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Seshadri constants of unisecant line bundles on ruled surfaces

Dedicated to Professor Dr. A. Zajtz

Abstract. The aim of this paper is to show that for any ruled surface X with a unisecant polarization $L \equiv C_0 + \mu_0 f$ the Seshadri constant of L at every point of X is equal 1.

1. Introduction

We investigate Seshadri constants of ample line bundles on ruled surfaces. In general it is difficult to calculate Seshadri constants and their values are known only in few examples. As for lower bounds, if a line bundle L is very ample, then we have always $\varepsilon(L;x) \geq 1$. Ein and Lazarsfeld proved that on any smooth surface X with arbitrary polarization L, the previous bound is somewhat surprisingly valid in almost every point of X, more exactly: we have $\varepsilon(L;x) \geq 1$ for x very general (see [2]). On the other hand Bauer proved that if L is very ample and there is a line passing through a point x, then $\varepsilon(L;x) = 1$ (see [1]). Here we investigate unisecant polarizations on ruled surfaces. In this situation there is a line passing through every point of x but the polarization is usually not very ample. Our main result states that nevertheless the Seshadri constant at every point is equal 1.

Main Theorem
If X is a ruled surface with a unisecant polarization $L \equiv C_0 + \mu_0 f$, then the Seshadri constant of L at every point of X is equal 1.

We follow the notation and terminology used by R. Hartshorne in [4]. All facts recalled in the introduction to theory of ruled surfaces are taken from [4] V section 2, and we use them here without proofs.

Throughout this paper we work over the field $\mathbb{C}$ of complex numbers. For any coherent sheaf on a (smooth, projective) variety X, we write $H^i(\mathcal{F})$ instead of $H^i(X, \mathcal{F})$, and we denote by $h^i(\mathcal{F})$ the dimension of the cohomology group

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$H^i(\mathcal{F})$. As customarily we use additive notation for tensor powers of line bundles.

2. Ruled surfaces

2.1. Basic definitions and properties

Definition 1

A geometrically ruled surface, or simply a ruled surface, is a surface X, together with a surjective morphism $\pi: X \longrightarrow C$ to a (nonsingular) curve C, such that for every point $y \in C$, the fibre X_y is isomorphic to $\mathbb{P}^1$, and such that π admits a section (i.e. a morphism $s: C \longrightarrow X$ such that $\pi \circ s = \mathrm{id}_C$).

Example 1

If C is a nonsingular curve, then $C \times \mathbb{P}^1$ with the first projection is a ruled surface.

Example 2

Let $\mathcal{E}$ be a vector bundle of rank 2 over a curve C. The associated projective space bundle $\mathbb{P}(\mathcal{E})$ with the projection morphism $\pi: \mathbb{P}(\mathcal{E}) \longrightarrow C$ is a ruled surface.

The following proposition shows that all ruled surfaces arise as in the above example.

Proposition 1 ([4] V, 2.2)

If $\pi: X \longrightarrow C$ is a ruled surface, then there exists a vector bundle $\mathcal{E}$ of rank 2 on C such that $X \cong \mathbb{P}(\mathcal{E})$ over C. If $\mathcal{E}$ and $\mathcal{E}'$ are two vector bundles of rank 2 on C, then $\mathbb{P}(\mathcal{E})$ and $\mathbb{P}(\mathcal{E}')$ are isomorphic as ruled surfaces over C if and only if there is an invertible sheaf $\mathcal{L}$ on C such that $\mathcal{E}' \cong \mathcal{E} \otimes \mathcal{L}$.

Remark 1

A surface X is called a birationally ruled surface if is birationally equivalent to $C \times \mathbb{P}^1$ for some curve C. Since $\mathbb{P}^2$ is birational to $\mathbb{P}^1 \times \mathbb{P}^1$, this means that every rational surface is a birationally ruled surface.

