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# On a class of Riemannian manifolds with harmonic Weyl conformal curvature tensor

Dedicated to Professor Dr. Andrzej Zajtz on his seventieth birthday

**Abstract**. The paper deals with the local structure of those n-dimensional ($n \geq 5$) Riemannian manifolds of harmonic conformal curvature $(M, g)$ which are not conformally flat and admit a non-homothetic conformal change of metric $g \mapsto \bar{g}$ such that $(M, \bar{g})$ is locally symmetric.

## 1. Introduction

An $n$-dimensional ($n \geq 4$) pseudo-Riemannian manifold $(M, g)$ is called conformally symmetric [2] if its Weyl conformal curvature tensor

$$\begin{aligned} C_{hijk} = R_{hijk} &- \frac{1}{n-2}(g_{ij}S_{hk} - g_{ik}S_{hj} + g_{hk}S_{ij} - g_{hj}S_{ik}) \\ &+ \frac{K}{(n-1)(n-2)}(g_{ij}g_{hk} - g_{hj}g_{ik}) \end{aligned} \tag{1}$$

is parallel, i.e., $C_{hijk,l} = 0$. Herewith and in the sequel we denote the curvature tensor, Ricci tensor and scalar curvature by $R$, $S$ and $K$ respectively, while the comma stands for covariant differentiation with respect to the Levi-Civita connection.

Clearly, the class of conformally symmetric manifolds contains all locally symmetric ones ($n \geq 4$) as well as all conformally flat manifolds of dimension $n \geq 4$. In the Riemannian case there are no more examples ([4], Theorem 2).

But in general, for each $n \geq 4$, there exist ([5]) conformally symmetric manifolds with metrics of indices from the range $\{1, 2, \ldots, n-1\}$ which are neither conformally flat nor locally symmetric.

It is not hard to check (see (5)) that for every conformally symmetric manifold the condition

$$S_{ij,l} - S_{il,j} = \frac{1}{2(n-1)}(K_{,l}\, g_{ij} - K_{,j}\, g_{il}) \tag{2}$$

holds.

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An $n$-dimensional ($n \geq 2$) pseudo-Riemannian manifold is said to be nearly conformally flat [3] (or nearly conformally symmetric [10]) if its Ricci tensor satisfies condition (2). Any conformally symmetric manifold is therefore nearly conformally flat. Moreover, condition (2) shows that any $n$-dimensional ($n \geq 2$) manifold of harmonic curvature ($S_{ij,k} = S_{ik,j}$) is also nearly conformally flat.

The existence of essentially nearly conformally flat metrics, i.e. nearly conformally flat metrics which are neither conformally flat nor of harmonic curvature, can be stated as follows:

Example 1 ([10], Example 1)
Let $M = \mathbb{R}^{n-1} \times \mathbb{R}^1_+$, ($n \geq 5$) be endowed with the metric $g$ given by

$$g_{\lambda\mu} dx^\lambda dx^\mu = ((n-1)x^n)^{\frac{2}{n-1}} f_{ij} dx^i dx^j + (dx^n)^2,$$

where $\lambda, \mu = 1, 2, \ldots, n$, $i, j = 1, 2, \ldots, n-1$, and $f$ is an arbitrary non-flat Ricci-flat metric on $\mathbb{R}^{n-1}$ (which evidently exists since $n \geq 5$ ). Then $(M, g)$ is essentially nearly conformally flat.

From Theorem 7 of [5] it follows that essentially nearly conformally flat manifolds cannot be conformally symmetric ones. Nearly conformally flat manifolds ($n \geq 4$) with positive definite metrics are also said to have harmonic Weyl tensor (i.e., $\delta C = 0$, see [1], p. 435) or to be of harmonic conformal curvature. Throughout this paper we shall use the latter name.

