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Estimate for initial MacLaurin coefficients of certain subclasses of bi-univalent functions of complex order associated with the Hohlov operator

Abstract. In this paper, we define a new subclass of bi-univalent functions involving a Hohlov operator in the open unit disk. For functions belonging to this class, we obtain estimates on the first two Taylor-Maclaurin coefficients $|a_2|$ and $|a_3|$.

1. Introduction

Let $\mathcal{A}$ denote the class of functions of the form

$$f(z) = z + \sum_{n=2}^{\infty} a_n z^n, \qquad (1)$$

which are analytic in the open unit disk $\mathbb{U} := \{z \in \mathbb{C} : |z| < 1\}$.

By $\mathcal{S}$ we will denote the subclass of all functions in $\mathcal{A}$ which are univalent in $\mathbb{U}$. Some of the important and well-investigated subclasses of the class $\mathcal{S}$ include, for example, the class $\mathcal{S}^*(\alpha)$ of *starlike functions of order* α in $\mathbb{U}$, and the class $\mathcal{K}(\alpha)$ of *convex functions of order* α in $\mathbb{U}$, with $0 \le \alpha < 1$.

It is well known that every function $f \in \mathcal{S}$ has an inverse f^{-1}, defined by

$$f^{-1}(f(z)) = z \qquad \text{for all } z \in \mathbb{U}$$

and

$$f(f^{-1}(w)) = w \qquad \text{for } |w| < r_0(f) \text{ and } r_0(f) \ge \frac{1}{4},$$

where

$$g(w) = f^{-1}(w) = w - a_2 w^2 + (2a_2^2 - a_3)w^3 - (5a_2^3 - 5a_2 a_3 + a_4)w^4 + \dots . \quad (2)$$

A function $f \in \mathcal{A}$ is said to be *bi-univalent* in $\mathbb{U}$ if $f(z)$ and $f^{-1}(w)$ are univalent in $\mathbb{U}$, and let Σ denote the class of *bi-univalent functions* in $\mathbb{U}$.

The *convolution* or *Hadamard product* of two functions $f, h \in \mathcal{A}$ is denoted by $f * h$, and is defined by

$$(f * h)(z) := z + \sum_{n=2}^{\infty} a_n b_n z^n,$$

where f is given by (1) and $h(z) = z + \sum_{n=2}^{\infty} b_n z^n$. Next, in our present investigation, we need to recall the *convolution operator* $\mathfrak{I}_{a,b,c}$ due to Hohlov [11, 10], which is a special case of the *Dziok-Srivastava operator* [6, 7].

For the complex parameters a, b and c ($c \neq 0, -1, -2, -3, \dots$), the *Gaussian hypergeometric function* $_2F_1(a, b, c; z)$ is defined as

$$_2F_1(a, b, c; z) := \sum_{n=0}^{\infty} \frac{(a)_n (b)_n}{(c)_n} \frac{z^n}{n!} = 1 + \sum_{n=2}^{\infty} \frac{(a)_{n-1}(b)_{n-1}}{(c)_{n-1}} \frac{z^{n-1}}{(n-1)!}, \quad z \in \mathbb{U}, \quad (3)$$

where $(\alpha)_n$ is the *Pochhammer symbol* (or the *shifted factorial*) given by

$$(\alpha)_n := \frac{\Gamma(\alpha + n)}{\Gamma(\alpha)} = \begin{cases} 1, & \text{if } n = 0, \\ \alpha(\alpha + 1)(\alpha + 2) \cdots (\alpha + n - 1), & \text{if } n = 1, 2, 3, \dots. \end{cases}$$

For the real positive values a, b and c, using the Gaussian hypergeometric function (3), Hohlov [11, 10] introduced the familiar convolution operator $\mathfrak{I}_{a,b,c} \colon \mathcal{A} \to \mathcal{A}$ by

$$\mathfrak{I}_{a,b,c} f(z) = [z \, _2F_1(a, b, c; z)] * f(z) = z + \sum_{n=2}^{\infty} \varphi_n a_n z^n, \qquad z \in \mathbb{U}, \qquad (4)$$

where

$$\varphi_n = \frac{(a)_{n-1}(b)_{n-1}}{(c)_{n-1}(n-1)!}, \qquad (5)$$

and the function f is of the form (1).