Let $\pi: X \longrightarrow C$ be a ruled surface over a curve C of a genus g. By Proposition 1, we can choose $\mathcal{E}_0$ a locally free sheaf of rank 2 on C such that $X \cong \mathbb{P}(\mathcal{E}_0)$. Moreover we can assume that $\mathrm{H}^0(\mathcal{E}_0) \neq 0$ but for all invertible sheaves $\mathcal{L}$ on C with $\deg \mathcal{L} < 0$, we have $\mathrm{H}^0(\mathcal{E}_0 \otimes \mathcal{L}) = 0$. A sheaf $\mathcal{E}_0$ with this property is called *normalized*.

In general $\mathcal{E}_0$ is not necessarily determined uniquely, but its invariant $e = -\deg(\mathcal{E}_0)$ is fixed.

EXAMPLE 3
Let C be a curve with positive genus, and $\mathcal{E} = \mathcal{O}_C \oplus \mathcal{L}$ where $\deg(\mathcal{L}) = 0$ but $\mathcal{L} \ncong \mathcal{O}_C$. In this case we have two choices of normalized $\mathcal{E}_0$, namely $\mathcal{E}$ and $\mathcal{E} \otimes \mathcal{L}^{-1}$.

Let $\mathfrak{e}$ be the divisor on C corresponding to the invertible sheaf $\bigwedge^2 \mathcal{E}_0$, then $e = -\deg(\mathfrak{e})$. Moreover, there exists a section $s_0 \colon C \longrightarrow X$ with the image C_0, such that $\mathcal{O}_X(C_0) \cong \mathcal{O}_X(1)$, where $\mathcal{O}_X(1)$ is the Serre line bundle on X (for more details see [4] V, 2.8).

PROPOSITION 2 ([4] V, 2.3)
Under above assumptions we have:

$$\mathrm{Pic}(X) \cong \mathbb{Z} \cdot C_0 \oplus \pi^* \mathrm{Pic}(C).$$

Also

$$\mathrm{Num}(X) \cong \mathbb{Z} \cdot C_0 \oplus \mathbb{Z} \cdot f,$$

where f is the class of a fiber. Moreover $C_0.f = 1$, $f^2 = 0$ and $C_0^2 = -e$ (see Proposition 3).

If $\mathfrak{b}$ is any divisor on C, then we denote the divisor $\pi^*\mathfrak{b}$ on X by $\mathfrak{b}f$. Thus from Proposition 2 we have that, any element of $\mathrm{Pic}(X)$ can be written as $aC_0 + \mathfrak{b}f$ with $a \in \mathbb{Z}$ and $\mathfrak{b} \in \mathrm{Pic}(C)$. Any element of $\mathrm{Num}(X)$ can be written as $aC_0 + bf$ with $a, b \in \mathbb{Z}$.

LEMMA 1 ([4] V, 2.20 and 2.11)
Using above notations

(1) *the canonical divisor K on X is given by*

$$K \sim -2C_0 + (\mathfrak{k} + \mathfrak{e})f$$

where $\mathfrak{k}$ is the canonical divisor on C.

(2) *For numerical equivalence, we have*

$$K \equiv -2C_0 + (2g - 2 - e)f$$

and therefore

$$K^2 = 8(1 - g).$$

PROPOSITION 3 ([4] V, 2.6 and 2.9)
Let $\mathcal{E}$ be a locally free sheaf of rank two on a curve C, and let X be the ruled surface $\mathbb{P}(\mathcal{E})$. Let $\mathcal{O}_X(1)$ be the invertible sheaf $\mathcal{O}_{\mathbb{P}(\mathcal{E})}(1)$. Then there exists a one-to-one correspondence between sections $s \colon C \longrightarrow X$ and surjections $\mathcal{E} \longrightarrow \mathcal{L} \longrightarrow 0$, where $\mathcal{L}$ is an invertible sheaf on C, given by $\mathcal{L} = s^*\mathcal{O}_X(1)$.

