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New method to solve certain differential equations

Abstract. A new method to solve stationary one-dimensional Schroedinger equation is investigated. Solutions are described by means of representation of circles with multiple winding number. The results are demonstrated using the well-known analytical solutions of the Schroedinger equation.

1. Introduction

One of methods to obtain the equation of a unit circle with winding number equal to n [1] is raising the equation

$$Q_0^2 + P_0^2 = 1 \tag{1}$$

to the n-th power, where $Q_0, P_0 \in \mathbb{R}$. For instance, in the case $n = 2$, by appropriate grouping of terms we get

$$\overline{Q}_1^2 + \overline{P}_1^2 = 1, \tag{2}$$

where

$$\overline{Q}_1^2 = (2Q_0^2 - 1)^2, \tag{3}$$

$$\overline{P}_1^2 = 4Q_0^2(1 - Q_0^2). \tag{4}$$

The results are illustrated in Figure 1.

Generally, a circle with winding number equal to n is represented by the sum of polynomials of $2(n+1)$ degree and proper number of zeros. Moreover, it can be seen that in this representation the domain of Q_0 does not change regardless of the winding number.

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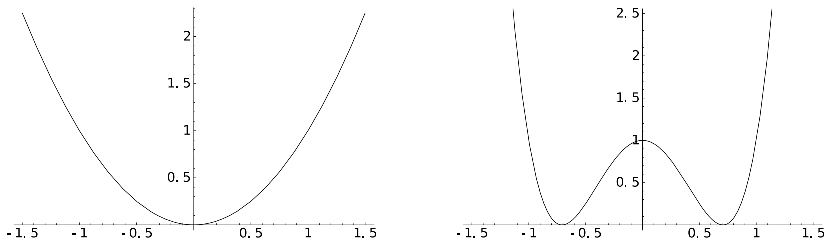


(a) Graph of the function $Q_0^2(Q_0)$. (b) Graph of the function $\overline{Q}_1^2(Q_0)$.

Figure 1: Graphs of the second order polynomials which form the equations of the circles with $n = 1$ (a) and $n = 2$ (b), respectively. Q_0 is on the horizontal axis.

2. Unit circle transformation

In the present paper, we use the transformations (3), (4) and others to construct solutions to the nonlinear Schroedinger equations. We are looking for a transformation $Q_0 \mapsto Q_1(Q_0)$ ($P_0 \mapsto P_1(Q_0)$), where the functions $Q_1(Q_0)$, $P_1(Q_0)$ have two zeros and three zeros respectively, satisfy the equation

$$Q_1^2 + P_1^2 = 1.$$

This equation, by analogy to (2), represents a unit circle with winding number $n = 2$ if we choose Q_1^2, P_1^2 in the form

$$Q_1^2 = \frac{(a_{11}Q_0 - b_{11})^2(a_{12}Q_0 - b_{12})^2}{A_{11}Q_0^2 + B_{11}Q_0 + C_{11}},$$

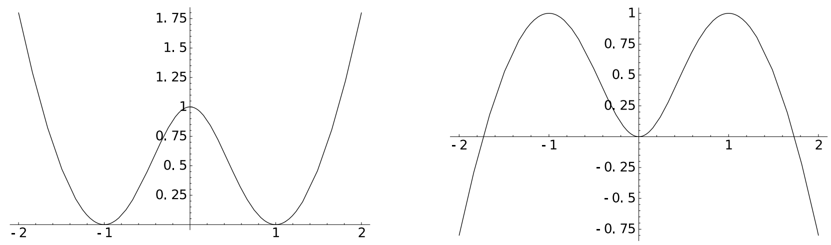
$$P_1^2 = \frac{(\alpha_{11}Q_0 - \beta_{11})^2(\alpha_1 + \beta_1 Q_0 + \gamma_1 Q_0^2)}{A_{11}Q_0^2 + B_{11}Q_0 + C_{11}},$$

where $A_{11} > 0$, $B_{11}^2 - 4A_{11}C_{11} < 0$ and $\beta_1^2 - 4\alpha_1\gamma_1 > 0$. For example, functions which represent the well-known analytical solutions of the quantum oscillator [2, 3],

$$Q_1^2 = \frac{(Q_0^2 - 1)^2}{1 + Q_0^2}, \tag{5}$$

$$P_1^2 = \frac{Q_0^2(3 - Q_0^2)}{1 + Q_0^2} \tag{6}$$

are illustrated in Figure 2.



(a) Graph of the function $Q_1^2(Q_0)$. (b) Graph of the function $P_1^2(Q_0)$.