Let $M$ be a manifold of of class $C^\infty$ endowed with a (not necessarily positive definite) metric $g$. If $\bar{g}$ is another metric on $M$ and there exists a smooth function $p$ on $M$ such that $\bar{g} = (\exp 2p)g$, then $g$ and $\bar{g}$ are said to be conformally related or conformal to each other, and such a change of metric $g \mapsto \bar{g}$ is called a conformal change. If $p = constant$, then the conformal change of the metric is called a homothety.

Nickerson initiated [8] investigations of Riemannian manifolds $(M, g)$ admitting a conformal change of metric $g \mapsto \bar{g}$ such that $(M, \bar{g})$ is locally symmetric.

The present paper deals with similar problems. It contains at generic points (Theorem 2) a full description of the local structure of those $n$-dimensional ($n \geq 5$) (Riemannian) manifolds of harmonic conformal curvature $(M, g)$ which are not conformally flat and admit a non-homothetic conformal change of metric $g \mapsto \bar{g}$ such that $(M, \bar{g})$ is locally symmetric. Theorem 2 bases on the following result:

Theorem 1
*Let $(M, g)$, $\dim M \geq 4$, be of harmonic conformal curvature. If $(M, g)$ is not conformally flat and it admits a non-homothetic conformal change of metric $g \mapsto \bar{g} = (\exp 2p)g$ such that $(M, \bar{g})$ is conformally symmetric, then $\dim M \geq 5$ and $(M, \bar{g})$ is a locally reducible locally symmetric manifold.*

Throughout this paper, all manifolds under consideration are assumed to be connected and of class $C^\infty$. Their metrics, unless stated otherwise, are assumed to be positive definite.

## 2. Preliminaries

In the sequel we need the following results:

Lemma 1
*The Weyl conformal curvature tensor satisfies the well-known equations:*

$$C_{hijl} = -C_{ihjl} = -C_{hilj} = C_{jlhi}\,, \tag{3}$$

$$C_{hijl} + C_{hjli} + C_{hlij} = 0, \qquad C^r{}_{ijr} = C^r{}_{irl} = C^r{}_{rjl} = 0, \tag{4}$$

$$C^r{}_{ijl,r} = \frac{n-3}{n-2}(S_{ij,l} - S_{il,j} - \frac{1}{2(n-1)}(K_{,l}g_{ij} - K_{,j}g_{il})). \tag{5}$$

Lemma 2 ([6], p. 89-90)
*Let $\bar{g}_{ij} = (\exp 2p)g_{ij}$ . Then we have:*

$$\bar{\Gamma}^i_{jk} = \Gamma^i_{jk} + \delta^i_j p_k + \delta^i_k p_j - p^i g_{jk}\,, \tag{6}$$

$$\bar{C}^h{}_{ijl} = C^h{}_{ijl}\,, \tag{7}$$

*where $\Gamma$ denotes Christoffel symbols, $p_i = p_{,i}$ and $p^h = g^{hr}p_r$.*

Lemma 3
*Let $\bar{g}_{ij} = (\exp 2p)g_{ij}$. Then we have:*

$$\bar{C}^r{}_{ijk;r} = C^r{}_{ijk,r} + (n-3)p_r C^r{}_{ijk}\,, \tag{8}$$

*where the semicolon denotes covariant differentiation with respect to $\bar{g}$.*

*Proof.* Differentiating (7) covariantly, using (6) and Lemma 1, we obtain

$$\begin{aligned}\bar{C}^h{}_{ijk;l} = C^h{}_{ijk,l} &+ \delta^h_l p_r C^r{}_{ijk} - 2p_l C^h{}_{ijk} - p^h C_{lijk} - p_i C^h{}_{ljk} - p_j C^h{}_{ilk} \\ &- p_k C^h{}_{ijl} + g_{il}p^r C^h{}_{rjk} + g_{jl}p^r C^h{}_{irk} + g_{kl}p^r C^h{}_{ijr}\,.\end{aligned} \tag{9}$$

Equation (8) follows from (9) and Lemma 1. This completes the proof.