Hohlov [11, 10] discussed some interesting geometrical properties exhibited by the operator $\mathfrak{I}_{a,b,c}$, and the three-parameter family of operators $\mathfrak{I}_{a,b,c}$ contains, as its special cases, most of the known linear integral or differential operators. In particular, if $b = 1$ in (4), then $\mathfrak{I}_{a,b,c}$ reduces to the *Carlson-Shaffer operator*. Similarly, it is easily seen that the *Hohlov operator* $\mathfrak{I}_{a,b,c}$ is also a generalization of the *Ruscheweyh derivative operator* as well as the *Bernardi-Libera-Livingston operator*. It is of interest to note that for $a = c$ and $b = 1$, $\mathfrak{I}_{a,1,a} f = f$ for all $f \in \mathcal{A}$.

Recently there has been triggering interest to study bi-univalent function class Σ and obtained non-sharp coefficient estimates on the first two coefficients $|a_2|$ and $|a_3|$ of (1) (see [3, 2, 4, 13, 15, 21]). No estimates for the general coefficient $|a_n|$, $n > 3$, was investigated up until the publication of the article [12] in 2013.

Many researchers (see [8, 9, 14, 19]) have recently introduced and investigated several interesting subclasses of the bi-univalent function class Σ and they have found non-sharp estimates on the first two Taylor-MacLaurin coefficients $|a_2|$ and $|a_3|$.

2. Definitions and preliminaries

In [16] Padmanabhan and Parvatham defined the classes of functions $\mathcal{P}_m(\beta)$ as follows.

DEFINITION 2.1 ([16])
Let $\mathcal{P}_m(\beta)$, with $m \geq 2$ and $0 \leq \beta < 1$, denote the class of univalent analytic functions P, normalized with $P(0) = 1$ and satisfying

$$\int_0^{2\pi} \left| \frac{\operatorname{Re} P(z) - \beta}{1 - \beta} \right| \mathrm{d}\,\theta \leq m\pi,$$

where $z = re^{i\theta} \in \mathbb{U}$.

For $\beta = 0$, we denote $\mathcal{P}_m := \mathcal{P}_m(0)$, hence the class $\mathcal{P}_m$ represents the class of functions p analytic in $\mathbb{U}$, normalized with $p(0) = 1$, and having the representation

$$p(z) = \int_0^{2\pi} \frac{1 - ze^{it}}{1 + ze^{it}} \, \mathrm{d}\,\mu(t),$$

where μ is a real-valued function with bounded variation, which satisfies

$$\int_0^{2\pi} d\mu(t) = 2\pi \qquad \text{and} \qquad \int_0^{2\pi} |d\mu(t)| \leq m, \quad m \geq 2.$$

Details referring the above integral representation could be found in [16, Lemma 1]. Remark that $\mathcal{P} := \mathcal{P}_2$ is the well-known class of *Carathéodory functions*, i.e. the normalized functions with positive real part in the open unit disk $\mathbb{U}$.

Motivated by the earlier work of Deniz [5], Peng et al. [18] (see also [17, 20]) and Goswami et al. [1], in the present paper we introduce new subclasses of the function class Σ of complex order $\gamma \in \mathbb{C}^* := \mathbb{C} \setminus \{0\}$, involving Hohlov operator $\mathcal{I}_{a,b,c}$, and we find estimates on the coefficients $|a_2|$ and $|a_3|$ for the functions that belong to these new subclasses of functions of the class Σ. Several related classes are also considered, and connection to earlier known results are made.

DEFINITION 2.2
For $0 \leq \lambda \leq 1$ and $0 \leq \beta < 1$, a function $f \in \Sigma$ is said to be in the class $\mathcal{S}_{\Sigma}^{a,b,c}(\gamma, \lambda, \beta)$ if the following two conditions are satisfied:

$$1 + \frac{1}{\gamma} \left[\frac{z(\mathcal{I}_{a,b,c}f(z))'}{(1 - \lambda)z + \lambda \mathcal{I}_{a,b,c}f(z)} - 1 \right] \in \mathcal{P}_m(\beta) \tag{6}$$

and

$$1 + \frac{1}{\gamma} \left[\frac{w(\mathcal{I}_{a,b,c}g(w))'}{(1 - \lambda)w + \lambda \mathcal{I}_{a,b,c}g(w)} - 1 \right] \in \mathcal{P}_m(\beta), \tag{7}$$

where $\gamma \in \mathbb{C}^*$, the function g is given by (2) and $z, w \in \mathbb{U}$.