Furthermore, if D is any section of X corresponding to a surjection $\mathcal{E} \longrightarrow \mathcal{L} \longrightarrow 0$, and if $\mathcal{L} = \mathcal{O}_C(\mathfrak{d})$, for some divisor $\mathfrak{d}$ on C, then $\deg(\mathfrak{d}) = C_0.D$, and $D \sim C_0 + (\mathfrak{d} - \mathfrak{e})f$. In particular, we have that $C_0^2 = \deg(\mathfrak{e}) = -e$.

From Proposition 2 and Proposition 3 it follows that C_0 is a curve on X with the minimum self-intersection. Next lemma gives us more information about a number of such curves.

Lemma 2 ([3], 2.8)
Let $\pi\colon X = \mathbb{P}(\mathcal{E}_0) \longrightarrow C$ be a ruled surface. Then $\mathrm{h}^0(\mathcal{O}_X(C_0)) = 2$ if $X \cong C \times \mathbb{P}^1$ and $\mathrm{h}^0(\mathcal{O}_X(C_0)) = 1$ in all other cases.

This means that the curve C_0 is unique in its class of linear equivalence, except when the ruled surface is the product $C \times \mathbb{P}^1$.

Decomposable ruled surfaces

Definition 2
A ruled surface $X \cong \mathbb{P}(\mathcal{E}_0)$ is called *decomposable* if $\mathcal{E}_0$ is a direct sum of two invertible sheaves.

Theorem 1 ([4] V, 2.12)
Let X be a ruled surface over a curve C of genus g, determined by a normalized locally free sheaf $\mathcal{E}_0$.

(1) *If $\mathcal{E}_0$ is decomposable, then $\mathcal{E}_0 \cong \mathcal{O}_C \oplus \mathcal{L}$ for some $\mathcal{L}$ with $\deg(\mathcal{L}) \leq 0$. Therefore $e \geq 0$. All values of $e \geq 0$ are possible.*

(2) *If $\mathcal{E}_0$ is indecomposable, then $-g \leq e \leq 2g - 2$.*

Let $X \cong \mathbb{P}(\mathcal{E}_0)$ be a decomposable ruled surface. Geometrically it means that X has two disjoint unisecant curves C_0 and C_1 (i.e. $C_i.f = 1$ for each fiber f). These curves are given by surjections $\mathcal{E}_0 \cong \mathcal{O}_C(\mathfrak{e}) \oplus \mathcal{O}_C \longrightarrow \mathcal{O}_C \longrightarrow 0$ and $\mathcal{E}_0 \cong \mathcal{O}_C(\mathfrak{e}) \oplus \mathcal{O}_C \longrightarrow \mathcal{O}_C(\mathfrak{e}) \longrightarrow 0$, respectively. Moreover from Proposition 3, it follows that $C_1 \sim C_0 - \mathfrak{e}f$.

3. Linear systems on ruled surfaces

We start by recalling some basic facts.

Theorem 2 ([4] V, 2.20 and 2.21)
Let X be a ruled surface over a curve C of genus g, with a fiber f, the section C_0 and $e = -\deg(\mathfrak{e}) = -C_0^2$.

(1) *If $Y \equiv aC_0 + bf$ is an irreducible curve different from C_0 and a fiber, then*

(a) $a > 0$ *and* $b \geq ae$ *for* $e \geq 0$,

(b) $(a = 1$ *and* $b \geq 0)$ *or* $(a \geq 2$ *and* $b \geq \frac{1}{2}ae)$ *for* $e < 0$.

(2) *A divisor* $D \equiv aC_0 + bf$ *is ample if and only if*

(a) $a > 0$ *and* $b > ae$ *for* $e \geq 0$,

(b) $a > 0$ *and* $b > \frac{1}{2}ae$ *for* $e < 0$.

Remark 2
There are no better numerical conditions characterizing irreducible curves on ruled surfaces as this property does not depend only on the numerical equivalence class of the considered line bundle.