Lemma 4 ([4], Theorem 2)
*Let $(M,g)$ be a Riemannian conformally symmetric manifold. If it is not conformally flat, then $(M,g)$ is locally symmetric.*

Lemma 5 ([11], Theorem 3)
*Let $(M,g)$ be a pseudo-Riemannian conformally symmetric manifold. If it admits a conformal change of metric $g \mapsto \bar{g}$ such that $(M,\bar{g})$ is conformally symmetric, then both $(M,g)$ and $(M,\bar{g})$ are conformally flat or the conformal change of metric is a homothety.*

Remark 1
It is known ([12], p. 286) that a Riemannian manifold is locally decomposable if and only if it admits a symmetric parallel tensor field of type $(0,2)$ which is not a multiple of the metric tensor.

If $(M,g)$ is locally decomposable and $\dim M = n$, then coordinates $(x^1,\ldots,x^{r_1},x^{r_1+1},\ldots,x^{r_1+r_2},\ldots,x^n)$ can be locally chosen so (see [13], p. 414 ) that its metric takes the form:

$$\begin{bmatrix} g_{i_1 j_1} & & & \\ & g_{i_2 j_2} & & \\ & & \ddots & \\ & & & g_{i_t j_t} \end{bmatrix}, \tag{10}$$

where $i_1,j_1 = 1,\ldots,r_1$, $i_2,j_2 = r_1+1,\ldots,r_1+r_2,\ldots$, $i_t,j_t = 1+\sum_{l=1}^{t-1} r_l,\ldots,n$, and the tensors $g_s$ $(s = 1,\ldots,t)$ given by $g_1 = [g_{i_1 j_1}(x^1,\ldots,x^{r_1})]$, $g_2 = [g_{i_2 j_2}(x^{r_1+1},\ldots,x^{r_1+r_2})],\ldots$ are irreducible. $M$ can be therefore locally written in the form $M_1 \times \ldots \times M_t$ and its metric is the direct sum of the metrics on $M_i$'s. Obviously, if one or more of the $M_i$'s are 1-dimensional, then, by a renumeration of coordinates, (10) can be modified so that $(M_1,g_1)$ is Euclidean and that all $g_k$ $(k = 2,\ldots,m \le t)$ are irreducible and no one of the $M_k$'s is 1-dimensional. Moreover, (10) implies

$$[g_{ij}] = \begin{bmatrix} g_{ab} & \\ & g_{AB} \end{bmatrix}, \tag{11}$$

where $a,b = 1,\ldots,r$, $A,B = r+1,\ldots,n$, $g_{ab}$ are functions of $x^1,\ldots,x^r$ only, and $g_{AB}$ depend on $x^{r+1},\ldots,x^n$ only. Clearly, in a matrix of the form (11) the tensors $g_1$ and $g_2$ can be reducible.

Lemma 6
*In the metric* (11)*, the only components of the Weyl conformal curvature tensor and its covariant derivative which may not vanish are those related to*

$$\begin{aligned} C_{abcd} = R_{abcd} &- \frac{1}{n-2}(g_{bc}S_{ad} - g_{bd}S_{ac} + g_{ad}S_{bc} - g_{ac}S_{bd}) \\ &+ \frac{Q+N}{(n-1)(n-2)}(g_{bc}g_{ad} - g_{ac}g_{bd}), \end{aligned} \tag{12}$$

$$C_{ABCD} = R_{ABCD} - \frac{1}{n-2}(g_{BC}S_{AD} - g_{BD}S_{AC} + g_{AD}S_{BC} - g_{AC}S_{BD}) + \frac{Q+N}{(n-1)(n-2)}(g_{BC}g_{AD} - g_{AC}g_{BD}), \tag{13}$$

$$C_{aABc} = -\frac{1}{n-2}\left(\left(S_{ac} - \frac{1}{n-1}Qg_{ac}\right)g_{AB} + \left(S_{AB} - \frac{1}{n-1}Ng_{AB}\right)g_{ac}\right), \tag{14}$$