DEFINITION 2.3

For $0 \leq \lambda \leq 1$ and $0 \leq \beta < 1$, a function $f \in \Sigma$ is said to be in the class $\mathcal{K}_{\Sigma}^{a,b,c}(\gamma, \lambda, \beta)$ if it satisfies the following two conditions:

$$1 + \frac{1}{\gamma}\left[\frac{z(\mathfrak{I}_{a,b,c}f(z))' + z^2(\mathfrak{I}_{a,b,c}f(z))''}{(1-\lambda)z + \lambda z(\mathfrak{I}_{a,b,c}f(z))'} - 1\right] \in \mathcal{P}_m(\beta) \qquad (8)$$

and

$$1 + \frac{1}{\gamma}\left[\frac{w(\mathfrak{I}_{a,b,c}g(w))' + w^2(\mathfrak{I}_{a,b,c}g(w))''}{(1-\lambda)w + \lambda w(\mathfrak{I}_{a,b,c}g(w))'} - 1\right] \in \mathcal{P}_m(\beta), \qquad (9)$$

where $\gamma \in \mathbb{C}^*$, the function g is given by (2) and $z, w \in \mathbb{U}$.

On specializing the parameters λ one can state various new subclasses of Σ.

In order to derive our main results, we shall need the following lemma.

LEMMA 2.1 ([1, Lemma 2.1])

Let the function $\Phi(z) = 1 + \sum\limits_{n=1}^{\infty} h_n z^n$, $z \in \mathbb{U}$ be such that $\Phi \in \mathcal{P}_m(\beta)$. Then

$$|h_n| \leq m(1-\beta), \qquad n \geq 1.$$

By employing the techniques used earlier by Deniz [5], in the following section we find estimates of the coefficients $|a_2|$ and $|a_3|$ for the functions of the above-defined subclasses $\mathcal{S}_{\Sigma}^{a,b,c}(\gamma, \lambda, \beta)$ and $\mathcal{K}_{\Sigma}^{a,b,c}(\gamma, \lambda, \beta)$ of the function class Σ.

3. Coefficient bounds for the function class $\mathcal{S}_{\Sigma}^{a,b,c}(\gamma, \lambda, \beta)$

We begin by finding the estimates on the coefficients $|a_2|$ and $|a_3|$ for functions belonging to the class $\mathcal{S}_{\Sigma}^{a,b,c}(\gamma, \lambda, \beta)$. Supposing that the functions $p, q \in \mathcal{P}_m(\beta)$, with

$$p(z) = 1 + \sum_{k=1}^{\infty} p_k z^k, \qquad z \in \mathbb{U}, \qquad (10)$$

$$q(z) = 1 + \sum_{k=1}^{\infty} q_k z^k, \qquad z \in \mathbb{U}, \qquad (11)$$

from Lemma 2.1 it follows that

$$|p_k| \leq m(1-\beta), \qquad (12)$$

$$|q_k| \leq m(1-\beta) \qquad \text{for all } k \geq 1. \qquad (13)$$

THEOREM 3.1

If the function f given by (1) belongs to the class $\mathcal{S}_{\Sigma}^{a,b,c}(\gamma, \lambda, \beta)$, then

$$|a_2| \leq \min\left\{\sqrt{\frac{m|\gamma|(1-\beta)}{|(\lambda^2 - 2\lambda)\varphi_2^2 + (3-\lambda)\varphi_3|}}; \frac{m|\gamma|(1-\beta)}{(2-\lambda)\varphi_2}\right\} \qquad (14)$$

and

$$|a_3| \leq \min\left\{ \frac{m|\gamma|(1-\beta)}{(3-\lambda)\varphi_3} + \frac{m|\gamma|(1-\beta)}{|(\lambda^2-2\lambda)\varphi_2^2 + (3-\lambda)\varphi_3|};\right.$$

$$\frac{m|\gamma|(1-\beta)}{(3-\lambda)\varphi_3}\left(1 + \frac{m|\gamma|(2\lambda-\lambda^2)(1-\beta)}{(2-\lambda)^2\varphi_2^2}\right); \tag{15}$$

$$\left.\frac{m|\gamma|(1-\beta)}{(3-\lambda)\varphi_3}\left(1 + m|\gamma|(1-\beta)\frac{|(\lambda^2-2\lambda)\varphi_2^2 + 2(3-\lambda)\varphi_3|}{(2-\lambda)^2\varphi_2^2}\right)\right\},$$

where φ_2 and φ_3 are given by (5).

