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A sharp companion of Ostrowski's inequality for the Riemann–Stieltjes integral and applications

Abstract. A sharp companion of Ostrowski's inequality for the Riemann-Stieltjes integral $\int_a^b f(t)\,du(t)$, where f is assumed to be of r-H-Hölder type on $[a,b]$ and u is of bounded variation on $[a,b]$, is proved. Applications to the approximation problem of the Riemann-Stieltjes integral in terms of Riemann-Stieltjes sums are also pointed out.

1. Introduction

In [12], Dragomir has proved an Ostrowski inequality for the Riemann-Stieltjes integral, as follows:

Theorem 1.1
Let $f\colon [a,b] \to \mathbb{R}$ be a r-H–Hölder type mapping, that is, it satisfies the condition

$$|f(x) - f(y)| \le H|x-y|^r, \qquad \forall x, y \in [a,b],$$

where, $H > 0$ and $r \in (0,1]$ are given, and $u\colon [a,b] \to \mathbb{R}$ is a mapping of bounded variation on $[a,b]$. Then we have the inequality

$$\left| f(x)(u(b)-u(a)) - \int_a^b f(t)\,du(t) \right| \le H\Big[\frac{b-a}{2} + \Big|x - \frac{a+b}{2}\Big|\Big]^r \cdot \bigvee_a^b(u)$$

for all $x \in [a,b]$, where, $\bigvee_a^b(u)$ denotes the total variation of u on $[a,b]$. Furthermore, the constant $\frac{1}{2}$ is the best possible in the sense that it cannot be replaced by a smaller one, for all $r \in (0,1]$.

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In [13], Dragomir has proved the dual case as follows:

Theorem 1.2
Let $f\colon [a,b] \to \mathbb{R}$ be a mapping of bounded variation on $[a,b]$ and $u\colon [a,b] \to \mathbb{R}$ be of r-H–Hölder type on $[a,b]$. Then we have the inequality

$$\begin{aligned}
&\left|(u(b)-u(a))f(x) - \int_a^b f(t)\,du(t)\right| \\
&\qquad \le H\Big[(x-a)^r \cdot \bigvee_a^x(f) + (b-x)^r \cdot \bigvee_x^b(f)\Big] \\
&\qquad \le H \begin{cases} [(x-a)^r + (b-x)^r][\frac{1}{2}\bigvee_a^b(f) + \frac{1}{2}|\bigvee_a^x(f) - \bigvee_x^b(f)|] \\ [(x-a)^{qr} + (b-x)^{qr}]^{\frac{1}{q}}[(\bigvee_a^x(f))^p + (\bigvee_x^b(f))^p]^{\frac{1}{p}} \\ [\frac{b-a}{2} + |x - \frac{a+b}{2}|]^r \cdot \bigvee_a^b(f). \end{cases}
\end{aligned}$$

In [7], Barnett et al. established some Ostrowski and trapezoid type inequalities for the Stieltjes integral $\int_a^b f(t)\,du(t)$ in the case of Lipschitzian integrators for both Hölder continuous and monotonic integrand. The dual case was also analyzed in the same paper. In [8], Cerone et al. proved some Ostrowski type inequalities for the Stieltjes integral where the integrand f is absolutely continuous while the integrator u is of bounded variation. For other results concerning inequalities for Stieltjes integrals, see [5, 9, 10, 11, 16, 18, 19, 21, 23, 24].

Motivated by [20], Dragomir in [15], established the following companion of the Ostrowski inequality for mappings of bounded variation.

Theorem 1.3
Let $f\colon [a,b] \to \mathbb{R}$ be a mapping of bounded variation on $[a,b]$. Then we have the inequalities:

$$\left|\frac{f(x) + f(a+b-x)}{2} - \frac{1}{b-a}\int_a^b f(t)\,dt\right| \le \left[\frac{1}{4} + \left|\frac{x - \frac{3a+b}{4}}{b-a}\right|\right] \cdot \bigvee_a^b(f)$$

for any $x \in [a, \frac{a+b}{2}]$, where $\bigvee_a^b(f)$ denotes the total variation of f on $[a,b]$. The constant $\frac{1}{4}$ is best possible.

