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Some results on pseudo-Q algebras

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Abstract. The notions of a dual pseudo-Q algebra and a dual pseudo-QC algebra are introduced. The properties and characterizations of them are investigated. Conditions for a dual pseudo-Q algebra to be a dual pseudo-QC algebra are given. Commutative dual pseudo-QC algebras are considered. The interrelationships between dual pseudo-Q/QC algebras and other pseudo algebras are visualized in a diagram.

1. Introduction

G. Georgescu and A. Iorgulescu [6] and independently J. Rachůnek [15], introduced pseudo-MV algebras which are a non-commutative generalization of MV-algebras. After pseudo-MV algebras, pseudo-BL algebras [7] and pseudo-BCK algebras [8] were introduced and studied by G. Georgescu and A. Iorgulescu. A. Walendziak [18] gave a system of axioms defining pseudo-BCK algebras. W.A. Dudek and Y.B. Jun defined pseudo-BCI algebras as an extension of BCI-algebras [5]. Y.H. Kim and K.S. So [11] discussed on minimal elements in pseudo-BCI algebras. G. Dymek studied p-semisimple pseudo-BCI algebras and then defined and investigated periodic pseudo-BCI algebras [3].

A. Walendziak [19] introduced pseudo-BCH algebras as an extension of BCH-algebras and studied ideals in such algebras.

The notion of BE-algebras was introduced by H.S. Kim and Y.H. Kim [10].

B.L. Meng [13] introduced the notion of CI-algebras as a generalization of BE-algebras and dual BCK/BCI/BCH-algebras. R.A. Borzooei et al. defined and studied pseudo-BE algebras which are a generalization of BE-algebras [1]. A. Rezaei et al. introduced the notion of pseudo-CI algebras as a generalization

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of pseudo-BE algebras and proved that the class of commutative pseudo-CI algebras coincides with the class of commutative pseudo-BCK algebras [16]. Recently, Y.B. Jun et al. defined and investigated pseudo-Q algebras [9] as a generalization of Q-algebras [14].

In this paper, we define dual pseudo-Q and dual pseudo-QC algebras. We investigate the properties and characterizations of them. Moreover, we provide some conditions for a dual pseudo-Q algebra to be a dual pseudo-QC algebra. We also consider commutative dual pseudo-QC algebras and prove that the class of such algebras coincides with the class of commutative pseudo-BCI algebras. Finally, the interrelationships between dual pseudo-Q/QC algebras and other pseudo algebras are visualized in a diagram.

2. Preliminaries

In this section, we review the basic definitions and some elementary aspects that are necessary for this paper.

Definition 2.1 ([5])
An algebra $\mathfrak{X} = (X; \rightarrow, \rightsquigarrow, 1)$ of type $(2, 2, 0)$ is called a *pseudo-BCI algebra* if it satisfies the following axioms: for all $x, y, z \in X$,

(psBCI$_1$) $(x \rightarrow y) \rightsquigarrow ((y \rightarrow z) \rightsquigarrow (x \rightarrow z)) = 1$,

(psBCI$_2$) $(x \rightsquigarrow y) \rightarrow ((y \rightsquigarrow z) \rightarrow (x \rightsquigarrow z)) = 1$,

(psBCI$_3$) $x \rightarrow ((x \rightarrow y) \rightsquigarrow y) = 1$ and $x \rightsquigarrow ((x \rightsquigarrow y) \rightarrow y) = 1$,

(psBCI$_4$) $x \rightarrow x = x \rightsquigarrow x = 1$,

(psBCI$_5$) $x \rightarrow y = y \rightsquigarrow x = 1 \implies x = y$,

(psBCI$_6$) $x \rightarrow y = 1 \iff x \rightsquigarrow y = 1$.

Every pseudo-BCI algebra $\mathfrak{X}$ satisfying, for every $x \in X$, condition

(psBCK) $x \rightarrow 1 = 1$

is said to be a *pseudo-BCK algebra* ([12]).

