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Local convergence of a multi-step high order method with divided differences under hypotheses on the first derivative

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Abstract. This paper is devoted to the study of a multi-step method with divided differences for solving nonlinear equations in Banach spaces. In earlier studies, hypotheses on the Fréchet derivative up to the sixth order of the operator under consideration is used to prove the convergence of the method. That restricts the applicability of the method. In this paper we extended the applicability of the sixth-order multi-step method by using only hypotheses on the first derivative of the operator involved. Our convergence conditions are weaker than the conditions used in earlier studies. Numerical examples where earlier results cannot be applied to solve equations but our results can be applied are also given in this study.

1. Introduction

Grau et. al. in [12], studied a sixth-order multi-step method defined for each $n = 0, 1, 2, \dots$ by

$$\begin{aligned} y_n &= x_n - A_n^{-1}F(x_n),\\ z_n &= y_n - B_n^{-1}F(y_n),\\ x_{n+1} &= z_n - B_n^{-1}F(z_n), \end{aligned} \tag{1}$$

where $A_n = [u_n, v_n; F]$, $B_n = 2[y_n, x_n; F] - [u_n, v_n; F]$, $u_n = x_n + F(x_n)$ and $v_n = x_n - F(x_n)$, for approximating a solution x^* of the equation

$$F(x) = 0, \tag{2}$$

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where $F\colon D \subseteq \mathcal{B}_1 \to \mathcal{B}_2$ is a Fréchet differentiable operator between Banach spaces $\mathcal{B}_1, \mathcal{B}_2$ and $[.,.;F]$ is a divided difference of order one on D^2. Due to the wide applications, finding a solution for (2) is an important problem in applied mathematics. Most of the solution methods for solving (2) are iterative and for iterative methods order of convergence is an important issue. Convergence analysis of higher order iterative methods require assumptions on the higher order Fréchet derivatives of the operator F. That restricts the applicability of these methods.

Notice that in [12] $\mathcal{B}_1 = \mathcal{B}_2 = \mathbb{R}^i$ (i a natural integer). However, we study method (2) in the more general setting of a Banach space. We also provide computable radius of convergence and error bounds on $\|x_n - x^*\|$ based on Lipschitz constants not given in [12]. The study of the local convergence in this way is also important because it provides the difficulty in choosing the initial points. Otherwise as in the earlier studies the choice of the initial point is a shot in the dark. Throughout this paper $L(\mathcal{B}_2, \mathcal{B}_1)$ denotes the set of bounded linear operators between $\mathcal{B}_1$ and $\mathcal{B}_2$ and $B(z,\rho)$, $\bar{B}(z,\rho)$ stand, respectively for the open and closed balls in $\mathcal{B}_1$ with center $z \in \mathcal{B}_1$ and of radius $\rho > 0$.

Convergence analysis in [12] is based on the assumptions on the Fréchet derivative F up to the order six. In this study we use only assumptions on the first Fréchet derivative of the operator F in our convergence analysis, so that the method (1) can be applied to solve equations but the earlier results cannot be applied [1, 2, 3, 4, 5, 18, 6, 7, 8, 10, 11, 12, 13, 14, 15, 16, 17, 9, 19, 20, 21, 22] (see Example 3.2).

The rest of the paper is structured as follows. In Section 2 we present the local convergence analysis of the method (1). We also provide a radius of convergence, computable error bounds and a uniqueness result. Numerical examples are given in the last section.