Figure 2: Graphs of the rational functions (see equations (5) and (6)) which form the equation of circle with $n = 2$. Q_0 is on the horizontal axis.

On the basis of the procedure described above we are able to get functions $Q_n(Q_0), P_n(Q_0)$ which satisfy a unit circle equation

$$Q_n^2 + P_n^2 = 1,$$

where winding number is equal to $n+1$. Thus,

$$Q_n^2 = \frac{(a_{n1}Q_0 - b_{n1})^2(a_{n2}Q_0 - b_{n2})^2 \dots (a_{nn+1}Q_0 - b_{nn+1})^2}{A_{n1}Q_0^{2n} + A_{n2}Q_0^{2n-1} + \dots + A_{n2n}Q_0 + A_{n2n+1}},$$

$$P_1^2 = \frac{(\alpha_{n1}Q_0 - \beta_{n1})^2 \dots (\alpha_{nn}Q_0 - \beta_{nn})^2(\alpha_n + \beta_n Q_0 + \gamma_n Q_0^2)}{A_{n1}Q_0^{2n} + A_{n2}Q_0^{2n-1} + \dots + A_{n2n}Q_0 + A_{n2n+1}},$$

where $A_{n1}Q_0^{2n} + A_{n2}Q_0^{2n-1} + \dots + A_{n2n}Q_0 + A_{n2n+1} > 0$ for $Q_0 \in \mathbb{R}$ and $\beta_n^2 - 4\alpha_n\gamma_n > 0$.

3. Schroedinger equation

In the present paper, we consider the Schroedinger equation in one dimension

$$\frac{d^2}{dx^2}\psi_n(x) + (E_n - V(x))\psi_n(x) = 0.$$

Then the function

$$W_n(x) = -\frac{\psi_n'(x)}{\psi_n(x)},$$

where prime denotes differentiation with respect to x, satisfies the corresponding Riccati equation [4, 5, 6]

$$W_n'(x) - W_n^2(x) = E_n - V(x).$$

Assuming that the function $W_0(x)$ has a zero inside an interval I and

$$W_0'(x) > 0 \qquad \text{for all } x \in I \subset \mathbb{R},$$

what is associated with normalization of the basic function ψ_0, we get the equation of the unit circle (1), where

$$Q_0^2 = \frac{W_0^2(x)}{W_0'(x)}, \qquad P_0^2 = \frac{E_0 - V(x)}{W_0'(x)}$$

and $E_0 - V(x) \geq 0$.

Now we can express the function W_1^2 in terms of W_0. From $\frac{Q_1^2}{P_1^2}$ we get

$$W_1^2 = \frac{(a_{11}Q_0 - b_{11})^2(a_{12}Q_0 - b_{12})^2}{(\alpha_{11}Q_0 - \beta_{11})^2}\frac{E_1 - E_0 + W_0'(x) - W_0^2(x)}{\alpha_1 + \beta_1 Q_0 + \gamma_1 Q_0^2}.$$

Taking into account the properties of the function W_1 we obtain

$$W_0' = A + BW_0 + CW_0^2,$$

which should not be treated as a differential equation. Hence, we get

$$W_0' = \frac{L_{n+2}(W_0)}{R_n(W_0)},$$

where L_{n+2} is a polynomial in W_0 of degree $n+2$ and R_n is a polynomial of degree n. Finally,

$$W_1^2 = \frac{(a_{11}Q_0 - b_{11})^2(a_{12}Q_0 - b_{12})^2}{(\alpha_{11}Q_0 - \beta_{11})^2}W_0'(x)$$

and in general

$$W_n^2 = \frac{(a_{n1}Q_0 - b_{n1})^2(a_{n2}Q_0 - b_{n2})^2 \ldots (a_{nn+1}Q_0 - b_{nn+1})^2}{(\alpha_{n1}Q_0 - \beta_{n1})^2 \ldots (\alpha_{nn}Q_0 - \beta_{nn})^2}W_0'(x),$$

what allows to find solutions of the Schroedinger equation using only W_0.

4. Conclusions

In this paper, we have proposed the new method of solving one-dimensional stationary Schroedinger equation. This method is closely related to the properties of the unit circle equation which not only allow to find general solutions of the Schroedinger equation but may also help in further investigations concerning the relationship between quantum integral conditions, which are known in quantum mechanics as the Bohr-Sommerfeld condition, and the area of a circle. Moreover, this method can be also applied to differential equations of the relativistic quantum mechanics [7].

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