structure on the variety. Here we prove that in case of algebraic surfaces the bound from [3, Theorem 2] is in fact the optimal one. One could hope that on the contrary a large Seshadri constant in a generic point prohibits in turn a fibration structure on the variety. We show that this need not to be the case and we answer to the negative two related questions from [5].

1. Seshadri fibrations

Let us first recall the following

Definition
Let X be a smooth projective variety, let L be a nef line bundle on X and let $x \in X$ be a fixed point. Then the real number

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$$\varepsilon(L,x) := \inf\left\{\frac{L.C}{\mathrm{mult}_x C} \mid C \text{ an irreducible curve passing through } x\right\}$$

is the *Seshadri constant* of L at x.

In case of surfaces, it is well known that $\varepsilon(L,x) \le \sqrt{L^2}$. Moreover, if the Seshadri constant is *submaximal* $\varepsilon(L,x) < \sqrt{L^2}$, then a theorem of Campana and Peternell (see [1]) assures that there exists a *Seshadri curve* C_x computing $\varepsilon(L,x)$ i.e.

$$\varepsilon(L,x) = \frac{L.C_x}{\mathrm{mult}_x C_x}\,.$$

Nakamaye shows [5, Corollary 3] that if the Seshadri constant at every point of X is sufficienly small, namely $\varepsilon(L,x) < \sqrt{\frac{1}{3}L^2}$, then Seshadri curves form a fibration on X i.e. they are fibers of a non-trivial morphism $f\colon X \longrightarrow Y$ onto a curve Y. We speak in this situation of a *Seshadri fibration* on X. Our main result strengthens that of Nakamaye.

Theorem
Let (X,L) be a polarized surface and suppose that

$$\varepsilon(L,x) < \sqrt{\frac{3}{4}L^2} \tag{1}$$

at every point $x \in X$. Then there exists a non-trivial morphism $f\colon X \longrightarrow Y$ to a curve Y whose fibers are Seshadri curves. Moreover the above bound is sharp.

The proof of our result builds upon the following Lemma due to Xu [7, Lemma 1].

Lemma
Let X be a smooth projective surface, let (C_t, x_t) be a one parameter family of pointed curves on X and let $m \ge 2$ be an integer such that $\mathrm{mult}_{x_t} C_t \ge m$. *Then*

$$C_t^2 \ge m(m-1)+1.$$

Proof of Theorem. Let C_x be a Seshadri curve at $x \in X$ and suppose that for $x \in X$ general we have $\mathrm{mult}_x C_x \ge m$.

If $m \ge 2$, then the assumption (1), the Hodge index Theorem and the Lemma yield

$$\frac{3}{4}m^2L^2 > (L.C_x)^2 \ge L^2C_x^2 \ge (m(m-1)+1)L^2,$$

which is easily seen to be equivalent to $(m-2)^2 < 0$, a contradiction.

If $m = 1$ then again by the assumption (1) and the Hodge index Theorem we get

$$\frac{3}{4}L^2 > (L.C_x)^2 \geq C_x^2 L^2.$$

Since C_x moves in a family we have $C_x^2 \geq 0$ and thus the above inequality implies $C_x^2 = 0$. Then a standard argument (see e.g. [5]) shows that for some positive integer $k > 0$ the linear system $|kC_x|$ gives the desired fibration.

2. Examples

The following examples show that our Theorem is optimal.

EXAMPLE 2.1
Let (X, L) be a smooth cubic in $\mathbb{P}^3$. Then

$$\varepsilon(L, x) = \frac{3}{2} = \sqrt{\frac{3}{4}L^2}$$

for $x \in X$ general. Indeed, the hyperlane tangent at x cuts on X a curve $C_x \in |L|$, which for x general is irreducible and has multiplicity 2. Of course, the curves C_x do not define a fibration on X as they belong to a very ample line series.

It may well happen that for $\varepsilon(L, x) = \sqrt{\frac{3}{4}L^2}$ one gets a Seshadri fibration on X.

EXAMPLE 2.2
Let (X, L) be the Hirzebruch $\mathbb{F}_1$ surface with polarization $L = 6C_0 + 7f$, where as usual f denotes the class of the fiber in the projective bundle $\mathbb{P}(\mathcal{O}_{\mathbb{P}^1} \oplus \mathcal{O}_{\mathbb{P}^1}(-1))$ and C_0 its zero section. In this case the fibers compute the Seshadri constant of L at every point of X and we have

$$\varepsilon(L, x) = 6 = \sqrt{\frac{3}{4}L^2}.$$

One might ask whether there is a converse to our Theorem i.e. if the existence of a Seshadri fibration imposes some constrains on the Seshadri constant at a general point. The following example shows that this is not the case, the Seshadri constant can be arbitrarily close to its maximal possible value $\sqrt{L^2}$.

