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Translation equation and the Jordan non-measurable continuous functions

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Abstract. A connection between the continuous translation equation and the Jordan non-measurable continuous functions is given.

It is well known that a continuous function is Lebesgue measurable. It is not true for the Jordan measurability (in short: measurability). We give an example of a non-measurable continuous function by the solution of the translation equation.

1. Continuous solutions of translation equation

Every continuous solution of the translation equation

$$F(F(x,t),s) = F(x,t+s), \tag{1}$$

where $F\colon I\times\mathbb{R}\to I$ and I is a non-degenerated interval, is of the form

$$F(x,t) = \begin{cases} h_n^{-1}[h_n(g(x))+t], & \text{for } g(x)\in I_n,\ t\in\mathbb{R},\\ g(x), & \text{for } g(x)\in g(I)\setminus\bigcup I_n,\ t\in\mathbb{R}, \end{cases} \tag{2}$$

where $g\colon I\to I$ is a continuous idempotent ($g\circ g = g$), $I_n\subset g(I)$ for $n\in N_1\subset\mathbb{N}$ are open and disjoint intervals and $h_n\colon I_n\to\mathbb{R}$ are homeomorphisms.

Indeed, it is proved in the book [3] that every continuous solution F_1 of the translation equation for which $F_1(x,0)=x$ is of the form (1) with $g(x)=x$. Let F be a continuous solution of the translation equation. The function $F_1 = F|_{F(I,\mathbb{R})\times\mathbb{R}}$

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is a continuous solution of the translation equation for which $F_1(x,0)=x$, since if $x=F(x_1,t_1)$ for some $(x_1,t_1)\in I\times\mathbb{R}$, then

$$F_1(x,0)=F_1(F(x_1,t_1),0)=F(F(x_1,t_1),0)=F(x_1,t_1)=x.$$

Moreover, $F(x,t)=F(F(x,t),0)=F_1(F(x,0),t)$ and $F(x,0)$ is a continuous idempotent.

2. Main considerations

Definition
A function $f\colon I_1\to I_2$, where I_1, I_2 are the intervals in $\mathbb{R}$, is said to be measurable if the set $\{x\in I_1:\ f(x)>a\}$ is measurable for every $a\in\mathbb{R}$.

Let $I\subset\mathbb{R}$ be a non-degenerated interval, $g\colon I\to I$ a continuous idempotent such that $g(I)$ is a non-degenerated bounded interval. Let C be a set of the Smith-Volterra-Cantor type in $g(I)$, i.e. let C be a non-measurable set obtained in $g(I)$ as a modification of the construction of the Cantor set in which $\frac{1}{4}$ is taken in place of $\frac{1}{3}$ ([1] p.191) (the Cantor set is here not good since it is of Jordan measure zero as a closed set of Lebesgue measure zero). Let I_n be the components of the open set $g(I)\setminus C$. Let F be the function given by the formula (2) with these intervals I_n and arbitrary homeomorphisms $h_n\colon I_n\to\mathbb{R}$. Fix an arbitrary $t_0\neq 0$. We will prove that the functions $f(x)=F(x,t_0)-g(x)$ and $-f$ are continuous and at least one of these functions is non-measurable.

Indeed, they are continuous since F and g are continuous functions. We have

1) $f(x)=0$ for $g(x)\in C$,
2) $f(x)\neq 0$ for $g(x)\in I_n$, $n\in N_1\subset\mathbb{N}$, otherwise we would have $g(x)=F(x,t_0)=h_n^{-1}[h_n(g(x))+t_0]$ and $h_n(g(x))=h_n(g(x))+t_0$, a contradiction.

Thus,

$$\begin{aligned}\bigcup I_n&=\{x\in\bigcup I_n:\ f(x)>0\}\cup\{x\in\bigcup I_n:\ f(x)<0\}\\&=g(I)\cap\{x\in I:\ f(x)>0\}\cup g(I)\cap\{x\in I:\ f(x)<0\}.\end{aligned}$$

The set $\bigcup I_n$ is non-measurable since $\bigcup I_n=g(I)\setminus C$, thus at least one of the sets $\{x\in I:\ f(x)>0\}$ and $\{x\in I:\ f(x)<0\}=\{x\in I:\ -f(x)>0\}$ is non-measurable. The proof is completed.

The type of monotonicity of homeomorphisms h_n decides partly which function: f or $-f$, is not measurable., e.g. if $t_0>0$ and every h_n is increasing, then

$$F(x,t_0)=h_n^{-1}[h_n(g(x))+t_0]>h_n^{-1}[h_n(g(x))]=g(x)$$

for $g(x)\in I_n$. Thus we have $f(x)>0$ for $g(x)\in\bigcup I_n$, hence the function f is non-measurable. This type of monotonicity of h_n may be of course different for different n.