2. Local convergence

The local convergence analysis of (1) is based on some parameters and scalar functions. Let $\alpha \geq 0$, $\beta \geq 0$ be parameters and $\omega_0\colon [0,+\infty)^2 \to [0,+\infty)$ be a continuous nondecreasing function satisfying $\omega_0(0,0) = 0$. Define parameter r_0 by

$$r_0 = \sup\{t \geq 0 :\ \omega_0(\alpha t, \beta t) < 1\}. \tag{3}$$

Let $\varphi_0\colon [0,r_0) \to [0,+\infty)$, $\omega, \omega_1\colon [0,r_0)^2 \to [0,+\infty)$ be continuous and nondecreasing functions. Define functions g_1 and h_1 on the interval $[0,r_0)$ by

$$g_1(t) = \frac{\omega_1(\varphi_0(t)t, \varphi_0(t)t)}{1 - \omega_0(\alpha t, \beta t)}$$

and

$$h_1(t) = g_1(t) - 1.$$

Suppose that

$$\omega_1(0,0) < 1. \tag{4}$$

We have by (4) that

$$h_1(0) = \frac{\omega_1(0,0)}{1-\omega_0(0,0)} - 1 < 0, \qquad (\omega_0(0,0)=0)$$

and by (3) $h_1(t) \to +\infty$ as $t \to r_0^-$. The intermediate value theorem assures the existence of a solution for equation $h_1(t) = 0$ in $(0, r_0)$. Denote by r_1 the smallest such solution. Define also functions p and h_p on $[0, r_0)$ by

$$p(t) = \omega_0(g_1(t)t, t) + \omega((g_1(t)+\alpha)t, \varphi_0(t)t)$$

and

$$h_p(t) = p(t) - 1.$$

Suppose that

$$\omega(0,0) < 1. \tag{5}$$

We get

$$h_p(0) = \omega_0(0,0) + \omega(0,0) - 1 < 0$$

and $h_p(t) \to +\infty$ as $t \to r_0^-$. Denote by r_p the smallest solution of equation $h_p(t) = 0$. Let $\varphi\colon [0, r_p) \to [0, +\infty)$ be a continuous and nondecreasing function. Define functions g_2 and h_2 on the interval $[0, r_p)$ by

$$g_2(t) = \Big(1 + \frac{\varphi(g_1(t)t)}{1-p(t)}\Big) g_1(t)$$

and

$$h_2(t) = g_2(t) - 1.$$

Suppose that

$$\Big(1 + \frac{\varphi(0)}{1-p(0)}\Big)\omega_1(0,0) < 1.$$

We obtain that $h_2(0) < 0$ and $h_2(t) \to +\infty$ as $t \to r_p^-$. Denote by r_2 the smallest solution of equation $h_2(t) = 0$ on the interval $(0, r_p)$. Let $\omega\colon [0, r_0)^2 \to [0, +\infty)$ be a continuous and nondecreasing function. Define functions g_3 and h_3 on the interval $[0, r_p)$ by

$$g_3(t) = \Big(1 + \frac{\varphi(g_2(t)t)}{1-p(t)}\Big) g_2(t)$$

and

$$h_3(t) = g_3(t) - 1.$$

Suppose that

$$\Big(1 + \frac{\varphi(0)}{1-p(0)}\Big)\Big(1+\varphi(0)\Big)\omega_1(0,0) < 1. \tag{6}$$

We get by (6) that $h_3(0) < 0$ and $h_3(t) \to +\infty$ as $t \to r_0^-$. Denote by r_3 the smallest solution of equation $h_3(t) = 0$. Define the radius of convergence r by

$$r = \min\{r_i,\ i = 1, 2, 3\}. \tag{7}$$

Then, for each $t \in [0, r)$ we have

$$0 \le g_i(t) < 1 \tag{8}$$

and

$$0 \le p(t) < 1 \tag{9}$$

Define parameter R^* by

$$R^* = \max\{\alpha r, \beta r, r\}. \tag{10}$$

The local convergence analysis of method (1) follows under the previous notation.