$$C_{ABCD,a} = \frac{1}{(n-1)(n-2)}Q_{,a}(g_{BC}g_{AD} - g_{AC}g_{BD})\,, \tag{15}$$

$$C_{abcd,A} = \frac{1}{(n-1)(n-2)}N_{,A}(g_{bc}g_{ad} - g_{ac}g_{bd}), \tag{16}$$

$$C_{aABc,d} = -\frac{1}{n-2}(S_{ac,d} - \frac{1}{n-1}Q_{,d}g_{ac})g_{AB}\,, \tag{17}$$

$$C_{aABc,D} = -\frac{1}{n-2}(S_{AB,D} - \frac{1}{n-1}N_{,D}g_{AB})g_{ac}\,, \tag{18}$$

$$C_{abcd,e} = R_{abcd,e} - \frac{1}{n-2}(g_{bc}S_{ad,e} - g_{bd}S_{ac,e} + g_{ad}S_{bc,e} - g_{ac}S_{bd,e}) + \frac{1}{(n-1)(n-2)}Q_{,e}(g_{bc}g_{ad} - g_{ac}g_{bd}), \tag{19}$$

$$C_{ABCD,E} = R_{ABCD,E} - \frac{1}{n-2}(g_{BC}S_{AD,E} - g_{BD}S_{AC,E} + g_{AD}S_{BC,E} - g_{AC}S_{BD,E}) + \frac{1}{(n-1)(n-2)}N_{,E}(g_{BC}g_{AD} - g_{AC}g_{BD}), \tag{20}$$

*where* $a, b, c, d, e = 1, 2, \ldots, r$*,* $A, B, C, D, E = r+1, \ldots, n$*, and* $Q$ *and* $N$ *denote the scalar curvatures of the metrics* $[g_{ab}]$ *and* $[g_{AB}]$*, respectively.*

The proof is obvious.

Lemma 7

*Let* $(M, g)$ *be conformally symmetric with a possibly indefinite metric. If the metric can be locally written in the form* (11)*, then* $(M, g)$ *is locally symmetric.*

The proof follows easily from equations (15)-(20).

The following two results seem to be well known:

Lemma 8
*Let $M = M_1 \times M_2$ be an $n$-dimensional $(n \geq 4)$ pseudo-Riemannian manifold, where $M_1$ and $M_2$ are of constant sectional curvature, $\dim M_1 = r \geq 1$, $\dim M_2 = s \geq 1$ and $r + s = n$.*[1] *Denote by $Q$ and $N$ the scalar curvatures of $M_1$ and $M_2$, respectively. Then $M$ is conformally flat if and only if the condition*

$$s(s-1)Q + r(r-1)N = 0 \tag{21}$$

*holds.*

Lemma 9
*Let $M = M_1 \times M_2 \times \ldots \times M_t$ be a pseudo-Riemannian manifold, $M_i$ $(\dim M_i = q_i \geq 1,\ i = 1, 2, \ldots, t)$ being Einstein manifolds with scalar curvatures $Q_i$. Denote by $Q$ the scalar curvature of $M$ and let $q = \dim M$. Then $M$ is Einsteinian if and only if*

$$\frac{1}{q} Q = \frac{1}{q_i} Q_i \qquad (i = 1, 2, \ldots, t).$$

Lemma 10
*Let $(M, g)$ be an $n$-dimensional $(n \geq 2)$ Riemannian locally symmetric manifold whose curvature tensor satisfies the condition*

$$v_r R^r{}_{ijl} = B_l g_{ij} - B_j g_{il} \tag{22}$$

*for some covector fields $v$ and $B$. If $(M, g)$ is locally irreducible and if at least one of the covector fields $v$ or $B$ does not identically vanish, then $(M, g)$ is of constant sectional curvature.*

