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$*$-g-frames in tensor products of Hilbert C^*-modules

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Abstract. In this paper, we study $*$-g-frames in tensor products of Hilbert C^*-modules. We show that a tensor product of two $*$-g-frames is a $*$-g-frame, and we get some result.

1. Introduction

Frames for Hilbert spaces were introduced in 1952 by Duffin and Schaefer [9]. They abstracted the fundamental notion of Gabor [11] to study signal processing. Many generalizations of frames were introduced, frames of subspaces [3], Pseudo-frames [16], oblique frames [6], g-frames [14], $*$-frame [2] in Hilbert C^*-modules. In 2000, Frank-Larson [10] introduced the notion of frames in Hilbert C^*-modules as a generalization of frames in Hilbert spaces. Recentely, A. Khosravi and B. Khosravi [14] introduced the g-frame theory in Hilbert C^*-modules, and Alijani, and Dehghan [2] introduced the g-frame theory in Hilbert C^*-modules. N. Bounader and S. Kabbaj [4] and A. Alijani [1] introduced the $*$-g-frames which are generalizations of g-frames in Hilbert C^*-modules. In this article, we study the $*$-g-frames in tensor products of Hilbert C^*-modules and $*$-g-frames in two Hilbert C^*-modules with different C^*-algebras. In section 2, we briefly recall the definitions and basic properties of C^*-algebra, Hilbert C^*-modules, frames, g-frames, $*$-frames and $*$-g-frames in Hilbert C^*-modules. In section 3, we investigate tensor product of Hilbert C^*-modules, we show that tensor product of $*$-g-frames for Hilbert C^*-modules $\mathcal{H}$ and $\mathcal{K}$, present $*$-g-frames for $\mathcal{H}\otimes\mathcal{K}$, and tensor product of their $*$-g-frame operators is the $*$-g-frame operator of the tensor product of $*$-g-frames. We also study $*$-g-frames in two Hilbert C^*-modules with different C^*-algebras.

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2. Preliminaries

Let I and J be countable index sets. In this section we briefly recall the definitions and basic properties of C^*-algebra, Hilbert C^*-modules, g-frame, $*$-g-frame in Hilbert C^*-modules. For information about frames in Hilbert spaces we refer to [5]. Our reference for C^*-algebras is [8, 7]. For a C^*-algebra $\mathcal{A}$, an element $a \in \mathcal{A}$ is positive ($a \geq 0$) if $a = a^*$ and $sp(a) \subset \mathbf{R}^+$. $\mathcal{A}^+$ denotes the set of positive elements of $\mathcal{A}$.

Definition 2.1 ([13])
Let $\mathcal{A}$ be a unital C^*-algebra and $\mathcal{H}$ be a left $\mathcal{A}$-module such that the linear structures of $\mathcal{A}$ and $\mathcal{H}$ are compatible. $\mathcal{H}$ is a pre-Hilbert $\mathcal{A}$-module if $\mathcal{H}$ is equipped with an $\mathcal{A}$-valued inner product $\langle .,. \rangle_{\mathcal{A}} : \mathcal{H} \times \mathcal{H} \to \mathcal{A}$ such that is sesquilinear, positive definite and respects the module action. In the other words,

(i) $\langle x, x\rangle_{\mathcal{A}} \geq 0$ for all $x \in \mathcal{H}$ and $\langle x, x\rangle_{\mathcal{A}} = 0$ if and only if $x = 0$,

(ii) $\langle ax + y, z\rangle_{\mathcal{A}} = a\langle x, y\rangle_{\mathcal{A}} + \langle y, z\rangle_{\mathcal{A}}$ for all $a \in \mathcal{A}$ and $x, y, z \in \mathcal{H}$,

(iii) $\langle x, y\rangle_{\mathcal{A}} = \langle y, x\rangle_{\mathcal{A}}^*$ for all $x, y \in \mathcal{H}$.

For $x \in \mathcal{H}$, we define $\|x\| = \|\langle x, x\rangle_{\mathcal{A}}\|^{\frac{1}{2}}$. If $\mathcal{H}$ is complete with $\|.\|$, it is called a Hilbert $\mathcal{A}$-module or a Hilbert C^*-module over $\mathcal{A}$. For every a in C^*-algebra $\mathcal{A}$, we have $|a| = (a^*a)^{\frac{1}{2}}$ and the $\mathcal{A}$-valued norm on $\mathcal{H}$ is defined by $|x| = \langle x, x\rangle_{\mathcal{A}}^{\frac{1}{2}}$ for $x \in \mathcal{H}$.