Theorem 2.1

Let $F\colon \Omega \subset \mathcal{B}_1 \to \mathcal{B}_2$ be a continuously Fréchet differentiable operator and let $[.,.;F]\colon \Omega^2 \to L(\mathcal{B}_1, \mathcal{B}_2)$ be a divided difference of order one on Ω^2 for F. Suppose there exists $x^* \in \Omega$ and function $\omega_0\colon [0,+\infty)^2 \to [0,+\infty)$ continuous and nondecreasing with $\omega_0(0,0) = 0$, such that for each $x, y \in \Omega$,

$$F(x^*) = 0, \quad F'(x^*) \text{ is invertible} \tag{11}$$

and

$$\|F'(x^*)^{-1}([x, y; F] - F'(x^*))\| \le \omega_0(\|x - x^*\|, \|y - x^*\|). \tag{12}$$

Let $\Omega_0 = \Omega \cap B(x^*, r_0)$. There exist $\alpha \ge 0$, $\beta \ge 0$, functions $\omega, \omega_1\colon [0, r_0)^2 \to [0,+\infty)$, $\varphi_0, \varphi\colon [0, r_0) \to [0,+\infty)$ continuous and nondecreasing such that for each $x, y, u \in \Omega_0$,

$$\|F'(x^*)^{-1}([x, y; F] - [u, x^*; F])\| \le \omega_1(\|x - u\|, \|y - x^*\|), \tag{13}$$

$$\|F'(x^*)^{-1}([x, y; F] - [u, z; F])\| \le \omega(\|x - u\|, \|y - z\|), \tag{14}$$

$$\|[x, x^*; F]\| \le \varphi_0(\|x - x^*\|), \tag{15}$$

$$\|F'(x^*)^{-1}[x, x^*; F]\| \le \varphi(\|x - x^*\|), \tag{16}$$

$$\bar{B}(x^*, R^*) \subseteq \Omega, \tag{17}$$

$$\|I + [x, x^*; F]\| \le \alpha,$$

$$\|I - [x, x^*; F]\| \le \beta$$

and conditions (5), (6) hold, where r_0, r, R^* are defined by (3), (7) and (10), respectively. Then, the sequence $\{x_n\}$ generated for $x_0 \in U(x^*, r) - \{x^*\}$ by method (1) is well defined, remains in $B(x^*, r)$ for each $n = 0, 1, 2, \ldots$ and converges to x^*. Moreover, the following estimates hold

$$\|y_n - x^*\| \le g_1(\|x_n - x^*\|)\|x_n - x^*\| \le \|x_n - x^*\| < r, \tag{18}$$

$$\|z_n - x^*\| \le g_2(\|x_n - x^*\|)\|x_n - x^*\| \le \|x_n - x^*\|, \tag{19}$$

$$\|x_{n+1} - x^*\| \le g_3(\|x_n - x^*\|)\|x_n - x^*\| \le \|x_n - x^*\|, \tag{20}$$

where the functions g_i for $i = 1, 2, 3$ are defined previously. Furthermore, if there exists $R_1 \ge r$ such that

$$\omega_0(R_1, 0) < 1 \quad \text{or} \quad \omega_0(0, R_1) < 1,$$

then the limit point x^* is the only solution of equation $F(x) = 0$ in $\Omega_1 := \Omega \cap B(x^*, R_1)$.

Proof. The estimates (18)–(20) shall be shown using induction. First we show that A_0 is invertible, so y_0 is then well defined by the first substep of method (1) for $n = 0$. Using (3), (11) and (12), we have that

$$\begin{aligned}
&\|F'(x^*)^{-1}(A_0 - F'(x^*))\| \\
&\qquad \le \omega_0(\|x_0 - x^* - F(x_0)\|, \|x_0 - x^* + F(x_0)\|) \\
&\qquad \le \omega_0(\|(I + [x_0, x^*; F])(x_0 - x^*)\|, \|(I - [x_0, x^*; F])(x_0 - x^*)\|) \\
&\qquad \le \omega_0(\alpha\|x_0 - x^*\|, \beta\|x_0 - x^*\|) \\
&\qquad \le \omega_0(\alpha r_0, \beta r_0) < 1.
\end{aligned} \tag{21}$$