*Proof.* Obviously, $(M, g)$ is an Einstein manifold with constant scalar curvature and its curvature tensor satisfies the condition

$$R^{hij}{}_k R_{hijl} = \tau g_{kl} \tag{23}$$

where $\tau = \mathit{constant}$. Transvecting (22) with $R^{lji}{}_q$ and making use of (23), we easily obtain

$$\tau v_q = \frac{2}{n} K B_q\,. \tag{24}$$

On the other hand, condition (22) yields

$$\frac{1}{n} K v_l = (n-1) B_l$$

which, together with (24), implies

[1] Throughout this paper, 1-dimensional manifolds are ass med to be of constant sectional curvature.

$$\left(\tau - \frac{2}{n^2(n-1)}K^2\right) v_l = 0. \tag{25}$$

Assume that $v_i$ does not vanish at least at some point of $M$. Since $\tau$ and $K$ are *constants*, (25) gives

$$\tau = \frac{2}{n^2(n-1)}K^2.$$

But, by (23), we have $R^{hihk}R_{hijk} = \| R \|^2 = n\tau$, which, together with the last result, implies

$$\| R \|^2 = \frac{2}{n(n-1)}K^2. \tag{26}$$

Now, let $T$ be given by

$$T_{hijl} = R_{hijl} - \frac{K}{n(n-1)}(g_{ij}g_{hl} - g_{hj}g_{il}).$$

Then, in view of (26), we get $\|T\|^2 = 0$. Thus, $T = 0$, which completes the proof in the case $v \neq 0$. If $B_i$ does not identically vanish, then the proof is quite similar.

## 3. Locally decomposable manifolds

We are now in a position to prove Theorem 1.

*Proof of Theorem 1.* By (2) and (5), we get

$$C^r{}_{ijl,r} = 0, \tag{27}$$

which, in view of (8) and (7), yields

$$p_r\bar{C}^r{}_{ijl} = 0. \tag{28}$$

But (28), together with

$$\bar{C}^{hij}{}_l\bar{C}_{hijk} = \mu\bar{g}_{lk}\,.$$

which holds for every 4-dimensional manifold [9], implies $\mu p_l = 0$. Hence, $\bar{C} = 0$ at some point. Since $\bar{C}$ is parallel, it vanishes therefore everywhere, a contradiction. Thus $\dim M \geq 5$.

Assume that $(M, \bar{g})$ is locally irreducible. By (7) and Lemma 4, $(M, \bar{g})$ is locally symmetric and, in consequence, it must be Einsteinian. Thus, in view of (28), we have

$$p_r\bar{R}^r{}_{ijl} = \frac{1}{n(n-1)}\bar{K}(p_l\bar{g}_{ij} - p_j\bar{g}_{il}),$$

which, by Lemma 10, shows that $(M, \bar{g})$ is of constant sectional curvature. Consequently, $(M, \bar{g})$ is conformally flat, a contradiction. The last remark completes the proof.

Lemma 11
*Let $(M, g)$, $\dim M \geq 5$, be of harmonic conformal curvature admitting a non-homothetic conformal change of metric $g \mapsto \bar{g} = (\exp 2p)g$ such that $(M, \bar{g})$ is conformally symmetric. Assume that $(M, \bar{g})$ is in some coordinate neighbourhood $U$ decomposable into $M_1 \times M_2$, $\dim M_1 = r \geq 1$, $\dim M_2 = s = n - r \geq 1$. (The metric of $(M, \bar{g})$ is therefore in $U$ of the form (11)). If one of the $M_i'$s, say $M_1$, is either irreducible or Euclidean and there exists a point in $U$ at which the gradient of $p$ does not vanish in the direction of $M_1$ (i.e. $p_a = p_{,a} \neq 0$ for some $a \in \{1, 2, \ldots, r\}$), then*

(i) *$M_1$ is of constant sectional curvature (with constant scalar curvature) and $M_2$ is Einsteinian with parallel curvature tensor.*

(ii) *Condition (21) holds, where $Q$ and $N$ denote the scalar curvatures of $M_1$ and $M_2$, respectively.*