Let I be the bounded interval and $g(x)=x$ in (2). In this case the function $F(x,0)=x$ is evidently measurable. Moreover, for every $t_0\neq 0$, the function $F(\cdot,t_0)\colon I\to I$ is measurable too: for every real number a the set $\{x\in I:\ F(x,t_0)>a\}$ is an interval, as $F(\cdot,t_0)$ is onto, continuous and increasing.

Conclusion
The difference of measurable functions (even continuous) may be non-measurable.

It is known that this situation is impossible for the Lebesgue measurable functions.

There exists a continuous solution F of (1) such that for all $t \in \mathbb{R}$, functions $F(\cdot, t)$ are non-measurable.

Indeed, we put $h(x) = d(x, C) + 1$ for $x \in [0, 1)$ and $h(x) = x$ for $x \in [1, 2]$, where C is the above set on the interval $[0, 1]$ and $d(x, C)$ is the distance between x and C. This function h is

1) continuous since the function $d(x, C)$ is continuous ([2] p.103),
2) non-measurable since the set $\{x \in [0, 2] : \ h(x) > 1\} = (I \backslash C) \cup (1, 2]$ is non-measurable,
3) an idempotent function since it is the identity function on the range of the function h $(h([0, 2]) = [1, 2])$.

Thus the function $F(x, t) = h(x)$ for $(x, t) \in [0, 2] \times \mathbb{R}$ is the solution of (1) and $F(\cdot, t) \colon [0, 2] \to [0, 2]$ is a continuous, non-measurable function for every $t \in \mathbb{R}$.

3. Remark

Proposition
There exists a solution F of (1) for which $F(\cdot, 0)$ is measurable and $F(\cdot, 1)$ is non-measurable.

Proof. Let $g_1 \colon (0, 1] \cap \mathbb{Q} \to (-\infty, 0] \cap \mathbb{Q}$, $g_2 \colon (0, 1] \backslash \mathbb{Q} \to (0, +\infty) \backslash \mathbb{Q}$, $g_3 \colon (1, 3) \cap \mathbb{Q} \to (0, +\infty) \cap \mathbb{Q}$ and $g_4 \colon (1, 3) \backslash \mathbb{Q} \to (-\infty, 0] \backslash \mathbb{Q}$ be bijections such that $g_4((1, 2) \backslash \mathbb{Q}) \subset (-1, 0]$. The function $g = g_1 \cup g_2 \cup g_3 \cup g_4$ is a bijection from $(0, 3)$ onto $\mathbb{R}$. This implies that the function $F(x, t) = g^{-1}[g(x) + t]$ is a solution of (1). The function $F(x, 0) = x$ is evidently measurable. We prove that the function $F(\cdot, 1)$ is non-measurable by proving that the set $S = \{x \in (0, 3) : \ F(x, 1) > 1\}$ is non-measurable. We have

i) $(1, 2) \cap \mathbb{Q} \subset S$ since if $x \in (1, 2) \cap \mathbb{Q}$, then $g(x) \in (0, +\infty) \cap \mathbb{Q}$, thus $g(x) + 1 \in (1, +\infty) \cap \mathbb{Q}$ and this yields that $F(x, 1) = g^{-1}[g(x) + 1] \in (1, 3) \cap \mathbb{Q}$,
ii) $[(1, 2) \setminus \mathbb{Q}] \cap S = \emptyset$. Indeed, suppose to the contrary that there exists an $x_0 \in (1, 2) \setminus \mathbb{Q}$ such that $F(x_0, 1) > 1$. We obtain $g(x_0) = g_4(x_0) \in (-\infty, 0] \setminus \mathbb{Q}$, thus $g(x_0)$ and $g(x_0) + 1$ are irrational numbers. Moreover, $g(x_0) \epsilon g_4((1, 2) \setminus \mathbb{Q}) \subset (-1, 0]$ hence $g(x_0) + 1 \in (0, 1]$ and since $g(x_0) + 1$ is an irrational number, we have $g(x_0) + 1 \in (0, 1] \setminus \mathbb{Q} \subset (0, +\infty) \setminus \mathbb{Q}$. From here $F(x_0, 1) = g^{-1}[g(x_0) + 1] = g_2^{-1}[g(x_0) + 1] \in (0, 1] \setminus \mathbb{Q}$. We obtain a contradiction since $F(x_0, 1) > 1$.

By i) and ii), the set S is non-measurable.

The function F from the above proof is evidently discontinuous since, e.g. the set $F((0,1],1)$ is not an interval.

Question
Does there exist a continuous solution of (1) which has the property as in the Proposition?

Such a solution, if it exists, must be of the form (2) with $N_1 \neq \emptyset$ and the function g which is not the identity function (see section 2).

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