From [4] it follows that a pseudo-BCI-algebra $\mathfrak{X} = (X; \rightarrow, \rightsquigarrow, 1)$ has the following property (for all $x, y \in X$)

(psEx) $x \rightarrow (y \rightsquigarrow z) = y \rightsquigarrow (x \rightarrow z)$.

Definition 2.2 ([17])
A (dual) pseudo-BCH algebra is an algebra $(X; \rightarrow, \rightsquigarrow, 1)$ of type $(2, 2, 0)$ verifying the axioms (psBCI$_4$)–(psBCI$_6$) and (psEx).

Remark 2.3
Obviously, every pseudo-BCI algebra is a pseudo-BCH algebra.

Definition 2.4 ([16])
An algebra $\mathfrak{X} = (X; \to, \rightsquigarrow, 1)$ of type $(2, 2, 0)$ is called a *pseudo-CI algebra* if, for all $x, y, z \in X$, it satisfies the following axioms:

(psCI$_1$) $x \to x = x \rightsquigarrow x = 1$,

(psCI$_2$) $1 \to x = 1 \rightsquigarrow x = x$,

(psCI$_3$) $x \to (y \rightsquigarrow z) = y \rightsquigarrow (x \to z)$,

(psCI$_4$) $x \to y = 1 \iff x \rightsquigarrow y = 1$.

Remark 2.5
Since every pseudo-BCH algebra satisfies (psCI$_1$)–(psCI$_4$), pseudo-BCH algebras are contained in the class of pseudo-CI algebras.

A pseudo-CI algebra $\mathfrak{X} = (X; \to, \rightsquigarrow, 1)$ verifying condition

(psBE) $x \to 1 = x \rightsquigarrow 1 = 1$,

for all $x \in X$, is said to be a *pseudo-BE algebra* (see [1]).

Proposition 2.6 ([2])
Any pseudo-BCK algebra is a pseudo-BE algebra.

In a pseudo-CI algebra $\mathfrak{X}$ we can introduce a binary relation "$\leq$" by

$$x \leq y \iff x \to y = 1 \iff x \rightsquigarrow y = 1 \qquad \text{for all } x, y \in X.$$

An algebra $\mathfrak{X} = (X; \to, \rightsquigarrow, 1)$ of type $(2, 2, 0)$ is called *commutative* if for all $x, y \in X$, it satisfies the following identities:

(i) $(x \to y) \rightsquigarrow y = (y \to x) \rightsquigarrow x$,

(ii) $(x \rightsquigarrow y) \to y = (y \rightsquigarrow x) \to x$.

From [2] (see Theorem 3.4) it follows that any commutative pseudo-BE algebra is a pseudo-BCK algebra. By Theorem 3.9 of [16], any commutative pseudo-CI algebra is a pseudo-BE algebra. Therefore we obtain

Proposition 2.7
Commutative pseudo-CI algebras coincide with commutative pseudo-BE algebras and with commutative pseudo-BCK algebras (hence also coincide with commutative pseudo-BCI algebras and with commutative pseudo-BCH algebras).

Definition 2.8 ([9])
An algebra $\mathfrak{X} = (X; *, \diamond, 0)$ of type $(2, 2, 0)$ is called a *pseudo-Q algebra* if, for all $x, y, z \in X$, it satisfies the following axioms:

(psQ$_1$) $x * x = x \diamond x = 0$,

(psQ$_2$) $x * 0 = x \diamond 0 = x$,

(psQ$_3$) $(x * y) \diamond z = (x \diamond z) * y$.

3. Dual pseudo-Q algebras

Definition 3.1
An algebra $\mathfrak{X} = (X; \to, \rightsquigarrow, 1)$ of type $(2, 2, 0)$ is called a *dual pseudo-Q algebra* if, for all $x, y, z \in X$, it verifies the following axioms:

(dpsQ$_1$) $x \to x = x \rightsquigarrow x = 1$,

(dpsQ$_2$) $1 \to x = 1 \rightsquigarrow x = x$,

(dpsQ$_3$) $x \to (y \rightsquigarrow z) = y \rightsquigarrow (x \to z)$.