*Proof.* Both $M_1$ and $M_2$ are locally symmetric since $M_1 \times M_2$ does so (cf. Lemma 7). On the other hand, because of (28), equations (12)-(14) imply

$$p_a \bar{C}^a{}_{BDd} = 0, \quad p_a \bar{C}^a{}_{bcd} = 0, \quad p_A \bar{C}^A{}_{BCD} = 0, \quad p_A \bar{C}^A{}_{bdE} = 0, \tag{29}$$

whence, by (14), $\bar{S}_{ab} = \frac{1}{r} Q \bar{g}_{ab}$, $\bar{\nabla} \bar{C} = 0$ and $p_a \neq 0$, we get

$$\bar{S}_{BD} = \frac{1}{n-1} \left( N - \frac{s-1}{r} Q \right) \bar{g}_{BD} \, . \tag{30}$$

The last result shows that $M_2$ is Einsteinian and that condition (21) holds. Moreover, (12) yields

$$\bar{C}_{abcd} = \bar{R}_{abcd} - \frac{1}{n-2} \left( \frac{2}{r} Q - \frac{Q+N}{n-1} \right) (\bar{g}_{ad}\bar{g}_{bc} - \bar{g}_{ac}\bar{g}_{bd}),$$

which, because of (29), implies

$$p_a \bar{R}^a{}_{bcd} - \frac{1}{n-2} \left( \frac{2}{r} Q - \frac{Q+N}{n-1} \right) (p_d \bar{g}_{bc} - p_c \bar{g}_{bd}) = 0. \tag{31}$$

But from the last result, we have

$$\frac{1}{r} Q - \frac{r-1}{n-2} \left( \frac{2}{r} Q - \frac{Q+N}{n-1} \right) = 0,$$

which shows that in the case $\dim M > 1$ equation (31) takes the form

$$p_a \bar{R}^a{}_{bcd} = \frac{1}{r(r-1)} Q(p_d \bar{g}_{bc} - p_c \bar{g}_{bd}).$$

Since $p_a$ does not identically vanish, the assertion is therefore an immediate consequence of Lemma 10. This completes the proof.

Lemma 12

*Let $(M, g)$, $\dim M \geq 5$ be of harmonic conformal curvature admitting a conformal change of metric $g \mapsto \bar{g} = (\exp 2p)g$ such that $(M, \bar{g})$ is conformally symmetric. Assume that $(M, \bar{g})$ is in some coordinate neighbourhood $U$ decomposable into $M_1 \times M_2 \times \ldots \times M_t$, where $M_1$ is Euclidean or irreducible and the others of the $M_j's$ $(j = 2, \ldots, t)$ are irreducible and no one of them is 1-dimensional (the metric of $(M, \bar{g})$ is therefore in $U$ of the form (10), cf. Remark 1). If $p \neq$ constant on $U$, then $(M, \bar{g})$ is conformally flat or there is only one of the $M_i's$ $(i = 1, \ldots, t)$ in the direction of which the gradient of $p$ does not identically vanish (i.e., there exists $s \in \{1, 2, \ldots, t\}$ such that for any $q \neq s$ the condition $p_{i_q} = p_{,i_q} = 0$ holds everywhere on $U$, where $(x^{i_q})$ denote coordinates in $M_i$.)*

*Proof.* Let $t \geq 3$. Suppose that among $M_i's$ there exist at least two such in the direction of which the gradient of $p$ does not identically vanish on $U$. Without loss of generality, we may assume (it is enough to change the numeration of coordinates if it is necessary) that the metric is in $U$ of the form (10) and that among $(x^{i_1})$ and $(x^{i_t})$ there exist at least two coordinates $x^{\lambda_1}$ and $x^{\lambda_t}$ such that both componets $p_{\lambda_1} = p_{,\lambda_t}$ and $p_{\lambda_t} = p_{,\lambda_t}$ do not vanish identically on $U$. Lemma 11 shows that both $M_1$ and $M_t$ are of *constant* sectional curvature (with constant scalar curvatures), $N_1 = M_2 \times \ldots \times M_t$ and $N_2 = M_1 \times \ldots \times M_{t-1}$ are Einsteinian and condition (21) holds.