Let $\mathcal{H}$ and $\mathcal{K}$ be two Hilbert $\mathcal{A}$-modules. A map $T \colon \mathcal{H} \to \mathcal{K}$ is said to be adjointable if there exists a map $T^* \colon \mathcal{K} \to \mathcal{H}$ such that $\langle Tx, y\rangle_{\mathcal{A}} = \langle x, T^*y\rangle_{\mathcal{A}}$ for all $x \in \mathcal{H}$ and $y \in \mathcal{K}$.

From now on, we assume that $\{V_i\}_{i\in I}$ and $\{W_j\}_{j\in J}$ are two sequences of Hilbert $\mathcal{A}$-modules. We also reserve the notation $End_{\mathcal{A}}^*(\mathcal{H}, \mathcal{K})$ for the set of all adjointable operators from $\mathcal{H}$ to $\mathcal{K}$ and $End_{\mathcal{A}}^*(\mathcal{H}, \mathcal{H})$ is abbreviated to $End_{\mathcal{A}}^*(\mathcal{H})$.

Definition 2.2 ([13])
Let $\mathcal{H}$ be a Hilbert $\mathcal{A}$-module. A family $\{x_i\}_{i\in I}$ of elements of $\mathcal{H}$ is a frame for $\mathcal{H}$, if there exist two positive constants A, B such that for all $x \in \mathcal{H}$,

$$A\langle x, x\rangle_{\mathcal{A}} \leq \sum_{i\in I} \langle x, x_i\rangle_{\mathcal{A}} \langle x_i, x\rangle_{\mathcal{A}} \leq B\langle x, x\rangle_{\mathcal{A}}. \tag{1}$$

The numbers A and B are called lower and upper bound of the frame, respectively. If $A = B = \lambda$, the frame is λ-tight. If $A = B = 1$, it is called a normalized tight frame or a Parseval frame. If the sum in the middle of (1) is convergent in norm, the frame is called standard.

Definition 2.3 ([14])
Let $\mathcal{H}$ and $\mathcal{K}$ be Hilbert $\mathcal{A}$-modules and for each $i \in I$, V_i be a closed submodule of $\mathcal{K}$. We call a sequence $\{\Lambda_i \in End_{\mathcal{A}}^*(\mathcal{H}, V_i) : \ i \in I\}$ a g-frame in Hilbert $\mathcal{A}$-module $\mathcal{H}$ with respect to $\{V_i : \ i \in I\}$ if there exist two positive constants C, D such that for all $x \in \mathcal{H}$,

$$C\langle x, x\rangle_{\mathcal{A}} \leq \sum_{i\in I} \langle \Lambda_i x, \Lambda_i x\rangle_{\mathcal{A}} \leq D\langle x, x\rangle_{\mathcal{A}}. \tag{2}$$

The numbers C and D are called lower and upper bound of the g-frame, respectively. If $C = D = \lambda$, the g-frame is λ-tight. If $C = D = 1$, it is called a g-Parseval frame. If the sum in the middle of (2) is convergent in norm, the g-frame is called standard.

Definition 2.4 ([2])
Let $\mathcal{H}$ be a Hilbert $\mathcal{A}$-module. A family $\{x_i\}_{i\in I}$ of elements of $\mathcal{H}$ is a $*$-frame for $\mathcal{H}$, if there exist strictly non-zero elements A, B in $\mathcal{A}$ such that for all $x \in \mathcal{H}$,

$$A\langle x, x\rangle_{\mathcal{A}} A^* \leq \sum_{i\in I} \langle x, x_i\rangle_{\mathcal{A}} \langle x_i, x\rangle_{\mathcal{A}} \leq B\langle x, x\rangle_{\mathcal{A}} B^*. \tag{3}$$

The numbers A and B are called lower and upper bound of the $*$-frame, respectively. If $A = B = \lambda$, the $*$-frame is λ-tight. If $A = B = 1$, it is called a normalized tight $*$-frame or a Parseval $*$-frame. If the sum in the middle of (3) is convergent in norm, the $*$-frame is called standard.