In a dual pseudo-Q algebra, we can introduce two binary relations $\leq_\to$ and $\leq_\rightsquigarrow$ by

$$x \leq_\to y \iff x \to y = 1 \quad \text{and} \quad x \leq_\rightsquigarrow y \iff x \rightsquigarrow y = 1.$$

Proposition 3.2
Let $\mathfrak{X} = (X; \to, \rightsquigarrow, 1)$ be a dual pseudo-Q algebra. Then $\mathfrak{X}$ is a pseudo-CI algebra if and only if $\leq_\to = \leq_\rightsquigarrow$.

Example 3.3

(i) Let $X = \{1, a, b, c, d\}$. Define binary operations $\to$ and $\rightsquigarrow$ on X by the following tables ([16]):

$\to$	1	a	b	c	d
1	1	a	b	c	d
a	1	1	c	c	1
b	1	d	1	1	d
c	1	d	1	1	d
d	1	1	c	c	1

and

$\rightsquigarrow$	1	a	b	c	d
1	1	a	b	c	d
a	1	1	b	c	1
b	1	d	1	1	d
c	1	d	1	1	d
d	1	1	b	c	1

.

Then $\mathfrak{X} = (X; \to, \rightsquigarrow, 1)$ is a dual pseudo-Q algebra which is not a pseudo-BCI algebra, since $b \neq c$ and $b \to c = c \rightsquigarrow b = 1$ (that is, (psBCI$_5$) does not hold in $\mathfrak{X}$).

(ii) Let $X = \{1, a, b, c\}$. Define binary operations $\to$ and $\rightsquigarrow$ on X by the following tables:

$\to$	1	a	b	c
1	1	a	b	c
a	1	1	b	c
b	1	1	1	1
c	1	1	a	1

and

$\rightsquigarrow$	1	a	b	c
1	1	a	b	c
a	1	1	c	c
b	1	1	1	c
c	1	1	c	1

.

Then $\mathfrak{X} = (X; \to, \rightsquigarrow, 1)$ is a dual pseudo-Q algebra which is not a pseudo-CI algebra, because $b \to c = 1$ but $b \rightsquigarrow c = c$.

By definition, we have

Proposition 3.4
Any pseudo-CI algebra is a dual pseudo-Q algebra.

Remark 3.5
The converse of Proposition 3.4 does not hold. See Example 3.3 (ii).

Proposition 3.6
Let $\mathfrak{X}$ be a dual pseudo-Q algebra. If one of the following identities:

(1) $(y \to x) \to x = y \to x$,

(2) $(y \to x) \rightsquigarrow x = y \rightsquigarrow x$,

(3) $(y \rightsquigarrow x) \to x = y \to x$,

(4) $(y \rightsquigarrow x) \rightsquigarrow x = y \to x$,

(5) $(y \rightsquigarrow x) \rightsquigarrow x = y \rightsquigarrow x$,

(6) $(y \rightsquigarrow x) \to x = y \rightsquigarrow x$,

(7) $(y \to x) \rightsquigarrow x = y \to x$,

(8) $(y \to x) \to x = y \rightsquigarrow x$

holds in $\mathfrak{X}$, then $\mathfrak{X}$ is a trivial algebra.

Proof. Suppose, for example, that (1) is satisfied. Let $x \in X$. Applying (dpsQ$_1$), (1) and (dpsQ$_2$) we have

$$1 = x \to x = (x \to x) \to x = 1 \to x = x.$$

Thus $\mathfrak{X}$ is a trivial algebra.

Proposition 3.7
Let $\mathfrak{X}$ be a dual pseudo-Q algebra. If one of the following identities:

(1) $(y \to x) \to x = x \to y$,

(2) $(y \to x) \rightsquigarrow x = x \rightsquigarrow y$,

(3) $(y \rightsquigarrow x) \to x = x \to y$,

(4) $(y \rightsquigarrow x) \rightsquigarrow x = x \to y$,

(5) $(y \rightsquigarrow x) \rightsquigarrow x = x \rightsquigarrow y$,

(6) $(y \rightsquigarrow x) \to x = x \rightsquigarrow y$,

(7) $(y \to x) \rightsquigarrow x = x \to y$,

(8) $(y \to x) \to x = x \rightsquigarrow y$

holds in $\mathfrak{X}$, then $\mathfrak{X}$ is a trivial algebra.