For recent results concerning the above companion of Ostrowski's inequality and other related results see [1, 2, 3, 4, 6, 14, 15, 17, 22].

In this paper, we establish a companion of Ostrowski's integral inequality for the Riemann-Stieltjes integral $\int_a^b f(t)\,du(t)$, where f is assumed to be of r-H-Hölder type on $[a,b]$ and u is of bounded variation on $[a,b]$. Applications to the approximation problem of the Riemann-Stieltjes integral in terms of Riemann-Stieltjes sums are also pointed out.

2. The results

The following companion of Ostrowski's inequality for Riemann-Stieltjes integral holds.

Theorem 2.1

Let $f\colon [a,b]\to\mathbb{R}$ be a r-H–Hölder type mapping, where, $H>0$ and $r\in(0,1]$ are given, and $u\colon [a,b]\to\mathbb{R}$ is a mapping of bounded variation on $[a,b]$. Then we have the inequality

$$\begin{aligned}\Big|f(x)\Big[u\Big(\frac{a+b}{2}\Big)-u(a)\Big]+f(a+b-x)\Big[u(b)-u\Big(\frac{a+b}{2}\Big)\Big]-\int_a^b f(t)\,du(t)\Big|\\ \le H\Big[\frac{b-a}{4}+\Big|x-\frac{3a+b}{4}\Big|\Big]^r\cdot\bigvee_a^b(u)\end{aligned}\tag{1}$$

for all $x\in[a,\frac{a+b}{2}]$, where $\bigvee_a^b(u)$ denotes the total variation of u on $[a,b]$. Furthermore, the constant $\frac14$ is the best possible in the sense that it cannot be replaced by a smaller one, for all $r\in(0,1]$.

Proof. Using the integration by parts formula for Riemann-Stieltjes integral, we have

$$\int_a^{\frac{a+b}{2}}[f(x)-f(t)]\,du(t)=f(x)\Big[u\Big(\frac{a+b}{2}\Big)-u(a)\Big]-\int_a^{\frac{a+b}{2}}f(t)\,du(t)$$

and

$$\begin{aligned}&\int_{\frac{a+b}{2}}^b[f(a+b-x)-f(t)]\,du(t)\\ &\qquad=f(a+b-x)\Big[u(b)-u\Big(\frac{a+b}{2}\Big)\Big]-\int_{\frac{a+b}{2}}^b f(t)\,du(t).\end{aligned}$$

Adding the above equalities, we have

$$\begin{aligned}&\int_a^{\frac{a+b}{2}}[f(x)-f(t)]\,du(t)+\int_{\frac{a+b}{2}}^b[f(a+b-x)-f(t)]\,du(t)\\ &\quad=f(x)\Big[u\Big(\frac{a+b}{2}\Big)-u(a)\Big]+f(a+b-x)\Big[u(b)-u\Big(\frac{a+b}{2}\Big)\Big]-\int_a^b f(t)\,du(t).\end{aligned}$$

It is well known that if $p\colon[c,d]\to\mathbb{R}$ is continuous and $\nu\colon[c,d]\to\mathbb{R}$ is of bounded variation, then the Riemann-Stieltjes integral $\int_c^d p(t)\,d\nu(t)$ exists and the following inequality holds

$$\Big|\int_c^d p(t)\,d\nu(t)\Big|\le\sup_{t\in[c,d]}|p(t)|\bigvee_c^d(\nu).\tag{2}$$