Definition 2.5 ([4])
Let $\mathcal{H}$ and $\mathcal{K}$ be Hilbert $\mathcal{A}$-modules and for each $i \in I$, V_i be a closed submodule of $\mathcal{K}$. We call a sequence $\{\Lambda_i \in End^*_{\mathcal{A}}(\mathcal{H}, V_i) : \ i \in I\}$ a $*$-g-frame in Hilbert $\mathcal{A}$-module $\mathcal{H}$ with respect to $\{V_i : \ i \in I\}$ if there exist strictly non-zero elements A, B in $\mathcal{A}$ such that for all $x \in \mathcal{H}$,

$$A\langle x, x\rangle_{\mathcal{A}} A^* \leq \sum_{i\in I} \langle \Lambda_i x, \Lambda_i x\rangle_{\mathcal{A}} \leq B\langle x, x\rangle_{\mathcal{A}} B^*. \tag{4}$$

The numbers A and B are called lower and upper bound of the $*$-g-frame, respectively. If $A = B = \lambda$, the $*$-g-frame is λ-tight. If $A = B = 1$, it is called a $*$-g-Parseval frame. If the sum in the middle of (4) is convergent in norm, the $*$-g-frame is called standard.
The $*$-g-frame operator S_Λ is defined by $S_\Lambda x = \sum_{i\in I} \Lambda_i^* \Lambda_i x$ for all $x \in \mathcal{H}$.

3. Main results

Suppose that $\mathcal{A}, \mathcal{B}$ are C^*-algebras and we take $\mathcal{A} \otimes \mathcal{B}$ as the completion of $\mathcal{A} \otimes_{alg} \mathcal{B}$ with the spatial norm. $\mathcal{A} \otimes \mathcal{B}$ is the spatial tensor product of $\mathcal{A}$ and $\mathcal{B}$, also suppose that $\mathcal{H}$ is a Hilbert $\mathcal{A}$-module and $\mathcal{K}$ is a Hilbert $\mathcal{B}$-module. We want to define $\mathcal{H} \otimes \mathcal{K}$ as a Hilbert $(\mathcal{A} \otimes \mathcal{B})$-module. Start by forming the algebraic tensor product $\mathcal{H} \otimes_{alg} \mathcal{K}$ of the vector spaces $\mathcal{H}$, $\mathcal{K}$ (over $\mathbb{C}$). This is a left module over $(\mathcal{A} \otimes_{alg} \mathcal{B})$ (the module action being given by $(a \otimes b)(x \otimes y) = ax \otimes by$ $(a \in \mathcal{A}, b \in \mathcal{B}, x \in \mathcal{H}, y \in \mathcal{K})$). For $(x_1, x_2 \in \mathcal{H}, y_1, y_2 \in \mathcal{K})$ we define

$$\langle x_1 \otimes y_1, x_2 \otimes y_2\rangle_{\mathcal{A}\otimes\mathcal{B}} = \langle x_1, x_2\rangle_{\mathcal{A}} \otimes \langle y_1, y_2\rangle_{\mathcal{B}}.$$

We also know that for $z = \sum_{i=1}^n x_i \otimes y_i$ in $\mathcal{H} \otimes_{alg} \mathcal{K}$ we have $\langle z, z\rangle_{\mathcal{A}\otimes\mathcal{B}} = \sum_{i,j} \langle x_i, x_j\rangle_{\mathcal{A}} \otimes \langle y_i, y_j\rangle_{\mathcal{B}} \geq 0$ and $\langle z, z\rangle_{\mathcal{A}\otimes\mathcal{B}} = 0$ iff $z = 0$. This extends by linearity to an $(\mathcal{A} \otimes_{alg} \mathcal{B})$-valued sesquilinear form on $\mathcal{H} \otimes_{alg} \mathcal{K}$, which makes $\mathcal{H} \otimes_{alg} \mathcal{K}$ into a semi-inner-product module over the pre-$\mathcal{C}^*$-algebra $(\mathcal{A} \otimes_{alg} \mathcal{B})$. The semi-inner-product on $\mathcal{H} \otimes_{alg} \mathcal{K}$ is actually an inner product, see [15]. Then $\mathcal{H} \otimes_{alg} \mathcal{K}$ is an inner-product module over the pre-$\mathcal{C}^*$-algebra $(\mathcal{A} \otimes_{alg} \mathcal{B})$, and we can perform the double completion discussed in chapter 1 of [15] to conclude that the completion $\mathcal{H} \otimes \mathcal{K}$ of $\mathcal{H} \otimes_{alg} \mathcal{K}$ is a Hilbert $(\mathcal{A} \otimes \mathcal{B})$-module. We call $\mathcal{H} \otimes \mathcal{K}$ the exterior tensor product of $\mathcal{H}$ and $\mathcal{K}$. With $\mathcal{H}$, $\mathcal{K}$ as above, we wish to investigate the adjointable operators on $\mathcal{H} \otimes \mathcal{K}$. Suppose that $S \in End^*_{\mathcal{A}}(\mathcal{H})$ and $T \in End^*_{\mathcal{B}}(\mathcal{K})$. We define a linear operator $S \otimes T$ on $\mathcal{H} \otimes \mathcal{K}$ by

