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Local analytic solutions of a functional equation

Dedicated to Professor Andrzej Zajtz
on the occasion of his 70th birthday

Abstract. All analytic solutions of the functional equation

$$|f(r\exp(i\theta))|^2 + |f(1)|^2 = |f(r)|^2 + |f(\exp(i\theta))|^2$$

in the annulus

$$P := \{z \in \mathbb{C} : \ 1-\epsilon < |z| < 1+\epsilon\}$$

and in the domain

$$D := \{z = re^{i\theta} \in \mathbb{C} : \ 1-\epsilon < r < 1+\epsilon, \ \theta \in (-\delta, \delta)\},$$

are found.

1. Introduction

Hiroshi Haruki in [1] studied the following functional equations

$$|f(r\exp(i\theta))|^2 + |f(1)|^2 = |f(r)|^2 + |f(\exp(i\theta))|^2, \tag{1}$$

and

$$|f(r\exp(i\theta))| = |f(r)|, \tag{2}$$

where $r > 0$, θ are real. Equation (1) can be obtained from (2). In fact, let us put $r = 1$ in (2). Then we have

$$|f(\exp(i\theta))| = |f(1)| \tag{3}$$

for $\theta \in \mathbb{R}$. Next squaring (2) and (3) and adding them together we infer (1). Thus (1) is a generalization of (2), i.e., if f is a solution of (2), then it is a solution of (1). In paper [1] H. Haruki showed that all analytic solutions in $\mathbb{C}\setminus\{0\}$ of (1) which are analytic at 0 or have a pole at this point can be written as follows

$$f(z) = Az^p + Bz^{-p}, \tag{4}$$

where A, B are complex constants and p is an integer.

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We are going to prove that the functions of the form (4) are unique analytic solutions of (1) in the annulus

$$P := \{z \in \mathbb{C}:\ 1-\epsilon < |z| < 1+\epsilon\},$$

where $0 < \epsilon \leq 1$ is a constant. We shall also find all analytic solutions of (1) in the domain

$$D := \{z = re^{i\theta} \in \mathbb{C}:\ 1-\epsilon < r < 1+\epsilon,\ \theta \in (-\delta, \delta)\},$$

where $0 < \epsilon \leq 1$ and $0 < \delta \leq \pi$ are given constants. Moreover, we shall determinate all analytic solutions in P and in D of (2) and of the equation

$$|f\left(r \exp\left(i\theta\right)\right)| = |f\left(\exp\left(i\theta\right)\right)|. \tag{5}$$

Of course, (1) is also a generalization of (5).

2. Solutions of (1), (2) and (5) in P

In this section we will be concerned with analytic solutions of equations (1), (2) and (5) in the annulus P.

Theorem 1
If f is an analytic solution of (1) in P, then there exist complex constants A, B and an integer p such that (4) is valid. Conversely, for every complex constants A, B and for every integer p, f given by (4) is a solution of (1).

Proof. It is easy to check that f given by (4) satisfies (1). The function $f(z) \equiv 0$ in P is a solution of (1) of the form (4). Suppose that an analytic function f is a solution of (1) and $f \not\equiv 0$. Of course,

$$f(re^{i\theta})\overline{f(re^{i\theta})} + |f(1)|^2 = |f(r)|^2 + |f(e^{i\theta})|^2 \tag{6}$$

for $\theta \in \mathbb{R}$ and $r \in (1-\epsilon, 1+\epsilon)$. Differentiating (6) at first with respect to r and then with respect to θ we successively infer

$$e^{i\theta} f'(re^{i\theta})\overline{f(re^{i\theta})} + e^{-i\theta} f(re^{i\theta})\overline{f'(re^{i\theta})} = \frac{d}{dr}|f(r)|^2$$

and

$$\begin{aligned} re^{2i\theta} f''(re^{i\theta})\overline{f(re^{i\theta})} - re^{-2i\theta} f(re^{i\theta})\overline{f''(re^{i\theta})} + e^{i\theta} f'(re^{i\theta})\overline{f(re^{i\theta})} \\ - e^{-i\theta} f(re^{i\theta})\overline{f'(re^{i\theta})} \\ = 0. \end{aligned}$$