Applying the inequality (2) for $\nu(t)=u(t)$, $p(t)=f(x)-f(t)$ for all $t\in[a,\frac{a+b}{2}]$;

and then for $p(t) = f(a+b-x) - f(t)$, $\nu(t) = u(t)$ for all $t \in (\frac{a+b}{2}, b]$, we get

$$\begin{aligned}
&\left|f(x)\Big[u\Big(\frac{a+b}{2}\Big) - u(a)\Big] + f(a+b-x)\Big[u(b) - u\Big(\frac{a+b}{2}\Big)\Big] - \int_a^b f(t)\,du(t)\right| \\
&\quad = \left|\int_a^{\frac{a+b}{2}} [f(x) - f(t)]\,du(t) + \int_{\frac{a+b}{2}}^b [f(a+b-x) - f(t)]\,du(t)\right| \\
&\quad \le \left|\int_a^{\frac{a+b}{2}} [f(x) - f(t)]\,du(t)\right| + \left|\int_{\frac{a+b}{2}}^b [f(a+b-x) - f(t)]\,du(t)\right| \\
&\quad \le \sup_{t\in[a,\frac{a+b}{2}]} |f(x) - f(t)| \cdot \bigvee_a^{\frac{a+b}{2}}(u) + \sup_{t\in[\frac{a+b}{2},b]} |f(a+b-x) - f(t)| \cdot \bigvee_{\frac{a+b}{2}}^{b}(u).
\end{aligned} \tag{3}$$

As f is of r-H–Hölder type, we have

$$\begin{aligned}
\sup_{t\in[a,\frac{a+b}{2}]} |f(x) - f(t)| &\le \sup_{t\in[a,\frac{a+b}{2}]} [H|x-t|^r] \\
&= H \max\left\{(x-a)^r, \Big(\frac{a+b}{2} - x\Big)^r\right\} \\
&= H\Big[\max\left\{(x-a), \Big(\frac{a+b}{2} - x\Big)\right\}\Big]^r \\
&= H\Big[\frac{b-a}{4} + \Big|x - \frac{3a+b}{4}\Big|\Big]^r
\end{aligned}$$

and

$$\begin{aligned}
\sup_{t\in[\frac{a+b}{2},b]} |f(a+b-x) - f(t)| &\le \sup_{t\in[\frac{a+b}{2},b]} [H|a+b-x-t|^r] \\
&= H \max\left\{\Big(a+b-x-\frac{a+b}{2}\Big)^r, (b-a-b+x)^r\right\} \\
&= H\Big[\max\left\{(x-a), \Big(\frac{a+b}{2} - x\Big)\right\}\Big]^r \\
&= H\Big[\frac{b-a}{4} + \Big|x - \frac{3a+b}{4}\Big|\Big]^r.
\end{aligned}$$

Therefore, by (3), we have

$$\begin{aligned}
&\left|f(x)\Big[u\Big(\frac{a+b}{2}\Big) - u(a)\Big] + f(a+b-x)\Big[u(b) - u\Big(\frac{a+b}{2}\Big)\Big] - \int_a^b f(t)\,du(t)\right| \\
&\quad \le H\Big[\frac{b-a}{4} + \Big|x - \frac{3a+b}{4}\Big|\Big]^r \cdot \bigvee_a^{\frac{a+b}{2}}(u) + H\Big[\frac{b-a}{4} + \Big|x - \frac{3a+b}{4}\Big|\Big]^r \cdot \bigvee_{\frac{a+b}{2}}^{b}(u) \\
&\quad = H\Big[\frac{b-a}{4} + \Big|x - \frac{3a+b}{4}\Big|\Big]^r \cdot \bigvee_a^b(u).
\end{aligned}$$

To prove the sharpness of the constant $\frac{1}{4}$ for any $r \in (0,1]$, assume that (1) holds with a constant $C > 0$, that is,

$$\left| f(x)\Big[u\Big(\frac{a+b}{2}\Big) - u(a)\Big] + f(a+b-x)\Big[u(b) - u\Big(\frac{a+b}{2}\Big)\Big] - \int_a^b f(t)\,du(t)\right| \\ \leq H\Big[C(b-a) + \Big|x - \frac{3a+b}{4}\Big|\Big]^r \cdot \bigvee_a^b(u).$$