Proof. The proof is similar to the proof of Proposition 3.6.

Proposition 3.8
In a dual pseudo-Q algebra $\mathfrak{X}$, for all $x, y, z \in X$, we have:

(1) *if $1 \leq_{\to} x$ or $1 \leq_{\rightsquigarrow} x$, then $x = 1$,*

(2) $x \leq_{\rightsquigarrow} y \to z \iff y \leq_{\to} x \rightsquigarrow z$,

(3) $x \to 1 = x \rightsquigarrow 1$,

(4) $(x \to y) \to 1 = (x \to 1) \rightsquigarrow (y \rightsquigarrow 1)$ *and* $(x \rightsquigarrow y) \rightsquigarrow 1 = (x \rightsquigarrow 1) \to (y \to 1)$,

(5) *if* $x \leq_{\to} y$, *then* $x \to 1 = y \to 1$,

(6) *if* $x \leq_{\rightsquigarrow} y$, *then* $x \rightsquigarrow 1 = y \rightsquigarrow 1$,

(7) $y \to ((y \to x) \rightsquigarrow x) = 1$ *and* $y \rightsquigarrow ((y \rightsquigarrow x) \to x) = 1$,

Proof. (1) Let $1 \leq_{\to} x$. Then $1 \to x = 1$. Now, by (dpsQ$_2$) we obtain $x = 1$. Similarly, if $1 \leq_{\rightsquigarrow} x$, then $x = 1$.

(2) Let $x, y, z \in X$. By (dpsQ$_3$),

$$x \rightsquigarrow (y \to z) = 1 \iff y \to (x \rightsquigarrow z) = 1.$$

Consequently, (2) holds.

(3) We have $x \to 1 = x \to (x \rightsquigarrow x) = x \rightsquigarrow (x \to x) = x \rightsquigarrow 1$.

(4) Let $x, y, z \in X$. Then

$$\begin{aligned}(x \to y) \to 1 &= (x \to y) \to [(x \to 1) \rightsquigarrow (x \to 1)]\\ &= (x \to 1) \rightsquigarrow [(x \to y) \to (x \to 1)]\\ &= (x \to 1) \rightsquigarrow [(x \to y) \to (x \to (y \rightsquigarrow y))]\\ &= (x \to 1) \rightsquigarrow [(x \to y) \to (y \rightsquigarrow (x \to y))]\\ &= (x \to 1) \rightsquigarrow [y \rightsquigarrow ((x \to y) \to (x \to y))]\\ &= (x \to 1) \rightsquigarrow (y \rightsquigarrow 1).\end{aligned}$$

The proof of the second part is similar.

(5) Let $x \leq_{\to} y$. Then $x \to y = 1$ and so $y \to 1 = y \rightsquigarrow 1 = y \rightsquigarrow (x \to y) = x \to (y \rightsquigarrow y) = x \to 1$. Thus $y \to 1 = x \to 1$.

(6) The proof is similar to the proof of (5).

(7) By (dpsQ$_3$) and (dpsQ$_1$) we get

$$y \to ((y \to x) \rightsquigarrow x) = (y \to x) \rightsquigarrow (y \to x) = 1$$

and

$$y \rightsquigarrow ((y \rightsquigarrow x) \to x) = (y \rightsquigarrow x) \to (y \rightsquigarrow x) = 1.$$

A dual pseudo-Q algebra $\mathfrak{X} = (X; \to, \rightsquigarrow, 1)$ satisfying the conditions (psBCI$_1$) and (psBCI$_2$) is said to be a *dual pseudo-QC algebra*. The following example shows that there exist pseudo-Q algebras which do not satisfy (psBCI$_1$) or (psBCI$_2$).