3.1. Elementary transformation of a ruled surface

Let $\pi: X \longrightarrow C$ be a geometrically ruled surface and let x be a point on X with $\pi(x) = P$. We denote by Pf the fiber through the point x.

Let $\sigma: X_x \longrightarrow X$ be the blow-up of X at x with the exceptional divisor $E = \sigma^{-1}(x)$. We have $\sigma^*(Pf) = \widetilde{Pf} + E$, where $\widetilde{Pf} = \overline{\sigma^{-1}(Pf \setminus \{x\})}$ is the *strict transform* of the fiber Pf. Since $\widetilde{Pf} \cong \mathbb{P}^1$, and $\widetilde{Pf}^2 = -1$, this means that we can blow-down the surface X_x along $\widetilde{Pf}$ (this follows from the Castelnuovo's criterion). We denote by $\tau: X_x \longrightarrow X'$ the blow-down of X_x along the exceptional curve $E' = \widetilde{Pf}$.

Definition 3
An elementary transformation of X at the point x is the birational map $\nu: X' \longrightarrow X$ where $\nu = \sigma \circ \tau^{-1}$. The surface X' is called the elementary transform of X at x. For a curve C on the surface X we define its strict transform as $C' = \tau_*(\widetilde{C})$.

Note that $(Pf)'$ is zero as τ contracts $\widetilde{PF}$ to a point. We observe furter that:

Remark 3
If X' is an elementary transform of X at x, then X is the elementary transform of X' at $\tau(y)$, where y is the intersection of the exceptional divisors E and E' on X_x. Moreover, if Pf' is the fiber through the point $\tau(y)$, then $\widetilde{Pf'} = E$.

Assume that $\pi: X \longrightarrow C$ is a geometrically ruled surface over a curve C of genus g with the invariant e. Let $\nu: X' \longrightarrow X$ be the elementary transformation of the surface X at a point x with $\pi(x) = P$. The question is: how the elementary transformation ν changes properties of X?

Proposition 4 ([3], 4.4)
With above assumptions we have:

(1) *If* $\mathfrak{b}$ *is a divisor on* C*, then* $\nu^*(\mathfrak{b}f) = \mathfrak{b}f'$.

(2) *If* D *is a curve on* X*, then* $\nu^* D = D' + (\operatorname{mult}_x D) \cdot Pf'$.

(3) *If* D *is a* n*-secant curve on* X *(i.e.* $D \equiv nC_0 + bf$ *for some* $b \in \mathbb{Z}$*) and* G *is a* m*-secant curve on* X*, then*

$$D'.G' = D.G + nm - n \cdot \operatorname{mult}_x G - m \cdot \operatorname{mult}_x D.$$

Therefore, if D *and* G *are unisecant curves on* X *then:*

(a) *if* $x \in D \cap G$*, then* $D'.G' = D.G - 1$.
(b) *if* $x \notin D \cup G$*, then* $D'.G' = D.G + 1$.
(c) *if* $x \in D$ *but* $x \notin G$*, then* $D'.G' = D.G$.

(4) *If* D *is a unisecant curve on* X*, then* $\nu_* \nu^* D = D + Pf$.

Let C_0 be the minimum self-intersection curve on X. We know that $C_0^2 = -e$ and for any other curve D on X, we have $D^2 \geq -e$. Moreover assume that $x \in C_0$. Let C_0' denote the strict transform of C_0 by the elementary transformation X at x. From Proposition 4 it follows that $C_0'^2 = C_0^2 - 1$, but for any other unisecant curve D we have $D'^2 \geq D^2 - 1$. It means that $D'^2 \geq C_0'^2$ and C_0' is the minimum self-intersection curve on X'. Since $C_0'^2 = -e-1$, then $e' = e+1$.