Choose $f(t) = t^r$, $r \in (0,1]$, $t \in [0,1]$ and $u\colon [0,1] \to [0,\infty)$ given by

$$u(t) = \begin{cases} 0, & t \in (0,1], \\ -1, & t = 0. \end{cases}$$

As

$$|f(x) - f(y)| = |x^r - y^r| \leq |x-y|^r, \qquad \forall x \in [0,1],\ r \in (0,1], \tag{4}$$

it follows that f is r-H–Hölder type with the constant $H = 1$.

By using the integration by parts formula for Riemann-Stieltjes integrals, we have

$$\int_0^1 f(t)\,du(t) = f(1)u(1) - f(0)u(0) - \int_0^1 u(t)\,df(t) = 0 \quad \text{and} \quad \bigvee_0^1(u) = 1.$$

Consequently, by (4), we get

$$|x^r| \leq \Big[C + \Big|x - \frac{1}{4}\Big|\Big]^r, \qquad \forall x \in \Big[0, \frac{1}{2}\Big].$$

For $x = \frac{1}{2}$, we get $\frac{1}{2^r} \leq (C + \frac{1}{4})^r$, which implies that $C \geq \frac{1}{4}$, and the theorem is completely proved.

The following inequalities are hold:

Corollary 2.2
Let f and u be as in Theorem 2.1. In (1) choose

1. $x = a$, *then we get the following trapezoid type inequality*

$$\left| f(a)\Big[u\Big(\frac{a+b}{2}\Big) - u(a)\Big] + f(b)\Big[u(b) - u\Big(\frac{a+b}{2}\Big)\Big] - \int_a^b f(t)\,du(t)\right| \\ \leq H\Big(\frac{b-a}{2}\Big)^r \cdot \bigvee_a^b(u).$$

2. $x = \frac{a+b}{2}$, *then we get the following mid-point type inequality*

$$\left| (u(b) - u(a))f\Big(\frac{a+b}{2}\Big) - \int_a^b f(t)\,du(t)\right| \leq H\Big(\frac{b-a}{2}\Big)^r \cdot \bigvee_a^b(u).$$

We may state the following Ostrowski type inequality:

Corollary 2.3
Let f be as in Theorem 2.1. Then we have

$$\left|\frac{f(x)+f(a+b-x)}{2}-\frac{1}{b-a}\int_a^b f(t)\,dt\right|\leq H\Big[\frac{b-a}{4}+\Big|x-\frac{3a+b}{4}\Big|\Big]^r$$

for all $x\in[a,\frac{a+b}{2}]$. The constant $\frac{1}{4}$ is the best possible in the sense that it cannot be replaced by a smaller one, for all $r\in(0,1]$.

Corollary 2.4
Let u be as in Theorem 2.1, and $f\colon[a,b]\to\mathbb{R}$ be an L-Lipschitzian mapping on $[a,b]$, that is,

$$|f(x)-f(y)|\leq L|x-y|,\qquad \forall x,y\in[a,b],$$

where $L>0$ is fixed. Then, for all $x\in[a,\frac{a+b}{2}]$, we have the inequality

$$\begin{aligned}\Big|f(x)\Big[u\Big(\frac{a+b}{2}\Big)-u(a)\Big]+f(a+b-x)\Big[u(b)-u\Big(\frac{a+b}{2}\Big)\Big]-\int_a^b f(t)\,du(t)\Big|\\ \leq L\Big[\frac{b-a}{4}+\Big|x-\frac{3a+b}{4}\Big|\Big]\cdot\bigvee_a^b(u).\end{aligned}$$

The constant $\frac{1}{4}$ is the best possible in the sense that it cannot be replaced by a smaller one.