$$S \otimes T(x \otimes y) = Sx \otimes Ty \qquad \text{for } x \in \mathcal{H}, y \in \mathcal{K}.$$

It is a routine verification that is $S^* \otimes T^*$ is the adjoint of $S \otimes T$, so in fact $S \otimes T \in End^*_{\mathcal{A} \otimes \mathcal{B}}(\mathcal{H} \otimes \mathcal{K})$. For more details see [8, 15]. We note that if $a \in \mathcal{A}^+$ and $b \in \mathcal{B}^+$, then $a \otimes b \in (\mathcal{A} \otimes \mathcal{B})^+$. Plainly if a, b are Hermitian elements of $\mathcal{A}$ and $a \geq b$, then for every positive element x of $\mathcal{B}$, we have $a \otimes x \geq b \otimes x$.

For the proof of our main results, we need the followings lemma and result.

Lemma 3.1 ([2])
If $\varphi\colon \mathcal{A} \to \mathcal{B}$ is a $*$-homomorphism between $\mathcal{C}^*$-algebras, then φ is increasing, that is, if $a \leq b$, then $\varphi(a) \leq \varphi(b)$.

Result 3.2 ([13])
If $Q \in End^*_{\mathcal{A}}(\mathcal{H})$ is an invertible $\mathcal{A}$-linear map then for all $z \in \mathcal{H} \otimes \mathcal{K}$ we have

$$\|Q^{*-1}\|^{-1} \cdot |z| \leq |(Q^* \otimes I)z| \leq \|Q\| \cdot |z|.$$

Theorem 3.3
Let $\mathcal{H}$ and $\mathcal{K}$ be two Hilbert C^*-modules over unitary C^*-algebras $\mathcal{A}$ and $\mathcal{B}$, respectively. Let $\{\Lambda_i\}_{i\in I} \subset End^*_{\mathcal{A}}(\mathcal{H}, V_i)$ and $\{\Gamma_j\}_{j\in J} \subset End^*_{\mathcal{B}}(\mathcal{K}, W_i)$ be two $*$-g-frames for $\mathcal{H}$ and $\mathcal{K}$ with $*$-g-frame operators S_Λ and S_Γ and $*$-g-frame bounds (A, B) and (C, D), respectively. Then $\{\Lambda_i \otimes \Gamma_j\}_{i\in I, j\in J}$ is a $*$-g-frame for Hibert $\mathcal{A}\otimes\mathcal{B}$-module $\mathcal{H}\otimes\mathcal{K}$ with $*$-g-frame operator $S_\Lambda \otimes S_\Gamma$ and lower and upper $*$-g-frame bounds $A \otimes C$ and $B \otimes D$, respectively.

Proof. By the definition of $*$-g-frames $\{\Lambda_i\}_{i\in I}$ and $\{\Gamma_j\}_{j\in J}$ we have

$$A\langle x, x\rangle_{\mathcal{A}} A^* \leq \sum_{i\in I} \langle \Lambda_i x, \Lambda_i x\rangle_{\mathcal{A}} \leq B\langle x, x\rangle_{\mathcal{A}} B^* \qquad \text{for all } x \in \mathcal{H}.$$

$$C\langle y, y\rangle_{\mathcal{B}} C^* \leq \sum_{j\in J} \langle \Gamma_j y, \Gamma_j y\rangle_{\mathcal{B}} \leq D\langle y, y\rangle_{\mathcal{B}} D^* \qquad \text{for all } y \in \mathcal{K}.$$