Let us multiply the obtained equality by r and replace $re^{i\theta}$ by z. Then

$$z^2 f''(z)\overline{f(z)} - \overline{z}^2 f(z)\overline{f''(z)} + zf'(z)\overline{f(z)} - \overline{z}f(z)\overline{f'(z)} = 0,$$

i.e.,

$$\Im[z^2 f''(z)\overline{f(z)} + zf'(z)\overline{f(z)}\,] = 0 \tag{7}$$

for all $z \in P$. Since $f \not\equiv 0$, we can find a disc $V \subset P$ such that $f(z) \neq 0$ for all $z \in V$. The equality $\overline{f(z)} = \frac{|f(z)|^2}{f(z)}$, valid in this disc, and (7) imply

$$\Im\left[\frac{z^2 f''(z) + zf'(z)}{f(z)}\right] = 0$$

for all $z \in V$. Since an analytic function preserves domains, there exists a real constant k such that

$$z^2 f''(z) + zf'(z) - kf(z) = 0 \tag{8}$$

for all $z \in V$. By the Identity Theorem formula (8) remains valid in P. (The above part of the proof is due to H. Haruki, see [1], pp. 130-131). We can find complex numbers a_n, $n \in \mathbb{Z}$ such that for all $z \in P$,

$$f(z) = \sum_{n=-\infty}^{\infty} a_n z^n.$$

Since

$$f'(z) = \sum_{n=-\infty}^{\infty} n a_n z^{n-1}, \qquad f''(z) = \sum_{n=-\infty}^{\infty} n(n-1) a_n z^{n-2}$$

we conclude that

$$0 = z^2 f''(z) + zf'(z) - kf(z) = \sum_{n=-\infty}^{\infty} [n(n-1) + n - k] a_n z^n,$$

whence

$$(n^2 - k)a_n = 0 \qquad \text{for all } n \in \mathbb{Z}. \tag{9}$$

We choose $p \in \mathbb{Z}$ such that $a_p \neq 0$. It is possible as $f \neq 0$. From (9) we get that $p^2 = k$ and

$$\left(n^2 - p^2\right) a_n = 0 \qquad \text{for all } n \in \mathbb{Z}.$$

So, if $n^2 \neq p^2$, then $a_n = 0$, whence it follows that $a_n = 0$ for all $n \neq p$ and $n \neq -p$. Thus

$$f(z) = a_p z^p + a_{-p} z^{-p}$$

for $z \in P$, as desired.

The following two lemmas are quite obvious.

Lemma 1
If the equality

$$Ae^{ia\theta} + \overline{A}e^{-ia\theta} = A + \overline{A}$$

holds true for all $\theta \in (-\delta, \delta)$, *where* A *is a complex constant,* $a \neq 0$ *is a real one, then* $A = 0$.

Lemma 2
If the equality

$$\alpha e^{a\theta} + \beta e^{-a\theta} = \alpha + \beta$$

holds true for all $\theta \in (-\delta, \delta)$, *where* $a \neq 0$, α, β *are real constants, then* $\alpha = \beta = 0$.

Now we will consider equation (2). As we mentioned above, every solution of (2) is a solution of (1). Thus if f is an analytic solution of (2), then f has to be of form (4) for some complex constants A, B and some integer p. Assume that $p \neq 0$. Substituting (4) to (2) we get

$$A\overline{B}e^{2ip\theta} + \overline{A}Be^{-2ip\theta} = A\overline{B} + \overline{A}B, \qquad \theta \in \mathbb{R}.$$

Lemma 1 yields $A = 0$ or $B = 0$. Thus we have

Theorem 2
If f *is an analytic solution of* (2) *in the annulus* P, *then there exist a complex constant* A *and an integer* p *such that*

$$f(z) = Az^p. \tag{10}$$

Conversely, for every complex constant A *and for every integer* p, *the function* f *given by* (10) *is a solution of* (2).

Theorem 3
Every analytic solution of (5) *in the annulus* P *is a constant function.*

Proof. Suppose that f is a solution of (5). Then f has to be of form (4). We may assume that $p \neq 0$. Combining (4) with (5) we obtain

$$|A|^2 r^{2p} + |B|^2 r^{-2p} = |A|^2 + |B|^2 \qquad \text{for all } r \in (1-\epsilon, 1+\epsilon).$$

Lemma 2 shows that $A = B = 0$, which completes the proof.