By (21) and the Banach perturbation lemma [2, 3], we deduce that A_0 is invertible and

$$\|A_0^{-1}F'(x^*)\| \le \frac{1}{1 - \omega_0(\alpha\|x_0 - x^*\|, \beta\|x_0 - x^*\|)}. \tag{22}$$

We can write by method (1) that

$$\begin{aligned}
y_0 - x^* &= x_0 - x^* - A_0^{-1}F(x_0) \\
&= A_0^{-1}F'(x^*)F'(x^*)^{-1}([u_0, v_0; F] - [x_0, x^*; F])\|x_0 - x^*\|.
\end{aligned} \tag{23}$$

In view of (7), (8) (for $i = 1$), (13), (15), (22) and (23), we get in turn that

$$\begin{aligned}
\|y_0 - x^*\| &\le \|A_0^{-1}F'(x^*)\|\|F'(x^*)^{-1}([u_0, v_0; F] - [x_0, x^*; F])\|\|x_0 - x^*\| \\
&\le \frac{\omega_1(\|u_0 - x_0\|, \|v_0 - x^*\|)\|x_0 - x^*\|}{1 - \omega_0(\alpha\|x_0 - x^*\|, \beta\|x_0 - x^*\|)} \\
&\le \frac{\omega_1(\|F(x_0)\|, \|F(x_0)\|)}{1 - \omega_0(\alpha\|x_0 - x^*\|, \beta\|x_0 - x^*\|)} \\
&\le \frac{\omega_1(\|[x_0, x^*; F](x_0 - x^*)\|, \|[x_0, x^*; F](x_0 - x^*)\|)}{1 - \omega_0(\alpha\|x_0 - x^*\|, \beta\|x_0 - x^*\|)} \\
&\le \frac{\omega_1(v_0(\|x_0 - x^*\|)\|x_0 - x^*\|, v_0(\|x_0 - x^*\|)\|x_0 - x^*\|)}{1 - \omega_0(\alpha\|x_0 - x^*\|, \beta\|x_0 - x^*\|)} \\
&= g_1(\|x_0 - x^*\|)\|x_0 - x^*\| \le \|x_0 - x^*\| \\
&< r,
\end{aligned} \tag{24}$$

which shows (18) for $n = 0$ and $y_0 \in B(x^*, r)$, where we also used

$$\begin{aligned}
\|u_0 - x^*\| &= \|x_0 - x^* + F(x_0)\| \\
&= \|(I + [x_0, x^*; F])(x_0 - x^*)\| \\
&\le \|I + [x_0, x^*; F]\|\|x_0 - x^*\| \\
&\le \alpha r
\end{aligned}$$

and $\|v_0 - x^*\| \le \|I - [x_0, x^*; F]\|\|x_0 - x^*\| \le \beta r$ so $u_0, v_0 \in B(x^*, r)$ (by (17)). Next, we must show B_0 is invertible, which will make z_0 well defined by the second substep of method (1) for $n = 0$. Using (3), (7), (9), (12), (14) and (24), we get in turn that

$$\begin{aligned}
&\|F'(x^*)^{-1}(B_0 - F'(x^*))\| \\
&\qquad \le \|F'(x^*)^{-1}([y_0, x_0; F] - F'(x^*))\| + \|F'(x^*)^{-1}([y_0, x_0; F] - [u_0, v_0; F])\| \\
&\qquad \le \omega_0(\|y_0 - x^*\|, \|x_0 - x^*\|) + \omega(\|y_0 - u_0\|, \|x_0 - v_0\|) \\
&\qquad \le \omega_0(g_1(\|x_0 - x^*\|)\|x_0 - x^*\|, \|x_0 - x^*\|) \\
&\qquad\quad + \omega(\|y_0 - x^*\| + \|u_0 - x^*\|, \|F(x_0)\|) \\
&\qquad \le \omega_0(g_1(\|x_0 - x^*\|)\|x_0 - x^*\|, \|x_0 - x^*\|) \\
&\qquad\quad + \omega(g_1(\|x_0 - x^*\| + \alpha)\|x_0 - x^*\|, \varphi_0(\|x_0 - x^*\|)\|x_0 - x^*\|) \\
&\qquad = p(\|x_0 - x^*\|) \le p(r) \\
&\qquad < 1,
\end{aligned}$$