Example 3.9

(i) Dual pseudo-Q algebra from Example 3.3 (ii) satisfies (psBCI_2) but it does not satisfy (psBCI_1), since

$$(a \to b) \rightsquigarrow ((b \to c) \rightsquigarrow (a \to c)) = b \rightsquigarrow (1 \rightsquigarrow c) = c \neq 1.$$

(ii) Let $X = \{1, a, b, c, d, e, f, g, h\}$. We define the binary operations $\to$ and $\rightsquigarrow$ on X as follows ([17]):

$\to$	1	a	b	c	d	e	f	g	h
1	1	a	b	c	d	e	f	g	h
a	1	1	1	1	d	e	f	g	h
b	1	c	1	1	d	e	f	g	h
c	1	c	b	1	d	e	f	g	h
d	d	d	d	d	1	g	h	e	f
e	e	e	e	e	h	1	g	f	d
f	f	f	f	f	g	h	1	d	e
g	h	h	h	h	e	f	d	1	g
h	g	g	g	g	f	d	e	h	1

and

$\rightsquigarrow$	1	a	b	c	d	e	f	g	h
1	1	a	b	c	d	e	f	g	h
a	1	1	1	1	d	e	f	g	h
b	1	c	1	1	d	e	f	g	h
c	1	c	b	1	d	e	f	g	h
d	d	d	d	d	1	h	g	f	e
e	e	e	e	e	g	1	h	d	f
f	f	f	f	f	h	g	1	e	d
g	h	h	h	h	f	d	e	1	g
h	g	g	g	g	e	f	d	h	1

.

Then $\mathfrak{X} = (X; \to, \rightsquigarrow, 1)$ is a dual pseudo-Q algebra which does not satisfy (psBCI_1) and (psBCI_2). Indeed,

$$(c \to a) \rightsquigarrow ((a \to b) \rightsquigarrow (c \to b)) = c \rightsquigarrow (1 \rightsquigarrow b) = c \rightsquigarrow b = b \neq 1$$

and

$$(c \rightsquigarrow a) \to ((a \rightsquigarrow b) \to (c \rightsquigarrow b)) = c \to (1 \to b) = c \to b = b \neq 1.$$

(iii) Let $X = \{1, a, b, c\}$. Define binary operations $\to$ and $\rightsquigarrow$ on $X$by the following tables:

$\to$	1	a	b	c
1	1	a	b	c
a	1	1	b	b
b	1	a	1	c
c	1	1	1	1

and

$\rightsquigarrow$	1	a	b	c
1	1	a	b	c
a	1	1	b	c
b	1	a	1	a
c	1	1	1	1

.

Then $\mathfrak{X} = (X; \to, \rightsquigarrow, 1)$ is a dual pseudo-QC algebra.

Lemma 3.10

Let $\mathfrak{X} = (X; \to, \rightsquigarrow, 1)$ be a dual pseudo-QC algebra and $x, y \in X$. Then $x \to y = 1$ if and only if $x \rightsquigarrow y = 1$.

Proof. Let $x \to y = 1$. Using (dpsQ_2) and (psBCI_1) we obtain

$$x \rightsquigarrow y = x \rightsquigarrow (1 \rightsquigarrow y) = (1 \to x) \rightsquigarrow ((x \to y) \rightsquigarrow (1 \to y)) = 1.$$

Similarly, if $x \rightsquigarrow y = 1$, then $x \to y = 1$.

From Lemma 3.10 we have

Proposition 3.11
Any dual pseudo-QC algebra is a pseudo-CI algebra.

Remark 3.12
The converse of Proposition 3.11 does not hold. See Example 3.9 (ii).

Proposition 3.13
Every pseudo-BCI algebra is a dual pseudo-QC algebra.

Proof. Let $\mathfrak{X}$ be a pseudo-BCI algebra. It is easy to see that $\mathfrak{X}$ satisfies (dpsQ$_1$)–(dpsQ$_3$), that is, it is a dual pseudo-Q algebra. Moreover, $\mathfrak{X}$ obviously satisfies (psBCI$_1$) and (psBCI$_2$). Consequently, $\mathfrak{X}$ is a dual pseudo-QC algebra.