Therefore, for all $x \in \mathcal{H}$ and all $y \in \mathcal{K}$,

$$\begin{aligned}(A\langle x, x\rangle_{\mathcal{A}} A^*) \otimes (C\langle y, y\rangle_{\mathcal{B}} C^*) &\leq \sum_{i\in I} \langle \Lambda_i x, \Lambda_i x\rangle_{\mathcal{A}} \otimes \sum_{j\in J} \langle \Gamma_j y, \Gamma_j y\rangle_{\mathcal{B}} \\ &\leq (B\langle x, x\rangle_{\mathcal{A}} B^*) \otimes (D\langle y, y\rangle_{\mathcal{B}} D^*)\end{aligned}$$

Then

$$\begin{aligned}(A \otimes C)(\langle x, x\rangle_{\mathcal{A}} \otimes \langle y, y\rangle_{\mathcal{B}})(A^* \otimes C^*) &\leq \sum_{i\in I, j\in J} \langle \Lambda_i x, \Lambda_i x\rangle_{\mathcal{A}} \otimes \langle \Gamma_j y, \Gamma_j y\rangle_{\mathcal{B}} \\ &\leq (B \otimes D)(\langle x, x\rangle_{\mathcal{A}} \otimes \langle y, y\rangle_{\mathcal{B}})(B^* \otimes D^*).\end{aligned}$$

Consequently,

$$\begin{aligned}(A\otimes C)\langle x\otimes y, x\otimes y\rangle_{\mathcal{A}\otimes\mathcal{B}}(A\otimes C)^* &\leq \sum_{i\in I, j\in J}\langle \Lambda_i x\otimes \Gamma_j y, \Lambda_i x\otimes\Gamma_j y\rangle_{\mathcal{A}\otimes\mathcal{B}}\\ &\leq (B\otimes D)\langle x\otimes y, x\otimes y\rangle_{\mathcal{A}\otimes\mathcal{B}}(B\otimes D)^*.\end{aligned}$$

Then for all $x\otimes y\in\mathcal{H}\otimes\mathcal{K}$ we have

$$\begin{aligned}&(A\otimes C)\langle x\otimes y, x\otimes y\rangle_{\mathcal{A}\otimes\mathcal{B}}(A\otimes C)^*\\ &\qquad\leq \sum_{i\in I, j\in J}\langle (\Lambda_i\otimes\Gamma_j)(x\otimes y), (\Lambda_i\otimes\Gamma_j)(x\otimes y)\rangle_{\mathcal{A}\otimes\mathcal{B}}\\ &\qquad\leq (B\otimes D)\langle x\otimes y, x\otimes y\rangle_{\mathcal{A}\otimes\mathcal{B}}(B\otimes D)^*.\end{aligned}$$

The last inequality is satisfied for every finite sum of elements in $\mathcal{H}\otimes_{alg}\mathcal{K}$ and then it is satisfied for all $z\in\mathcal{H}\otimes\mathcal{K}$. It shows that $\{\Lambda_i\otimes\Gamma_j\}_{i\in I, j\in J}$ is $*$-g-frame for Hibert $\mathcal{A}\otimes\mathcal{B}$-module $\mathcal{H}\otimes\mathcal{K}$ with lower and upper $*$-g-frame bounds $A\otimes C$ and $B\otimes D$, respectively.

By the definition of $*$-g-frame operator S_Λ and S_Γ we have

$$S_\Lambda x = \sum_{i\in I}\Lambda_i^*\Lambda_i x \qquad \text{for all } x\in\mathcal{H}$$

and

$$S_\Gamma y = \sum_{j\in J}\Gamma_j^*\Gamma_j y \qquad \text{for all } y\in\mathcal{K}.$$

Therefore

$$\begin{aligned}(S_\Lambda\otimes S_\Gamma)(x\otimes y) &= S_\Lambda x\otimes S_\Gamma y\\ &= \sum_{i\in I}\Lambda_i^*\Lambda_i x\otimes\sum_{j\in J}\Gamma_j^*\Gamma_j y\\ &= \sum_{i\in I, j\in J}\Lambda_i^*\Lambda_i x\otimes\Gamma_j^*\Gamma_j y\\ &= \sum_{i\in I, j\in J}(\Lambda_i^*\otimes\Gamma_j^*)(\Lambda_i x\otimes\Gamma_j y)\\ &= \sum_{i\in I, j\in J}(\Lambda_i^*\otimes\Gamma_j^*)(\Lambda_i\otimes\Gamma_j)(x\otimes y)\\ &= \sum_{i\in I, j\in J}(\Lambda_i\otimes\Gamma_j)^*)(\Lambda_i\otimes\Gamma_j)(x\otimes y).\end{aligned}$$