In this way we gave the idea of the following

Theorem 3 ([3], 4.9)
Let $\pi \colon \mathbb{P}(\mathcal{E}_0) \longrightarrow C$ *be a ruled surface. Fix a point* x *on the minimum self-intersection curve* C_0 *on* X*, with* $\pi(x) = P$*. Let* X' *denote the elementary transform of* X *at* x*. Then* X' *is a ruled surface corresponding to a normalized sheaf* $\mathcal{E}_0'$ *with* $\bigwedge^2 \mathcal{E}_0' \cong \mathcal{O}_C(\mathfrak{e}')$ *satisfying* $\mathfrak{e}' \sim \mathfrak{e} - P$ $(e' = e+1)$*. Furthermore, the minimum self-intersection curve on* X' *is* C_0'.

Let X_0 be an indecomposable ruled surface over a curve C of genus g and invariant e. If we apply an elementary transformation to X at a point on C_0, then we obtain a ruled surface X_1 with invariant $e_1 = e+1$ (from Theorem 3). We can take n such transformations so that $e_n = e+n > 2g-2$. This means that after n steps the surface X_n is decomposable (see Theorem 1). Applying Remark 3 to surfaces X and X_n, we have that X can be obtained from X_n by elementary transformations. We proved the following

Remark 4 ([3], 4.10)
Any indecomposable ruled surface is obtained from a decomposable one by a finite number of elementary transformations.

We can say more, namely

Remark 5 ([3], 4.11)
Any ruled surface over the curve C is obtained from $C \times \mathbb{P}^1$ applying a finite number of elementary transformations.

From Remark 5 follows that every ruled surface is birationally ruled surface (compare with Remark 1).

Remark 3 and Theorem 3 give us useful tools to study numerical properties of transformed divisors.

Proposition 5
Let $\nu: X' \longrightarrow X$ be the elementary transformation at $x \in C_0$.

(a) *Let D be a divisor on X. If $D \equiv aC_0 + bf$ with integers a and b, then $\nu^* D \equiv aC_0' + (a+b)f'$, where C_0' and f' generate $Num(X')$.*

(b) *Let Y be a divisor on X'. If $Y \equiv pC_0' + qf'$ with integers p and q, then $\nu_* Y \equiv pC_0 + qf$.*

Proof. We are using the notation introduced in the definition of an elementary transformation and in the previous propositions.

Part (a). Let

$$\nu^* D \equiv pC_0' + qf', \qquad \text{with } p, q \in \mathbb{Z}. \tag{1}$$

From Proposition 2 we have that for any fiber f'

$$(\nu^* D).f' = p,$$

but

$$(\nu^* D).f' = (\tau_* \sigma^* D).f' = (\sigma^* D).(\tau^* f').$$

Let $\tau(y) \in f'$. In our notation it means that $f' = Pf'$. Then

$$\begin{aligned}(\nu^* D).f' &= (\sigma^* D).(\widetilde{Pf'} + E') = (\sigma^* D).E + (\sigma^* D).\widetilde{Pf} \\ &= (\sigma^* D).(\sigma^*(Pf)) - (\sigma^* D).E = D.(Pf) \\ &= a.\end{aligned}$$

If $\tau(y) \notin f'$, then

$$(\nu^* D).f' = (\sigma^* D).(\tau^* f') = (\sigma^* D).\widetilde{f'} = (\sigma^* D).\widetilde{f} = (\sigma^* D).(\sigma^* f) = D.f = a.$$

In this way we proved $p = a$.

To show that it holds $q = a + b$, it is enough to test the intersection product $(\nu^* D).C_0'$.