Corollary 2.5
In Theorem 2.1, if u is monotonic on $[a,b]$, and f is of r-H–Hölder type. Then, for all $x\in[a,\frac{a+b}{2}]$, we have the inequality

$$\begin{aligned}\Big|f(x)\Big[u\Big(\frac{a+b}{2}\Big)-u(a)\Big]+f(a+b-x)[u(b)-u\Big(\frac{a+b}{2}\Big)\Big]-\int_a^b f(t)\,du(t)\Big|\\ \leq H\Big[\frac{b-a}{4}+\Big|x-\frac{3a+b}{4}\Big|\Big]^r\cdot|u(b)-u(a)|.\end{aligned}$$

Corollary 2.6
Let f be of r-H–Hölder type and $g\colon[a,b]\to\mathbb{R}$ be continuous on $[a,b]$. Then we have the inequality

$$\begin{aligned}\Big|f(x)\int_a^{\frac{a+b}{2}}g(s)\,ds+f(a+b-x)\int_{\frac{a+b}{2}}^b g(s)\,ds-\int_a^b f(t)g(t)\,dt\Big|\\ \leq H\Big[\frac{b-a}{4}+\Big|x-\frac{3a+b}{4}\Big|\Big]^r\|g\|_1,\end{aligned}$$

for all $x\in[a,\frac{a+b}{2}]$, where $\|g\|_1=\int_a^b|g(t)|dt$.

Proof. Define the mapping $u\colon [a,b] \to \mathbb{R}$, $u(t) = \int_a^t g(s)\,ds$. Then u is differentiable on (a,b) and $u'(t) = g(t)$. Using the properties of the Riemann-Stieltjes integral, we have

$$\int_a^b f(t)\,du(t) = \int_a^b f(t)g(t)\,dt$$

and

$$\bigvee_a^b(u) = \int_a^b |u'(t)|\,dt = \int_a^b |g(t)|\,dt.$$

Remark 1

In Corollary 2.6, if f is symmetric about the x-axis, i.e. $f(a+b-x) = f(x)$, then we have

$$\left|f(x)\int_a^b g(s)\,ds - \int_a^b f(t)g(t)\,dt\right| \le H\Big[\frac{b-a}{4} + \Big|x - \frac{3a+b}{4}\Big|\Big]^r \|g\|_1$$

for all $x \in [a, \frac{a+b}{2}]$. For instance, choose $x = \frac{a+b}{2}$, then we get

$$\left|f\Big(\frac{a+b}{2}\Big)\int_a^b g(s)\,ds - \int_a^b f(t)g(t)\,dt\right| \le H\Big(\frac{b-a}{2}\Big)^r \|g\|_1.$$

3. An approximation for the Riemann-Stieltjes integral

Let $I_n : a = x_0 < x_1 < \ldots < x_n = b$ be a division of the interval $[a,b]$, $h_i = x_{i+1} - x_i$, $(i = 0,1,2,\ldots,n-1)$ and $\nu(h) := \max\{h_i|\ i = 0,1,2,\ldots,n-1\}$. Define the general Riemann-Stieltjes sum

$$\begin{aligned} S(f,u,I_n,\xi) = \sum_{i=0}^{n-1} \Big\{ & f(\xi_i)[u(x_i + x_{i+1}2) - u(x_i)] \\ & + f(x_i + x_{i+1} - \xi_i)\Big[u(x_{i+1}) - u\Big(\frac{x_i + x_{i+1}}{2}\Big)\Big]\Big\}. \end{aligned} \tag{5}$$

In the following, we establish some upper bounds for the error approximation of the Riemann-Stieltjes integral $\int_a^b f(t)\,du(t)$ by its Riemann-Stieltjes sum $S(f,u,I_n,\xi)$.