Now by the uniqueness of $*$-g-frame operator, the last expression is equal to $S_{\Lambda\otimes\Gamma}(x\otimes y)$. Consequently we have $(S_\Lambda\otimes S_\Gamma)(x\otimes y) = S_{\Lambda\otimes\Gamma}(x\otimes y)$. The last equality is satisfied for every finite sum of elements in $\mathcal{H}\otimes_{alg}\mathcal{K}$ and then it is satisfied for all $z\in\mathcal{H}\otimes\mathcal{K}$. It shows that $(S_\Lambda\otimes S_\Gamma)(z) = S_{\Lambda\otimes\Gamma}(z)$. So $S_{\Lambda\otimes\Gamma} = S_\Lambda\otimes S_\Gamma$.

Theorem 3.4
If $Q \in End^*_{\mathcal{A}}(\mathcal{H})$ is invertible and $\{\Lambda_i\}_{i\in I} \subset End^*_{\mathcal{A}\otimes\mathcal{B}}(\mathcal{H}\otimes\mathcal{K})$ is a $*$-g-frame for $\mathcal{H}\otimes\mathcal{K}$ with lower and upper $*$-g-frame bounds A and B respectively, and $*$-g-frame operator S, then $\{\Lambda_i(Q^*\otimes I)\}_{i\in I}$ is a $*$-g-frame for $\mathcal{H}\otimes\mathcal{K}$ with lower and upper $*$-g-frame bounds $\|Q^{*-1}\|^{-1}A$ and $\|Q\|B$ respectively, and $*$-g-frame operator $(Q\otimes I)S(Q^*\otimes I)$.

Proof. Since $Q \in End^*_{\mathcal{A}}(\mathcal{H})$, $Q\otimes I \in End^*_{\mathcal{A}\otimes\mathcal{B}}(\mathcal{H}\otimes\mathcal{K})$ with inverse $Q^{-1}\otimes I$. It is obvious that the adjoint of $Q\otimes I$ is $Q^*\otimes I$. An easy calculation shows that for every elementary tensor $x\otimes y$,

$$\begin{aligned}\|(Q\otimes I)(x\otimes y)\|^2 &= \|Q(x)\otimes y\|^2 = \|Q(x)\|^2\|y\|^2 \leq \|Q\|^2\|x\|^2\|y\|^2\\ &= \|Q\|^2\|x\otimes y\|^2.\end{aligned}$$

So $Q\otimes I$ is bounded, and therefore it can be extended to $\mathcal{H}\otimes\mathcal{K}$. Similarly for $Q^*\otimes I$, hence $Q\otimes I$ is $\mathcal{A}\otimes\mathcal{B}$-linear, adjointable with adjoint $Q^*\otimes I$. Hence for every $z\in\mathcal{H}\otimes\mathcal{K}$ we have by result 3.2,

$$\|Q^{*-1}\|^{-1}\cdot|z| \leq |(Q^*\otimes I)z| \leq \|Q\|\cdot|z|.$$

By the definition of $*$-g-frames we have

$$A\langle z,z\rangle_{\mathcal{A}\otimes\mathcal{B}}A^* \leq \sum_{i\in I}\langle\Lambda_i z,\Lambda_i z\rangle_{\mathcal{A}\otimes\mathcal{B}} \leq B\langle z,z\rangle_{\mathcal{A}\otimes\mathcal{B}}B^*.$$

Then

$$\begin{aligned}A\langle(Q^*\otimes I)z,(Q^*\otimes I)z\rangle_{\mathcal{A}\otimes\mathcal{B}}A^* &\leq \sum_{i\in I}\langle\Lambda_i(Q^*\otimes I)z,\Lambda_i(Q^*\otimes I)z\rangle_{\mathcal{A}\otimes\mathcal{B}}\\ &\leq B\langle(Q^*\otimes I)z,(Q^*\otimes I)z\rangle_{\mathcal{A}\otimes\mathcal{B}}B^*.\end{aligned}$$