Since $x \in C_0$, then $\tau(y) \notin C_0'$. Moreover from Theorem 3 it follows that

$$C_0'^2 = C_0^2 - 1. \tag{2}$$

By Proposition 2, conditions (1) and (2)

$$(\nu^* D).C'_0 = pC'^2_0 + q = pC^2_0 - p + q. \tag{3}$$

On another hand

$$\begin{aligned}(\nu^* D).C'_0 &= (\sigma^* D).(\tau^* C'_0) = (\sigma^* D).\widetilde{C'_0} = (\sigma^* D).\widetilde{C_0}\\ &= (\sigma^* D).(\sigma^* C_0) - (\sigma^* D).E = D.C_0\\ &= aC^2_0 + b.\end{aligned} \tag{4}$$

Applying the equality $p = a$ for conditions (3) and (4) we have $q = a + b$.

Part (b). Let Pf' denote, as before, the fiber through $\tau(y)$. Moreover assume that

$$\nu_* Y \equiv aC_0 + bf. \tag{5}$$

The idea of the proof for this part is the same as in the part (a). In particular, it is not difficult to see that $a = p$. We concentrate more on the second intersection product i.e. $(\nu_* Y).C_0$.

From conditions (2) and (5) it follows

$$(\nu_* Y).C_0 = aC^2_0 + b = aC'^2_0 + a + b. \tag{6}$$

We have also

$$\begin{aligned}(\nu_* Y).C_0 &= (\sigma_*(\tau^* Y)).C_0 = (\tau^* Y).(\sigma^* C_0) = (\tau^* Y).(\widetilde{C_0} + E)\\ &= (\tau^* Y).\widetilde{C'_0} + (\tau^* Y).\widetilde{Pf'}\\ &= (\tau^* Y).(\tau^* C'_0) + (\tau^* Y).(\tau^* Pf' - E')\\ &= Y.C'_0 + Y.Pf'\\ &= pC'^2_0 + q + p.\end{aligned} \tag{7}$$

Applying the equality $a = p$ to (6) and (8) we see that $b = q$.

PROPOSITION 6

For any n-secant curve D on X its strict transform D' on X' is still an n-secant curve.

Proof. Let $D \equiv nC_0 + bf$ be an n-secant curve on X and let $\nu\colon X' \longrightarrow X$ be an elementary transformation at a point x. From Proposition 4 it follows that

$$D' = \nu^* D - (\operatorname{mult}_x D) \cdot Pf'.$$

Hence by Proposition 5 we have:

(a) if $x \in C_0$, then $D' \equiv nC'_0 + (n + b - \operatorname{mult}_x D)f'$;

(b) if $x \notin C_0$, then $D' \equiv nC'_0 + (b - \operatorname{mult}_x D)f'$.

Let $G \equiv C_0 + \mu_0 f$ be an ample divisor on X. The question is: what happens with ampleness of the strict transform G' ? Is G' still ample? In general G' need not to be ample. More precisely we can formulate the following

PROPOSITION 7
With above assumptions, the strict transform G' is ample except when

(i) *in the case $e > 0$ we have $G \equiv C_0 + (e+1)f$ and we apply an elementary transformation at a point $x \in C_0$ which is also a base point of $|G|$,*

(ii) *in the case $e < 0$ and e odd we have $G \equiv C_0 + \frac{1}{2}(e+1)f$ and we apply an elementary transformation at a base point of $|G|$.*

Proof. Let X be a ruled surface with invariant e, and let $D \in |G|$. As before, by $\nu : X' \longrightarrow X$ we denote the elementary transformation at a point $x \in X$.

Case (1). If $x \in C_0$, then by Theorem 3 the surface X' is ruled with invariant $e' = e + 1$. Moreover by Proposition 5 the strict transform $D' \equiv C_0' + (\mu_0 + 1 - \operatorname{mult}_x D)f'$.