so B_0 is invertible and

$$\|B_0^{-1}F'(x^*)\| \le \frac{1}{1 - p(\|x_0 - x^*\|)}. \tag{25}$$

It follows that z_0 and x_1 are well defined by method (2). Then, by the second substep of method (1) for $n = 0$, (1), (8) (for $i = 2$), (16), (24) and (25), we have in turn that

$$\begin{aligned}
\|z_0 - x^*\| &\le \|y_0 - x^*\| + \|B_0^{-1}F'(x^*)\|\|F'(x^*)^{-1}F(y_0)\| \\
&\le \Big(1 + \frac{\varphi(\|y_0 - x^*\|)}{1 - p(\|x_0 - x^*\|)}\Big)\|y_0 - x^*\| \\
&\le \Big(1 + \frac{\varphi(g_1(\|x_0 - x^*\|)\|x_0 - x^*\|)}{1 - p(\|x_0 - x^*\|)}\Big) g_1(\|x_0 - x^*\|)\|x_0 - x^*\| \\
&= g_2(\|x_0 - x^*\|)\|x_0 - x^*\| \le \|x_0 - x^*\| \\
&< r,
\end{aligned}$$

so (19) holds for $n = 0$ and $z_0 \in B(x^*, r)$. Then, from the last substep of method (1) for $n = 0$, (8) (for $i = 3$) and (25), we get in turn that

$$\begin{aligned}
\|x_1 - x^*\| &\le \|z_0 - x^*\| + \|B_0^{-1}F'(x^*)\|\|F'(x^*)^{-1}F(z_0)\| \\
&\le \Big(1 + \frac{\varphi(\|z_0 - x^*\|)}{1 - p(\|x_0 - x^*\|)}\Big)\|z_0 - x^*\| \\
&\le \Big(1 + \frac{\varphi(g_2(\|x_0 - x^*\|)\|x_0 - x^*\|)}{1 - p(\|x_0 - x^*\|)}\Big) g_2(\|x_0 - x^*\|)\|x_0 - x^*\| \\
&= g_3(\|x_0 - x^*\|)\|x_0 - x^*\| \le \|x_0 - x^*\| \\
&< r,
\end{aligned}$$

which shows (20) and $x_1 \in B(x^*, r)$. The induction for (18)–(20) is completed in an analogous way, if we replace $x_0, y_0, u_0, v_0, z_0, x_1$ by $x_k, y_k, u_k, v_k, z_k, x_{k+1}$, respectively in the previous estimates. Then, from the estimate

$$\|x_{k+1} - x^*\| \le c\|x_k - x^*\| < r,$$

where $c = g_3(\|x_0 - x*\|) \in [0, 1)$, we deduce that $\lim_{k\to\infty} x_k = x^*$ and $x_{k+1} \in B(x^*, r)$. The uniqueness part is shown by assuming $y^* \in \Omega_1$ with $F(y^*) = 0$.

Define linear operator T by $T = [y^*, x^*; F]$. Using (12) and (21), we have in turn that

$$\|F'(x^*)^{-1}(T - F'(x^*))\| \leq \omega_0(0, \|y^* - x^*\|) \leq \omega_0(0, R_1) < 1,$$

so T is invertible. If then follows from the identity $0 = F(y^*) - F(x^*) = T(y^* - x^*)$ that $x^* = y^*$.