So

$$\begin{aligned}\|Q^{*-1}\|^{-1}A\langle z,z\rangle_{\mathcal{A}\otimes\mathcal{B}}(\|Q^{*-1}\|^{-1}A)^* &\leq \sum_{i\in I}\langle\Lambda_i(Q^*\otimes I)z,\Lambda_i(Q^*\otimes I)z\rangle_{\mathcal{A}\otimes\mathcal{B}}\\ &\leq \|Q\|B\langle z,z\rangle_{\mathcal{A}\otimes\mathcal{B}}(\|Q\|B)^*.\end{aligned}$$

Now

$$\begin{aligned}(Q\otimes I)S(Q^*\otimes I) &= (Q\otimes I)\big(\sum_{i\in I}\Lambda_i^*\Lambda_i\big)(Q^*\otimes I)\\ &= \sum_{i\in I}(Q\otimes I)\Lambda_i^*\Lambda_i(Q^*\otimes I)\\ &= \sum_{i\in I}(\Lambda_i(Q^*\otimes I))^*\Lambda_i(Q^*\otimes I).\end{aligned}$$

Which completes the proof.

Theorem 3.5

Let $(\mathcal{H}, \mathcal{A}, \langle .,. \rangle_{\mathcal{A}})$ and $(\mathcal{H}, \mathcal{B}, \langle .,. \rangle_{\mathcal{B}})$ be two Hilbert $\mathcal{C}^*$-modules and let $\varphi\colon \mathcal{A} \to \mathcal{B}$ be a $*$-homomorphism and θ be a map on $\mathcal{H}$ such that $\langle \theta x, \theta y\rangle_{\mathcal{B}} = \varphi(\langle x, y\rangle_{\mathcal{A}})$ for all $x, y \in \mathcal{H}$. Also, suppose that $\{\Lambda_i\}_{i\in I} \subset End^*_{\mathcal{A}}(\mathcal{H}, V_i)$ (where V_i is a closed submodule of $\mathcal{H}$ for each i in I) is a $*$-g-frame for $(\mathcal{H}, \mathcal{A}, \langle .,. \rangle_{\mathcal{A}})$ with $*$-g-frame operator $S_{\mathcal{A}}$ and lower and upper $*$-g-frame bounds A, B, respectively. If θ is surjective and $\theta\Lambda_i = \Lambda_i\theta$ for each i in I, then $\{\Lambda_i\}_{i\in I}$ is a $*$-g-frame for $(\mathcal{H}, \mathcal{B}, \langle .,. \rangle_{\mathcal{B}})$ with $*$-g-frame operator $S_{\mathcal{B}}$ and lower and upper $*$-g-frame bounds $\varphi(A)$, $\varphi(B)$ respectively, and $\langle S_{\mathcal{B}}\theta x, \theta y\rangle_{\mathcal{B}} = \varphi(\langle S_{\mathcal{A}} x, y\rangle_{\mathcal{A}})$.

Proof. Let $y \in \mathcal{H}$ then there exists $x \in \mathcal{H}$ such that $\theta x = y$ (θ is surjective). By the definition of $*$-g-frames we have

$$A\langle x, x\rangle_{\mathcal{A}} A^* \le \sum_{i\in I} \langle \Lambda_i x, \Lambda_i x\rangle_{\mathcal{A}} \le B\langle x, x\rangle_{\mathcal{A}} B^*.$$

By lemma 3.1 we obtain

$$\varphi(A\langle x, x\rangle_{\mathcal{A}} A^*) \le \varphi\big(\sum_{i\in I} \langle \Lambda_i x, \Lambda_i x\rangle_{\mathcal{A}}\big) \le \varphi(B\langle x, x\rangle_{\mathcal{A}} B^*).$$

The definition of $*$-homomorphism yields

$$\varphi(A)\varphi(\langle x, x\rangle_{\mathcal{A}})\varphi(A^*) \le \sum_{i\in I} \varphi(\langle \Lambda_i x, \Lambda_i x\rangle_{\mathcal{A}}) \le \varphi(B)\varphi(\langle x, x\rangle_{\mathcal{A}})\varphi(B^*).$$