Denote by $Q_i$ the scalar curvature of $M_i$ and let $q_i = \dim M_i$. Then, in view of (21), we have

$$q_1(1-q_1) \sum_{i=2}^{t} Q_i + Q_1 \sum_{i=2}^{t} q_i \left(1 - \sum_{i=2}^{t} q_i\right) = 0, \tag{32}$$

$$Q_t \sum_{i=1}^{t-1} q_i \left(1 - \sum_{i=1}^{t-1} q_i\right) + q_t(1-q_t) \sum_{i=1}^{t-1} Q_i = 0. \tag{33}$$

On the other hand, Lemma 9 implies

$$Q_i = \frac{1}{q_t} Q_t q_i \quad (i = 2, \ldots, t), \qquad Q_i = \frac{1}{q_1} Q_1 q_i \quad (i = 1, \ldots, t-1), \tag{34}$$

which, together with (32) and (33), yields

$$q_1(1-q_1)Q_t + q_tQ_1\left(1-\sum_{i=2}^{t} q_i\right) = 0,$$

$$q_t(1-q_t)Q_1 + q_1Q_t\left(1-\sum_{i=1}^{t-1} q_i\right) = 0.$$

Consequently, we have

$$Q_1\left(\left(1-\sum_{i=2}^{t} q_i\right)\left(1-\sum_{i=1}^{t-1} q_i\right) - (1-q_1)(1-q_t)\right) = 0,$$

whence it follows $Q_1 = Q_t = 0$. Thus, both $M_1$ and $M_t$ are Euclidean, a contradiction. Assume now that $t = 2$ and that there exist $p_b$, $b \in \{1, 2, \ldots, r\}$ and $p_B$, $B \in \{r+1, \ldots, n\}$, which do not vanish on $U$ (cf. Remark 1). Then, by Lemma 11, both $M_1$ and $M_2$ are of constant sectional curvature and condition (21) holds. Hence, in view of Lemma 8, $M_1 \times M_2$ is conformally flat. Since $\bar{C}$ is parallel and it vanishes on $U$, it does so everywhere. Consequently, $(M, \bar{g})$ is conformally flat. The last remark completes the proof.

## 4. A local structure result

We are now in a position to prove the following result:

Theorem 2

(i) *Let $(M_1, g_1)$ be of constant sectional curvature $K$ ($K$ = constant), $F$ a positive non-constant function on $M_1$, and $(M_2, g_2)$ a locally symmetric Einstein manifold whose scalar curvature $N = -s(s-1)K$, where $s = \dim M_2$. If $(M_2,\, g_2)$ is not of constant sectional curvature and $\dim M_1 + \dim M_2 \geq 5$, then $M = M_1 \times M_2$ with the warped product metric $g = F^2g_1 + F^2g_2 = (\exp 2\log F)(g_1 \oplus g_2)$ is of harmonic conformal curvature and it admits a non-homothetic conformal change of metric $g \mapsto \bar{g}$ such that $(M, \bar{g})$ is locally symmetric. Moreover, $(M, g)$ is neither conformally flat nor locally symmetric.*

(ii) *Let $(M, g)$, $\dim M \geq 4$ be of harmonic conformal curvature admitting a non-homothetic conformal change of metric $g \mapsto \bar{g} = (\exp 2p)g$ such that $(M, \bar{g})$ is conformally symmetric. If $(M, g)$ is not conformally flat, then $\dim M \geq 5$ and for each point $x \in M$ satisfying $(\operatorname{grad} p)(x) \neq 0$ coordinates can be chosen in a neighbourhood of $x$ so that the metric of $M$ takes the above stated warped product form with properties described in* (i).