From Theorem 2 it follows that:

(a) for $e \geq 0$ we have $\mu_0 \geq e + 1$ and
$$\mu_0 + 1 - \operatorname{mult}_x D \geq e' + 1 - \operatorname{mult}_x D,$$
hence D' is not ample if $\mu_0 = e + 1$ and $x \in D$;

(b) for $e < 0$ and e even, $\mu_0 \geq \frac{1}{2}e + 1$ and
$$\mu_0 + 1 - \operatorname{mult}_x D \geq \frac{1}{2}(e' + 1) + 1 - \operatorname{mult}_x D,$$
then D' always is ample;

(c1) for $e = -1$ we have $\mu_0 \geq 0$ and
$$\mu_0 + 1 - \operatorname{mult}_x D \geq 1 - \operatorname{mult}_x D;$$

(c2) for $e < -1$ and e odd, $\mu_0 \geq \frac{1}{2}(e + 1)$ and
$$\mu_0 + 1 - \operatorname{mult}_x D \geq \frac{1}{2}e' + 1 - \operatorname{mult}_x D.$$

It means that D' is not ample if $\mu_0 = \frac{1}{2}(e + 1)$ and $x \in D$.

Case (2). If $x \notin C_0$, then by Theorem 3 and Remark 3 the surface X' is the ruled surface with invariant $e' = e - 1$. By Proposition 5 the strict transform $D' \equiv C_0' + (\mu_0 - \operatorname{mult}_x D)f'$.

Using the same technique we have:

(a1) for $e > 0$ the coefficient

$$\mu_0 - \mathrm{mult}_x D \geq e' + 2 - \mathrm{mult}_x D,$$

(a2) for $e = 0$ we have $\mu_0 \geq 1$ and $\mu_0 - \mathrm{mult}_x D > -\frac{1}{2}$,

it means that for $e \geq 0$ the strict transform D' is always ample;

(b) for $e < 0$ and e even,

$$\mu_0 - \mathrm{mult}_x D \geq \frac{1}{2}e' + \frac{3}{2} - \mathrm{mult}_x D,$$

and D' is always ample;

(c) for $e < 0$ and e odd

$$\mu_0 - \mathrm{mult}_x D \geq \frac{1}{2}e' + 1 - \mathrm{mult}_x D,$$

and D' is not ample if $\mu_0 = \frac{1}{2}(e+1)$ and $x \in D$.

4. Seshadri constants

The concept of Seshadri constants was introduced by Damailly. He associated a real number $\varepsilon(L;x)$ with an ample line bundle L at a point x of an algebraic variety X. This number in effect measures how much of positivity of L can be concentrated at x.

In this section we calculate Seshadri constant for a ruled surface X with a unisecant polarization i.e. an ample line bundle of type $L \equiv C_0 + \mu_0 f$.

Let us recall the definition and some properties of Seshadri constants.

Definition 4
Let L be a nef line bundle on a smooth projective variety X. Fix a point x on X. Let $\sigma\colon X_x \longrightarrow X$ be the blowing-up of X at the point x with the exceptional divisor $E = \sigma^{-1}(x)$. The Seshadri constant of L at x is the non-negative real number

$$\varepsilon(L;x) = \sup\{\varepsilon \in \mathbb{R} \mid \ \sigma^* L - \varepsilon E \text{ is nef}\}.$$

From Kleiman's nefness criterion it follows that $\varepsilon(L;x) \leq \sqrt[\dim X]{L^{\dim X}}$. If the value of $\varepsilon(L;x)$ is less than the previous upper bound, then we say that the Seshadri constant is *L-submaximal* (or simply *submaximal*).

Remark 6
We can define the Seshadri constant of L at x as

$$\varepsilon(L;x) = \inf_{D \ni x} \left\{ \frac{L.D}{\operatorname{mult}_x D} \right\}$$

where the infimum is taken over all (irreducible) curves D (see [5] 5.1.5).

If $\frac{L.D}{\operatorname{mult}_x D} = \varepsilon(L;x)$, then we say that the curve D *computes* the Seshadri constant.

Assume moreover that L is an ample line bundle. For a fixed point $x \in X$, we denote by $\mathfrak{m}_x \subset \mathcal{O}_X$ its maximal ideal.