Remark 3.14
In a dual pseudo-QC algebra, $\leq_{\rightarrow}=\leq_{\rightsquigarrow}$. Set $\leq=\leq_{\rightarrow}$ $(=\leq_{\rightsquigarrow})$.

Proposition 3.15
Let $\mathfrak{X}$ be a dual pseudo-QC algebra and $x, y, z \in X$. Then:

(1) *if $x \leq y$, then $y \rightarrow z \leq x \rightarrow z$ and $y \rightsquigarrow z \leq x \rightsquigarrow z$,*

(2) *if $x \leq y$, then $z \rightarrow x \leq z \rightarrow y$ and $z \rightsquigarrow x \leq z \rightsquigarrow y$.*

Proof. (1) Let $x \leq y$. Then $x \rightarrow y = 1$. By (dpsQ$_2$) and (psBCI$_1$) we have

$$\begin{aligned}(y \rightarrow z) \rightsquigarrow (x \rightarrow z) &= 1 \rightsquigarrow ((y \rightarrow z) \rightsquigarrow (x \rightarrow z))\\ &= (x \rightarrow y) \rightsquigarrow ((y \rightarrow z) \rightsquigarrow (x \rightarrow z))\\ &= 1.\end{aligned}$$

Hence $y \rightarrow z \leq x \rightarrow z$. The proof of the second part is similar.

(2) Let $x \leq y$. Hence $x \rightarrow y = 1$. Applying (dpsQ$_2$) and (psBCI$_1$) we obtain

$$\begin{aligned}(z \rightarrow x) \rightarrow (z \rightarrow y) &= 1 \rightsquigarrow ((z \rightarrow x) \rightarrow (z \rightarrow y))\\ &= (x \rightarrow y) \rightsquigarrow ((z \rightarrow x) \rightarrow (z \rightarrow y))\\ &= (z \rightarrow x) \rightarrow ((x \rightarrow y) \rightsquigarrow (z \rightarrow y))\\ &= 1.\end{aligned}$$

Hence $z \rightarrow x \leq z \rightarrow y$. Similarly, $z \rightsquigarrow x \leq z \rightsquigarrow y$.

Theorem 3.16
Let $\mathfrak{X}$ be a dual pseudo-Q algebra. Then $\mathfrak{X}$ is a pseudo-QC algebra if and only if it satisfies the following implications:

($*$) $y \leq_{\rightarrow} z \Longrightarrow x \rightarrow y \leq_{\rightsquigarrow} x \rightarrow z$,

($**$) $y \leq_{\rightsquigarrow} z \Longrightarrow x \rightsquigarrow y \leq_{\rightarrow} x \rightsquigarrow z$.

Proof. If $\mathfrak{X}$ is a pseudo-QC algebra, then it satisfies ($*$) and ($**$) by Proposition 3.15. Conversely, suppose that implications ($*$) and ($**$) hold for all $x, y, z \in X$. By Proposition 3.8 (7), $y \leq_{\rightarrow} (y \rightarrow z) \rightsquigarrow z$. Using ($*$) we get $x \rightarrow y \leq_{\rightsquigarrow} x \rightarrow ((y \rightarrow z) \rightsquigarrow z)$. Hence $(x \rightarrow y) \rightsquigarrow (x \rightarrow ((y \rightarrow z) \rightsquigarrow z)) = 1$. Applying (psEx) we obtain $(x \rightarrow y) \rightsquigarrow ((y \rightarrow z) \rightsquigarrow (x \rightarrow z)) = 1$, that is, (psBCI$_1$) holds. Similarly, using ($**$) we have (psBCI$_2$).

Proposition 3.17
Let $\mathfrak{X}$ be a dual pseudo-QC algebra. Then $\mathfrak{X}$ is a pseudo-BCI algebra if and only if it verifies (psBCI$_5$)*.*

Proof. Let $\mathfrak{X}$ be a dual pseudo-QC algebra satisfying (psBCI$_5$). Clearly, $\mathfrak{X}$ verifies (psBCI$_1$), (psBCI$_2$) and (psBCI$_4$). The axiom (psBCI$_3$) follows from Proposition 3.8 (7). By Lemma 3.10, (psBCI$_6$) holds in $\mathfrak{X}$. Therefore, $\mathfrak{X}$ is a pseudo-BCI algebra.