3. Solutions of (1), (2) and (5) in D

In this part of the paper we shall find all analytic solutions of equations (1), (2) and (5) in the domain $D := \{re^{i\theta} : \ 1-\epsilon < r < 1+\epsilon, \ \theta \in (-\delta, \delta)\}$, where $0 < \epsilon \leq 1$ and $0 < \delta \leq \pi$. In the sequel z^a denotes the principal branch of the power in D and $\log z$ is the principal branch of the logarithm of z, i.e., $z^a = \exp(a \log z)$ and $\log z = \log|z| + i \arg z$ for $z \in D$, where $\arg z \in (-\delta, \delta)$.

THEOREM 4
If an analytic function f satisfies (1) *in D, then there exist complex constants A, B and $a \in \mathbb{R}$ or $a \in i\mathbb{R}$ such that*

$$f(z) = Az^a + Bz^{-a}. \tag{11}$$

Conversely, every function f of form (11) *with arbitrary complex constants A, B and arbitrary real or purely imaginary constant a is a solution of* (1).

Proof. We may repeat the argument of the proof of Theorem 1. Thus we observe that if an analytic function f satisfies (1) in D , then it has to be a solution of the differential equation

$$z^2 f''(z) + zf'(z) - kf(z) = 0, \qquad z \in D, \tag{12}$$

where k is a real constant. Let

$$G = \{\log z : \ z \in D\}.$$

Of course, G is a domain. We define a function $g : G \longrightarrow \mathbb{C}$ as follows

$$g(u) := f(e^u).$$

g is analytic, $f(z) = g(\log z)$ for $z \in D$ and

$$e^u f'(e^u) = g'(u), \quad e^{2u} f''(e^u) = g''(u) - g'(u), \qquad u \in G. \tag{13}$$

It follows from (12) that

$$e^{2u} f''(e^u) + e^u f'(e^u) - kf(e^u) = 0 \qquad \text{for all } u \in G,$$

whence by (13)

$$g''(u) - kg(u) = 0, \qquad u \in G.$$

Solving this differential equation we get

$$g(u) = Ae^{au} + Be^{-au},$$

where A, B are suitable complex constants and $a^2 = k$. So a is a real constant or $a = ic$, where $c \in \mathbb{R}$. Putting $u = \log z$ we obtain (11). The first assertion of the theorem follows.

For the second conclusion, let us take arbitrarily $a \in \mathbb{R}$, $A, B \in \mathbb{C}$ and let f be given by (11). We observe that

$$f(re^{i\theta}) = Ar^a e^{i\theta a} + Br^{-a} e^{-i\theta a}, \qquad f(e^{i\theta}) = Ae^{i\theta a} + Be^{-i\theta a},$$
$$f(r) = Ar^a + Br^{-a}, \qquad f(1) = A + B.$$

Thus

$$
\begin{aligned}
&|f(re^{i\theta})|^2+|f(1)|^2\\
&\quad=(Ar^ae^{i\theta a}+Br^{-a}e^{-i\theta a})(\overline{A}r^ae^{-i\theta a}+\overline{B}r^{-a}e^{i\theta a})+(A+B)(\overline{A}+\overline{B})\\
&\quad=|A|^2r^{2a}+|B|^2r^{-2a}+A\overline{B}e^{2i\theta a}+\overline{A}Be^{-2i\theta a}+|A|^2+|B|^2+A\overline{B}+\overline{A}B
\end{aligned}
$$

and

$$
\begin{aligned}
&|f(e^{i\theta})|^2+|f(r)|^2\\
&\quad=(Ae^{i\theta a}+Be^{-i\theta a})(\overline{A}e^{-i\theta a}+\overline{B}e^{i\theta a})+(Ar^a+Br^{-a})(\overline{A}r^a+\overline{B}r^{-a})\\
&\quad=|A|^2+|B|^2+A\overline{B}e^{2i\theta a}+\overline{A}Be^{-2i\theta a}+|A|^2r^{2a}+|B|^2r^{-2a}+A\overline{B}+\overline{A}B.
\end{aligned}
$$