Proof. Since $f \in \mathcal{S}_\Sigma^{a,b,c}(\gamma,\lambda,\beta)$ from (6) and (7) it follows that

$$1 + \frac{1}{\gamma}\left[\frac{z\left(\mathfrak{I}_{a,b,c}f(z)\right)'}{(1-\lambda)z + \lambda\mathfrak{I}_{a,b,c}f(z)} - 1\right]$$

$$= 1 + \frac{2-\lambda}{\gamma}\varphi_2 a_2 z + \left[\frac{\lambda^2-2\lambda}{\gamma}\varphi_2^2 a_2^2 + \frac{3-\lambda}{\gamma}\varphi_3 a_3\right]z^2 + \dots \tag{16}$$

$$=: p(z)$$

and

$$1 + \frac{1}{\gamma}\left[\frac{w\left(\mathfrak{I}_{a,b,c}g(w)\right)'}{(1-\lambda)w + \lambda\mathfrak{I}_{a,b,c}g(w)} - 1\right]$$

$$= 1 - \frac{2-\lambda}{\gamma}\varphi_2 a_2 w + \left[\frac{\lambda^2-2\lambda}{\gamma}\varphi_2^2 a_2^2 + \frac{3-\lambda}{\gamma}\varphi_3(2a_2^2 - a_3)\right]w^2 + \dots \tag{17}$$

$$=: q(w),$$

where $p, q \in \mathcal{P}_m(\beta)$, and are of the form (10) and (11), respectively. Now, equating the coefficients in (16) and (17) we get

$$p_1 = \frac{2-\lambda}{\gamma}\varphi_2 a_2, \tag{18}$$

$$p_2 = \frac{\lambda^2-2\lambda}{\gamma}\varphi_2^2 a_2^2 + \frac{3-\lambda}{\gamma}\varphi_3 a_3, \tag{19}$$

$$q_1 = -\frac{2-\lambda}{\gamma}\varphi_2 a_2 \tag{20}$$

and

$$q_2 = \frac{\lambda^2-2\lambda}{\gamma}\varphi_2^2 a_2^2 + \frac{3-\lambda}{\gamma}\varphi_3(2a_2^2 - a_3). \tag{21}$$

From (18) and (20) we find that

$$a_2 = \frac{\gamma p_1}{(2-\lambda)\varphi_2} = \frac{-\gamma q_1}{(2-\lambda)\varphi_2}, \tag{22}$$

which implies

$$|a_2| \leq \frac{|\gamma|m(1-\beta)}{(2-\lambda)\varphi_2}. \tag{23}$$

Adding (19) and (21), by using (22) we obtain

$$[2(\lambda^2 - 2\lambda)\varphi_2^2 + 2(3-\lambda)\varphi_3]a_2^2 = \gamma(p_2 + q_2).$$

Now, by using (12) and (13), we get

$$|a_2|^2 \leq \frac{m|\gamma|(1-\beta)}{|(\lambda^2 - 2\lambda)\varphi_2^2 + (3-\lambda)\varphi_3|}, \tag{24}$$

hence

$$|a_2| \leq \sqrt{\frac{m|\gamma|(1-\beta)}{|(\lambda^2 - 2\lambda)\varphi_2^2 + (3-\lambda)\varphi_3|}},$$

which gives the bound on $|a_2|$ as asserted in (14).