The converse is obvious.

Proposition 3.18
Let $\mathfrak{X}$ be a dual pseudo-QC algebra and $x, y, z \in X$ such that $x \leq y$ and $y \leq z$. Then $x \leq z$.

Proof. Applying (dpsQ$_2$) and (psBCI$_1$) we get

$$\begin{aligned} x \to z &= 1 \rightsquigarrow (x \to z) \\ &= 1 \rightsquigarrow (1 \rightsquigarrow (x \to z)) \\ &= (x \to y) \rightsquigarrow ((y \to z) \rightsquigarrow (x \to z)) \\ &= 1, \end{aligned}$$

and therefore $x \leq z$.

Corollary 3.19
If a dual pseudo-QC algebra $\mathfrak{X}$ satisfies the condition (psBCI$_5$)*, then $(X; \leq)$ is a poset.*

Theorem 3.20
If $\mathfrak{X}$ is a commutative dual pseudo-QC algebra, then it is a pseudo-BCI algebra.

Proof. It is sufficient to prove that (psBCI$_5$) holds in $\mathfrak{X}$. Let $x, y \in X$ and $x \to y = y \rightsquigarrow x = 1$. Then

$$x = 1 \to x = (y \rightsquigarrow x) \to x = (x \rightsquigarrow y) \to y = 1 \to y = y.$$

Therefore, $\mathfrak{X}$ satisfies (psBCI$_5$). Thus $\mathfrak{X}$ is a pseudo-BCI algebra.

From Theorem 3.20 it follows

Corollary 3.21
Commutative dual pseudo-QC algebras coincide with commutative pseudo-BCI algebras.

4. Conclusion

Denote by **psBCK**, **psBCI**, **psBCH**, **psCI**, **psBE**, **dpsQ**, and **dpsQC** the classes of pseudo-BCK, pseudo-BCI, pseudo-BCH, pseudo-CI, pseudo-BE, dual pseudo-Q, and dual pseudo-QC algebras respectively. By definition, **psBCK** $\subset$ **psBCI** and **psBE** $\subset$ **psCI** $\subset$ **dpsQ**. From Remarks 2.3 and 2.5 we obtain **psBCI**

$\subset$ **psBCH** $\subset$ **psCI**. Moreover, that **psBCI** $\subset$ **dpsQC** $\subset$ **psCI** follows from Propositions 3.13 and 3.11.

By Proposition 2.7 and Corollary 3.21, commutative pseudo-QC algebras coincide with commutative algebras pseudo-BCK, -BCI, -BCH, -CI, -BE.

Now, in the following diagram we summarize the results of this paper and the previous results in this filed. An arrow indicates proper inclusion, that is, if **X** and **Y** are classes of algebras, then $\mathbf{X} \to \mathbf{Y}$ denotes $\mathbf{X} \subset \mathbf{Y}$. The mark $\mathbf{X} \overset{C}{\to} \mathbf{Y}$ means that every commutative algebra of **X** belongs to **Y**.

$$
\begin{tikzcd}
 & \mathbf{dpsQ} & \\
 & \mathbf{psCI} \arrow[u] \arrow[r, shift left, "C"] \arrow[d, shift left, "C"] \arrow[ddl, shift left, "C"] & \mathbf{psBE} \arrow[l, shift left] \arrow[dddl, shift right, "C"'] \\
 & \mathbf{psBCH} \arrow[u, shift left] \arrow[d, shift left, "C"] & \\
\mathbf{dpsQC} \arrow[r, shift left, "C"] \arrow[uur, shift left] & \mathbf{psBCI} \arrow[l, shift left] \arrow[u, shift left] \arrow[d, shift left, "C"] & \\
 & \mathbf{psBCK} \arrow[u, shift left] \arrow[uuur, shift right] &
\end{tikzcd}
$$

Problem 4.1
Is it true that every commutative dual pseudo-Q algebra is a pseudo-BCK algebra?

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