By the relation between θ and φ we get

$$\varphi(A)\langle \theta x, \theta x\rangle_{\mathcal{B}}\varphi(A)^* \le \sum_{i\in I} \langle \theta\Lambda_i x, \theta\Lambda_i x\rangle_{\mathcal{B}} \le \varphi(B)\langle \theta x, \theta x\rangle_{\mathcal{B}}\varphi(B)^*.$$

By the relation between θ and Λ_i we have

$$\varphi(A)\langle \theta x, \theta x\rangle_{\mathcal{B}}\varphi(A)^* \le \sum_{i\in I} \langle \Lambda_i\theta x, \Lambda_i\theta x\rangle_{\mathcal{B}} \le \varphi(B)\langle \theta x, \theta x\rangle_{\mathcal{B}}\varphi(B)^*.$$

Then

$$\varphi(A)\langle y, y\rangle_{\mathcal{B}}(\varphi(A))^* \le \sum_{i\in I} \langle \Lambda_i y, \Lambda_i y\rangle_{\mathcal{B}} \le \varphi(B)\langle y, y\rangle_{\mathcal{B}}(\varphi(B))^*.$$

for all $y \in \mathcal{H}$. On the other hand,

$$\begin{aligned}
\varphi(\langle S_{\mathcal{A}} x, y\rangle_{\mathcal{A}}) &= \varphi\big(\big\langle \sum_{i\in I} \Lambda_i^*\Lambda_i x, y\big\rangle_{\mathcal{A}}\big) = \sum_{i\in I} \varphi(\langle \Lambda_i x, \Lambda_i y\rangle_{\mathcal{A}}) \\
&= \sum_{i\in I} \langle \theta\Lambda_i x, \theta\Lambda_i y\rangle_{\mathcal{B}} = \sum_{i\in I} \langle \Lambda_i\theta x, \Lambda_i\theta y\rangle_{\mathcal{B}} \\
&= \big\langle \sum_{i\in I} \Lambda_i^*\Lambda_i\theta x, \theta y\big\rangle_{\mathcal{B}} = \langle S_{\mathcal{B}}\theta x, \theta y\rangle_{\mathcal{B}}.
\end{aligned}$$

Which completes the proof.

In the following, we give an example of the function φ in the precedent theorem.

Example 3.6 ([12])
Let X and Y be two locally compact Hausdorff spaces. Let H be a Hilbert space. Let T be a surjective linear isometry from $C_0(X, H)$ onto $C_0(Y, H)$, then there exists a homeomorphism $\phi\colon Y \to X$ and for every $y \in Y$ there is a unitary operator $h(y)\colon H \to H$ such that

$$Tf(y) = h(y)f(\phi(y)).$$

In this case, we have

$$\begin{aligned}\langle Tf, Tg\rangle(y) &= \langle Tf(y), Tg(y)\rangle = \langle h(y)f(\phi(y)), h(y)g(\phi(y))\rangle \\ &= \langle f(\phi(y)), g(\phi(y))\rangle = \langle f, g\rangle \circ \phi(y).\end{aligned}$$

Then

$$\langle Tf, Tg\rangle = \langle f, g\rangle \circ \phi.$$

Let $\varphi\colon C_0(X) \to C_0(Y)$ be the $*$-isomorphism defined by $\varphi(\psi) = \psi\phi$. Then

$$\langle Tf, Tg\rangle = \varphi(\langle f, g\rangle).$$

The example 3.6 is a consequence of Banach-Stone's Theorem.

Example 3.7
Let $\mathcal{A}$ be a C^*-algebra, then

- $\mathcal{A}$ itself is a Hilbert $\mathcal{A}$-module with the inner product $\langle a, b\rangle_r := a^*b$ for $a, b \in \mathcal{A}$,
- $\mathcal{A}$ itself is a Hilbert $\mathcal{A}$-module with the inner product $\langle a, b\rangle_l := ab^*$ for $a, b \in \mathcal{A}$.

Let $\theta\colon \mathcal{A} \to \mathcal{A}$ be the invertible map defined by $\theta(a) = a^*$ and we take φ equal to the identity of $L(\mathcal{A})$. Then

$$\langle \theta a, \theta b\rangle_l = \theta a(\theta b)^* = a^*b = \langle a, b\rangle_r = \varphi(\langle a, b\rangle_r).$$

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