Definition 5
We say that the complete linear system $|L|$ separates s-jets at x, if the natural map

$$H^0(L) \longrightarrow H^0\left(L \otimes \mathcal{O}_X/\mathfrak{m}_x^{s+1}\right)$$

taking sections of L to their s-jets is surjective. By $s(L,x)$ we denote the maximal number such that $|L|$ separates s-jets at x.

Using above terminology we have the following

Proposition 8 ([5], 5.1.17)
For an ample line bundle L on X

$$\varepsilon(L;x) = \limsup_{k \to \infty} \frac{s(kL,x)}{k}.$$

Theorem 4 (Main theorem)
If X is a ruled surface with an invariant e and a polarization $L \equiv C_0 + \mu_0 f$, then for every point $x \in X$ the Seshadri constant $\varepsilon(L;x) = 1$.

Proof. Since $L \equiv C_0 + \mu_0 f$ is ample, then from Theorem 2 it follows that:

(a) $\mu_0 \geq e + 1$ for $e \geq 0$;

(b) $\mu_0 \geq \frac{1}{2}e + 1$ for $e < 0$ and e even;

(c) $\mu_0 \geq \frac{1}{2}(e+1)$ for $e < 0$ and e odd.

Fix a point $x \in X$. Let $D \equiv aC_0 + bf$ with $a, b \in \mathbb{Z}$, be an irreducible curve on X different from C_0 and a fiber f. By m we denote the multiplicity of D at the point x. Since D is a-secant it must be $m \leq a$.

To calculate the Seshadri constant in cases (a) and (b), it is enough to study the Seshadri quotients i.e. $\frac{L.D}{m} = \frac{b + a(\mu_0 - e)}{m}$.

By assumption D is an irreducible curve. By Theorem 2 we have two cases to consider.

Case $a > 0$ and $b \geq ae$.
This implies

$$\frac{L.D}{m} \geq \frac{a(e+1)}{m} \geq e+1 \geq 1.$$

Case ($a = 1$ and $b > 0$) or ($a \geq 2$ and $b \geq \frac{1}{2}ae$).
If $a = 1$ then also $m = 1$ and we have

$$\frac{L.D}{m} \geq 1 - \frac{1}{2}e > 1.$$

Also it is easy to check that for $a \geq 2$ and $b \geq \frac{1}{2}ae$ the Seshadri quotient satisfies

$$\frac{L.D}{m} \geq \frac{a}{m} \geq 1.$$

Thus we showed that $\varepsilon(L;x) \geq 1$.

For any ruled surface X and a unisecant line bundle L we have $\frac{L.Pf}{\mathrm{mult}_x Pf} = 1$, where Pf is the fiber through x. Thus $\varepsilon(L;x) = 1$ and Pf computes the Seshadri constant.

Using the same method in the case (c) we have:
for $a = 1$ and $b > 0$ the quotient

$$\frac{L.D}{m} \geq \frac{1}{2} - \frac{1}{2}e \geq 1,$$

but for $a \geq 2$ and $b \geq \frac{1}{2}ae$ it follows only that:

$$\frac{L.D}{m} \geq \frac{1}{2},$$

and it means that we still do not know the value of the Seshadri constant at the point x.

To prove that $\varepsilon(L;x) = 1$ for $e < 0$ and e odd, we use a different method.

Let p be a point on X such that $p \neq x$ and p not a base point of $|L|$. Apply the elementary transformation at the point p. Since $x \neq p$, then $\nu^{-1}(x) = x$. Moreover from Proposition 7 it follows that L' is an ample line bundle on the surface X' with even invariant e'. By (2) we have that $\varepsilon(L';x) = 1$. Separating s-jets at x is a local property of L at x and the elementary transformation change the surface X only in the neighborhood of the fiber through p. It means that we can choose p such that $s(L;x) = s(L';x)$. Note that there is an obvious isomorphism $H^0(X,L) \cong H^0(X',L')$. Then Proposition 8 implies $\varepsilon(L;x) = \varepsilon(L';x) = 1$.

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