*Proof.* (i) Obviously, coordinates can be locally chosen so that the metric of $M$ can be written as

$$[g_{ij}] = e^{-2p} \begin{bmatrix} \bar{g}_{ab} & \\ & \bar{g}_{AB} \end{bmatrix}, \tag{35}$$

where $\bar{g}_{ab}(x^1, \ldots, x^r)$ denote the components of $g_1$, $\bar{g}_{AB}(x^{r+1}, \ldots, x^n)$ the components of $g_2$, $r = \dim M_1 = n - s$ and $p = -\log F \neq \textit{constant}$ is a function of $x^1, \ldots, x^r$ only. Thus,

$$[\bar{g}_{ij}] = e^{2p}[g_{ij}]. \tag{36}$$

Since $(M_1, g_1)$ is of constant sectional curvature (with constant scalar curvature), $(M_2, g_2)$ is a locally symmetric Einstein manifold and condition (21) holds, $(M, \bar{g})$ is locally symmetric and equations (12)-(14) imply $\bar{C}_{abcd} = \bar{C}_{aABc} = 0$ and

$$\bar{C}_{ABCD} = \bar{R}_{ABCD} - \frac{1}{s(s-1)} N(\bar{g}_{BC}\bar{g}_{AD} - \bar{g}_{BD}\bar{g}_{AC}). \tag{37}$$

Moreover, one can easily check that condition (28) is satisfied. Hence, in view of (36), (7), (8) and $\bar{\nabla}\bar{C} = 0$, we obtain (27), which shows that $(M, g)$ is of harmonic conformal curvature. On the other hand, by virtue of (36) and the definition of $g$ it follows that $(M, g)$ admits a non-homothetic conformal change of metric $g \mapsto \bar{g}$ such that $(M, \bar{g})$ is locally symmetric. Because of (37) and (7), $(M, g)$ is not conformally flat and, by Lemma 5, it cannot be locally symmetric. The last remark completes the proof of (i).

(ii) Theorem 1 shows that $\dim M \geq 5$ and $(M, \bar{g})$ is a locally reducible locally symmetric manifold. Consequently, its metric has locally the form (10), where $(M_1, g_1)$ is either Euclidean or irreducible, and any other $(M_i, g_i)$ is neither reducible nor of dimension 1.

Denote by $U$ a coordinate neighourhood in which $\operatorname{grad} p$ does not vanish identically. Whithout loss of generality we may assume (it is enough to renumerate the coordinates if it is necassary) that $(M_1, g_1)$ with $g_1$ given by $[\bar{g}_{ab}]$, $a, b = 1, 2, \ldots, r$, is either Euclidean or irreducible, and that $p_c(x) \neq 0$ for some $x \in U$ and $c \in \{1, \ldots, r\}$. Obviously, the metric (10) can be written in the form (11), whence, by Lemma 11, it follows that $(M_1, g_1)$ is of constant sectional curvature (with constant scalar curvature), $(M_2, g_2)$ is Einsteinian with parallel curvature tensor and condition (21) holds. Moreover, Lemma 12 shows that $p$ is a non-constant function of $x^1, \ldots, x^r$ only.

Assume now that $(M_2, g_2)$ is of constant sectional curvature. Then, because of (37) and $\bar{C}_{abcd} = \bar{C}_{aABc} = 0$, the Weyl conformal curvature tensor $\bar{C}$ would vanish on $U$. Since $\bar{C}$ is parallel by assumption, it would vanish everywhere on $M$, a contradiction. Hence, in view of (35), $(M, g)$ has in $U$ the required warped product form with $F \neq \textit{constant}$ and properties described in (i). This completes the proof.

Remark 2

The main part of Theorem 2(i) is due to A. Derdziński (cf. [1], p. 442).

Remark 3
In the case $B \neq 0$ the assertion of Lemma 10 follows also from a result of Grycak (cf. [7], Theorem 1).

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