Next, in order to find the upper-bound for $|a_3|$, by subtracting (21) from (19) we get

$$2(3-\lambda)\varphi_3 a_3 = \gamma(p_2 - q_2) + 2(3-\lambda)\varphi_3 a_2^2. \tag{25}$$

It follows from (12), (13), (24) and (25) that

$$|a_3| \leq \frac{m|\gamma|(1-\beta)}{(3-\lambda)\varphi_3} + \frac{m|\gamma|(1-\beta)}{|(\lambda^2 - 2\lambda)\varphi_2^2 + (3-\lambda)\varphi_3|}.$$

From (18) and (19) we have

$$a_3 = \frac{1}{(3-\lambda)\varphi_3}\left(\gamma p_2 - \frac{\gamma^2(\lambda^2 - 2\lambda)p_1^2}{(2-\lambda)^2\varphi_2^2}\right),$$

hence

$$|a_3| \leq \frac{m|\gamma|(1-\beta)}{(3-\lambda)\varphi_3}\left(1 + \frac{m|\gamma(\lambda^2 - 2\lambda)|(1-\beta)}{(2-\lambda)^2\varphi_2^2}\right).$$

Further, from (18) and (21) we deduce that

$$|a_3| \leq \frac{m|\gamma|(1-\beta)}{(3-\lambda)\varphi_3}\left(1 + m|\gamma|(1-\beta)\frac{|(\lambda^2 - 2\lambda)\varphi_2^2 + 2(3-\lambda)\varphi_3|}{(2-\lambda)^2\varphi_2^2}\right)$$

and thus we obtain the conclusion (15) of our theorem.

REMARK 3.1
For $a = c$ and $b = 1$, we have $\varphi_n = 1$ for all $n \geq 1$. Taking $\gamma = 1$ and $m = 2$ in Theorem 3.1, for the special cases $\lambda = 1$ and $\lambda = 0$ we obtain more accurate results corresponding to the results obtained in [20, 19].

4. Coefficient bounds for the function class $\mathcal{K}_\Sigma^{a,b,c}(\gamma, \lambda, \beta)$

THEOREM 4.1
If the function f given by (1) belongs to the class $\mathcal{K}_\Sigma^{a,b,c}(\gamma, \lambda, \beta)$ then

$$|a_2| \leq \min\left\{\sqrt{\frac{m|\gamma|(1-\beta)}{|4(\lambda^2 - 2\lambda)\varphi_2^2 + 3(3-\lambda)\varphi_3|}}; \frac{m|\gamma|(1-\beta)}{2(2-\lambda)\varphi_2}\right\} \tag{26}$$

and

$$|a_3| \leq \min\left\{\frac{m|\gamma|(1-\beta)}{3(3-\lambda)\varphi_3}\left(1 + \frac{m|\gamma|(2\lambda - \lambda^2)(1-\beta)}{(2-\lambda)^2\varphi_2^2}\right);\right.$$

$$\frac{m|\gamma|(1-\beta)}{3(3-\lambda)\varphi_3} + \frac{m|\gamma|(1-\beta)}{|4(\lambda^2 - 2\lambda)\varphi_2^2 + 3(3-\lambda)\varphi_3|};$$

$$\left.\frac{m|\gamma|(1-\beta)}{3(3-\lambda)\varphi_3} + \frac{m^2|\gamma|^2(1-\beta)^2}{3(3-\lambda)\varphi_3}\left(\frac{\lambda}{\lambda - 2} + \frac{3(3-\lambda)\varphi_3}{2(2-\lambda)^2\varphi_2^2}\right)\right\}, \tag{27}$$

where φ_2 and φ_3 are given by (5).

Proof. For $f \in \mathcal{K}_\Sigma^{a,b,c}(\gamma, \lambda, \beta)$, from the definition relations (8) and (9) it follows that

$$1 + \frac{1}{\gamma}\left[\frac{z(\mathfrak{I}_{a,b,c}f(z))' + z^2(\mathfrak{I}_{a,b,c}f(z))''}{(1-\lambda)z + \lambda z(\mathfrak{I}_{a,b,c}f(z))'} - 1\right]$$

$$= 1 + \frac{2(2-\lambda)}{\gamma}\varphi_2 a_2 z + \left[\frac{4(\lambda^2 - 2\lambda)}{\gamma}\varphi_2^2 a_2^2 + \frac{3(3-\lambda)}{\gamma}\varphi_3 a_3\right]z^2 + \ldots \tag{28}$$

$$=: p(z)$$

and

$$1 + \frac{1}{\gamma}\left[\frac{w(\mathfrak{I}_{a,b,c}g(w))' + w^2(\mathfrak{I}_{a,b,c}g(w))''}{(1-\lambda)w + \lambda z(\mathfrak{I}_{a,b,c}g(w))'} - 1\right]$$