EXAMPLE 2.3
Let $f\colon X \longrightarrow \mathbb{P}^2$ be the blow up of $\mathbb{P}^2$ at a point $P \in \mathbb{P}^2$ with the exceptional divisor E and let $L = H - \lambda E$ be a $\mathbb{Q}$-line bundle with $H = f^*\mathcal{O}_{\mathbb{P}^2}(1)$. Then

$$\varepsilon(L, x) = 1 - \lambda$$

for $x \in X \setminus E$. Indeed, the quotient $1 - \lambda$ is computed by the proper transform of the line through P and x.

Moreover, if $C \subset X$ is another irreducible curve through x, then C is of the form $C = dH - mE$ with $m \leq d-1$ and $m + \operatorname{mult}_x C \leq d$. Then

$$\frac{L.C}{\operatorname{mult}_x C} = \frac{d - m\lambda}{\operatorname{mult}_x C} \geq \left\{ \begin{array}{ll} \frac{m(1-\lambda)}{\operatorname{mult}_x C} & \text{if } m \geq \operatorname{mult}_x C \\ \frac{d-\lambda \operatorname{mult}_x C}{\operatorname{mult}_x C} & \text{if } m \leq \operatorname{mult}_x C \end{array} \right\} \geq 1 - \lambda.$$

Hence $\varepsilon(L, x) = \sqrt{\frac{(1-\lambda)^2}{1-\lambda^2} L^2}$.

3. Answers

Finally, we answer two questions from Nakamaye's paper. First we address the question if the existence on a surface X of a nef real class χ with $\chi^2 = 0$ implies that X admits a surjective morphism to a curve [5, Question 9].

Example 3.1
Let (X, Θ) be a principally polarized abelian surface with the endomorphism ring $\operatorname{End}(X) \cong \mathbb{Z}[\sqrt{d}]$, where d is a square free positive integer. Let $M \in \operatorname{NS}(X)$ be a line bundle with $M^2 = -2d$ corresponding to the endomorphism $\sqrt{d}$ under the group homomorphism

$$\operatorname{NS}(X) \ni N \longrightarrow \varphi_\Theta^{-1} \circ \varphi_N \in \operatorname{End}(X).$$

Then Θ and M form an orthogonal basis of $\operatorname{NS}(X)$ and a line bundle $a\Theta + bM$ is ample if and only if

$$(a\Theta + bM)^2 > 0 \quad \text{and} \quad (a\Theta + bM).\Theta > 0.$$

It follows that

$$\operatorname{Nef}(X) = \mathbb{R}_{\geq 0} \cdot (\sqrt{d}\Theta + M) + \mathbb{R}_{\geq 0} \cdot (\sqrt{d}\Theta - M)$$

and $\sqrt{d}L \pm M$ are the only real nef classes with selfintersection 0. This shows that there is no non-trivial morphism from the abelian surface X to a curve, which in turn answers the above question negatively.

The last problem concerns pairs (X, L) consisting of a smooth projective surface X and an ample line bundle L with selfintersection 1. On such a surface $\varepsilon(L, x) = 1$ for x very general by the result of Ein and Lazarsfeld [4]. Nakamaye asks, obviously motivated by $\mathbb{P}^2$, if the Seshadri curves computing $\varepsilon(L, x)$ on X can be forced to form a fibration when one passes to a blow up of X at a single point. The answer to this question is also negative.

Example 3.2
Let (X, Θ) be a principally polarized abelian surface with Picard number $\rho(X) = 1$. Let $f\colon Y \longrightarrow X$ be the blow up of X at a point $P \in X$ with

the exceptional divisor E. Then $L = f^*\Theta - E$ is ample and $L^2 = 1$. Its Seshadri constant at every point x away of E is computed by the proper transform of a translate of the Θ-divisor passing through x and P. Indeed, there are exactly two such translates and since $\rho(Y) = 2$ the claim follows from [6, Proposition 1.8]. Of course there doesn't exist any point $y \in Y$ such that the translates of the Θ-divisor form a fibration on the blow up of Y at y.

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