Remark 2.2
Method (1) is not changing if we use the new instead of the old conditions [10, 11]. Moreover, for the error bounds in practice we can use the computational order of convergence (COC) [22]

$$\xi = \frac{\ln \frac{\|x_{n+2} - x^*\|}{\|x_{n+1} - x^*\|}}{\ln \frac{\|x_{n+1} - x^*\|}{\|x_n - x^*\|}} \qquad \text{for each } n = 1, 2, \ldots$$

or the approximate computational order of convergence (ACOC)

$$\xi^* = \frac{\ln \frac{\|x_{n+2} - x_{n+1}\|}{\|x_{n+1} - x_n\|}}{\ln \frac{\|x_{n+1} - x_n\|}{\|x_n - x_{n-1}\|}} \qquad \text{for each } n = 0, 1, 2, \ldots$$

instead of the error bounds obtained in Theorem 2.1.

3. Numerical Examples

The numerical examples are presented in this section. We choose

$$[x, y; F] = \int_0^1 F'(y + \theta(x - y))\, d\theta.$$

In the first example, we compute the convergence radius and (COC) not given in [12].

Example 3.1
Let $\mathcal{B}_1 = \mathcal{B}_2 = \mathbb{R}^3$, $\Omega = \bar{U}(0, 1)$, $x^* = (0, 0, 0)^T$. Define function F on D for $w = (x, y, z)^T$ by

$$F(w) = \Big(e^x - 1, \frac{e - 1}{2} y^2 + y, z\Big)^T.$$

Then, $x^* = (0, 0, 0)^T$ and the Fréchet-derivative is given by

$$F'(v) = \begin{bmatrix} e^x & 0 & 0 \\ 0 & (e-1)y + 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}.$$

Notice that using the (10) conditions, we get $\omega_0(s, t) = \frac{L_0}{2}(s+t)$, $\omega_1(s, t) = \frac{Ls + L_0 t}{2}$, $\omega(s, t) = \frac{1}{2}L(s+t)$, $\varphi_0(t) = \varphi(t) = \frac{1}{2}(1 + e^{\frac{1}{L_0}})$, $\alpha = \beta = 1 + \frac{1}{2}(1 + e^{\frac{1}{L_0}})$, $L_0 = e - 1$, $L = e$. The parameters are

$$r_1 = 0.1524, \quad r_2 = 0.7499, \quad r_3 = 0.0578 = r, \quad \xi = 4.9984.$$

The work in [10, 11, 12] cannot be used in the next example, since $\mathcal{B}_1 = \mathcal{B}_2 \neq \mathbb{R}^i$. This example is also used to show how to compute the convergence radii in abstract space setting.

Example 3.2
Let $\mathcal{B}_1 = \mathcal{B}_2 = C[0,1]$, $\Omega = \bar{U}(x^*, 1)$ and consider the nonlinear integral equation of the mixed Hammerstein-type [4, 6, 20] defined by

$$x(s) = \int_0^1 K(s,t)\frac{x(t)^2}{2}\,dt,$$

where the kernel K is the Green's function defined on the interval $[0,1]\times[0,1]$ by

$$K(s,t) = \begin{cases} (1-s)t, & t \leq s, \\ s(1-t), & s \leq t. \end{cases}$$

The solution $x^*(s) = 0$ is the same as the solution of equation (2), where $F\colon C[0,1] \to C[0,1])$ is defined by

$$F(x)(s) = x(s) - \int_0^1 K(s,t)\frac{x(t)^2}{2}\,dt.$$

Notice that

$$\left\| \int_0^1 K(s,t)\,dt \right\| \leq \frac{1}{8}.$$

Then, we have that

$$F'(x)y(s) = y(s) - \int_0^1 K(s,t)x(t)\,dt,$$

and $F'(x^*(s)) = I$,

$$\|F'(x^*)^{-1}(F'(x) - F'(y))\| \leq \frac{1}{8}\|x - y\|.$$

We can choose $\omega_0(t,s) = \omega_1(t,s) = \omega(s,t) = \frac{t+s}{16}$, $\varphi_0(t) = \varphi(t) = \frac{9}{16}$ and $\alpha = \beta = \frac{25}{16}$. The parameters are

$$r_1 = 0.6124, \quad r_2 = 0.1898, \quad r_3 = 0.1214 = r.$$

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