Now we assume that $a=ic$, where $c\in R$. Then

$$
\begin{aligned}
f\left(re^{i\theta}\right)&=Ae^{ic(\log r+i\theta)}+Be^{-ic(\log r+i\theta)}\\
&=Ae^{-c\theta}e^{ic\log r}+Be^{c\theta}e^{-ic\log r},\\
f\left(e^{i\theta}\right)&=Ae^{-c\theta}+Be^{c\theta},\\
f(r)&=Ae^{ic\log r}+Be^{-ic\log r},\\
f(1)&=A+B.
\end{aligned}
$$

These formulas lead to

$$
\begin{aligned}
&|f(re^{i\theta})|^2+|f(1)|^2\\
&\quad=(Ae^{-c\theta}e^{ic\log r}+Be^{c\theta}e^{-ic\log r})(\overline{A}e^{-c\theta}e^{-ic\log r}+\overline{B}e^{c\theta}e^{ic\log r})+|A+B|^2\\
&\quad=|A|^2e^{-2c\theta}+|B|^2e^{2c\theta}+A\overline{B}e^{2ic\log r}+\overline{A}Be^{-2ic\log r}\\
&\qquad+|A|^2+|B|^2+A\overline{B}+\overline{A}B
\end{aligned}
$$

and

$$
\begin{aligned}
|f(e^{i\theta})|^2+|f(r)|^2&=(Ae^{-c\theta}+Be^{c\theta})(\overline{A}e^{-c\theta}+\overline{B}e^{c\theta})\\
&\quad+(Ae^{ic\log r}+Be^{-ic\log r})(\overline{A}e^{-ic\log r}+\overline{B}e^{ic\log r})\\
&=|A|^2e^{-2c\theta}+|B|^2e^{2c\theta}+A\overline{B}+\overline{A}B+|A|^2+|B|^2\\
&\quad+A\overline{B}e^{2ic\log r}+\overline{A}Be^{-2ic\log r}.
\end{aligned}
$$

So in both cases the function f given by (11) satisfies (1), as required.

Theorem 5

All analytic solutions of (2) *in* D *are of the form*

$$
f(z)=Az^a, \tag{14}
$$

where A *is a complex constant and* a *is a real one.*

Proof. Suppose that f is a non-constant analytic solution of (2) in D. Since (1) is a generalization of (2) we can apply Theorem 4. Thus there exist complex constants A, B and real or purely imaginary $a\neq 0$ such that f is given by (11). At first we assume that a is real. Substituting (11) in (2) after some easy calculations we obtain

$$\overline{A}B\exp(-2ia\theta) + A\overline{B}\exp(2ia\theta) = \overline{A}B + A\overline{B}$$

for $\theta \in (-\delta, \delta)$. Lemma 1 yields $A = 0$ or $B = 0$ and f is of the form (14), as required.

Now, we assume that $a = ic$, where c is real. Replacing in (2), $f(z)$ by (11) we infer the equality

$$|A|^2\exp(-2c\theta) + |B|^2\exp(2c\theta) = |A|^2 + |B|^2.$$

This together with Lemma 2 yields $A = B = 0$.

Theorem 6
All analytic solutions of (5) *in D are given by the formula*

$$f(z) = Az^{ic}, \tag{15}$$

where A is a complex constant and c is a real one.

Proof. We argue as in the preceding proof. Suppose that f is a non-constant analytic solution of (5) in D. f has to be given by (11). Assume that a is a real constant. Substituting (11) in (5) we get

$$|A|^2r^{2a} + |B|^2r^{-2a} = |A|^2 + |B|^2$$

for all $r \in (1-\epsilon, 1+\epsilon)$. From Lemma 2 we infer that $A = B = 0$. It remains to consider $a = ic$, where c is real. Again substituting (11) in (5) we can obtain

$$A\overline{B}\exp(2ic\log r) + \overline{A}B\exp(-2ic\log r) = A\overline{B} + \overline{A}B.$$

The above formula and Lemma 1 yield (15).

References

[1] H. Haruki, *A new functional characterizing generalized Joukowski transformations*, Aequationes Math. **32** (1987), 327-335.

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