$$= 1 - \frac{2(2-\lambda)}{\gamma}\varphi_2 a_2 w \tag{29}$$

$$+ \left[\frac{4(\lambda^2 - 2\lambda)}{\gamma}\varphi_2^2 a_2^2 + \frac{3(3-\lambda)}{\gamma}\varphi_3(2a_2^2 - a_3)\right]w^2 + \ldots$$

$$=: q(w),$$

where $p, q \in \mathcal{P}_m(\beta)$, and are of the form (10) and (11), respectively. Now, equating the coefficients in (28) and (29), we get

$$p_1 = \frac{2}{\gamma}(2-\lambda)\varphi_2 a_2, \tag{30}$$

$$p_2 = \frac{1}{\gamma}[4(\lambda^2 - 2\lambda)\varphi_2^2 a_2^2 + 3(3-\lambda)\varphi_3 a_3], \tag{31}$$

$$q_1 = -\frac{2}{\gamma}(2-\lambda)\varphi_2 a_2$$

and

$$q_2 = \frac{1}{\gamma}[4(\lambda^2 - 2\lambda)\varphi_2^2 a_2^2 + 3(3-\lambda)(2a_2^2 - a_3)\varphi_3]. \tag{32}$$

From (30) we get

$$a_2 = \frac{p_1\gamma}{2(2-\lambda)\varphi_2}, \tag{33}$$

further, by adding (31) and (32), and using (33) we have

$$a_2^2 = \frac{(p_2 + q_2)\gamma}{8(\lambda^2 - 2\lambda)\varphi_2^2 + 6(3-\lambda)\varphi_3}. \tag{34}$$

Now, from (30) and (34), according to Lemma 2.1, we easily deduce the inequality (26).

Next, in order to find the upper-bound for $|a_3|$, from (31), by using (33) we have

$$a_3 = \frac{p_2\gamma}{3(3-\lambda)\varphi_3} - \frac{(\lambda^2 - 2\lambda)p_1^2\gamma^2}{3(2-\lambda)^2(3-\lambda)\varphi_3}.$$

Subtracting (32) from (31) we obtain

$$-6(3-\lambda)\varphi_3 a_3 + 6(3-\lambda)a_2^2\varphi_3 = (p_2 - q_2)\gamma$$

and using (34) we deduce

$$a_3 = \frac{(p_2 + q_2)\gamma}{8(\lambda^2 - 2\lambda)\varphi_2^2 + 6(3-\lambda)\varphi_3} - \frac{(p_2 - q_2)\gamma}{6(3-\lambda)\varphi_3}.$$

Finally, from (32) we compute

$$4\left(\lambda^2 - 2\lambda\right)\varphi_2^2 a_2^2 + 6(3-\lambda)a_2^2\varphi_3 - \gamma q_2 = 3(3-\lambda)\varphi_3 a_3$$

and replacing in this the value of a_2 with those given by (33) we get

$$a_3 = \frac{1}{3(3-\lambda)\varphi_3}\left(\frac{\lambda}{\lambda-2} + \frac{3(3-\lambda)\varphi_3}{2(2-\lambda)^2\varphi_2^2}\right)p_1^2\gamma^2 - \frac{q_2\gamma}{3(3-\lambda)\varphi_3}.$$

Proceeding on lines similarly to the proof of Theorem 3.1 and applying the Lemma 2.1, we get the desired estimate given in (27).

REMARK 4.1

(i) If $a = 1$, $b = 1 + \delta$, $c = 2 + \delta$, with $\operatorname{Re}\delta > -1$, then the operator $I_{a,b,c}$ turns into well-known *Bernardi operator*, that is

$$B_f(z) := \mathcal{I}_{a,b,c}f(z) = \frac{1+\delta}{z^\delta}\int_0^z t^{\delta-1}f(t)\,\mathrm{d}t.$$

(ii) Moreover, the operators $\mathcal{I}_{1,1,2}$ and $\mathcal{I}_{1,2,3}$ are the well-known *Alexander and Libera operators*, respectively.

(iii) Further, if we take $b = 1$ in (4), then $\mathcal{I}_{a,1,c}$ immediately yields the *Carlson-Shaffer operator*, that is $L(a,c) := \mathcal{I}_{a,1,c}$.

Remark that, various other interesting corollaries and consequences of our main results, which are asserted by Theorem 3.1 and Theorem 4.1 above, can be derived similarly. The details involved may be left as exercises for the interested reader.

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