Theorem 3.1

Let $u\colon [a,b] \to \mathbb{R}$ be a mapping of bounded variation on $[a,b]$ and $f\colon [a,b] \to \mathbb{R}$ be of r-H–Hölder type on $[a,b]$. Then

$$\int_a^b f(t)\,du(t) = S(f,u,I_n,\xi) + R(f,u,I_n,\xi),$$

where $S(f,u,I_n,\xi)$ is given in (5) and the remainder $R(f,u,I_n,\xi)$ satisfies the bound

$$\begin{aligned} |R(f,u,I_n,\xi)| &\le H\Big[\frac{1}{4}\nu(h) + \max_{i=0,1,\ldots,n-1}\Big|\xi_i - \frac{3x_i + x_{i+1}}{4}\Big|\Big]^r \cdot \bigvee_a^b(u) \\ &\le H\Big[\frac{1}{2}\nu(h)\Big]^r \cdot \bigvee_a^b(u). \end{aligned} \tag{6}$$

Proof. Applying Theorem 2.1 on the intervals $[x_i, x_{i+1}]$, we may state that

$$\begin{aligned}&\Big|f(\xi_i)\Big[u\Big(\frac{x_i+x_{i+1}}{2}\Big)-u(x_i)\Big]\\&\qquad+f(x_i+x_{i+1}-\xi_i)\Big[u(x_{i+1})-u\Big(\frac{x_i+x_{i+1}}{2}\Big)\Big]-\int_{x_i}^{x_{i+1}}f(t)\,du(t)\Big|\\&\qquad\le H\Big[\frac{1}{4}h_i+\Big|\xi_i-\frac{3x_i+x_{i+1}}{4}\Big|\Big]^r\cdot\bigvee_{x_i}^{x_{i+1}}(u)\end{aligned}$$

for all $i\in\{0,1,2,\cdots,n-1\}$.

Summing the above inequality over i from 0 to $n-1$ and using the generalized triangle inequality, we deduce

$$\begin{aligned}&|R(f,u,I_n,\xi)|\\&\quad=\sum_{i=0}^{n-1}\Big|\Big\{f(\xi_i)\Big[u\Big(\frac{x_i+x_{i+1}}{2}\Big)-u(x_i)\Big]\\&\quad\quad+f(x_i+x_{i+1}-\xi_i)\Big[u(x_{i+1})-u\Big(\frac{x_i+x_{i+1}}{2}\Big)\Big]-\int_{x_i}^{x_{i+1}}f(t)\,du(t)\Big\}\Big|\\&\quad\le H\sum_{i=0}^{n-1}\Big[\frac{1}{4}h_i+\Big|\xi_i-\frac{3x_i+x_{i+1}}{4}\Big|\Big]^r\cdot\bigvee_{x_i}^{x_{i+1}}(u)\\&\quad\le H\sup_{i=0,1,\dots,n-1}\Big[\frac{1}{4}h_i+\Big|\xi_i-\frac{3x_i+x_{i+1}}{4}\Big|\Big]^r\cdot\sum_{i=0}^{n-1}\bigvee_{x_i}^{x_{i+1}}\Big(u\Big).\end{aligned}$$

However,

$$\sup_{i=0,1,\dots,n-1}\Big[\frac{1}{4}h_i+\Big|\xi_i-\frac{3x_i+x_{i+1}}{4}\Big|\Big]^r\le\Big[\frac{1}{4}\nu(h)+\sup\Big|\xi_i-\frac{3x_i+x_{i+1}}{4}\Big|\Big]^r$$

and

$$\sum_{i=0}^{n-1}\bigvee_{x_i}^{x_{i+1}}(u)=\bigvee_a^b(u),$$

which completely proves the first inequality in (6).

For the second inequality, we observe that

$$\Big|\xi_i-\frac{3x_i+x_{i+1}}{4}\Big|\le\frac{1}{4}h_i$$

for all $i\in\{0,1,2,\dots,n-1\}$, which completes the proof.

Corollary 3.2

In Theorem 3.1, additionally, if f is symmetric about the x-axis, then we have $S(f,u,I_n,\xi)$ reduced to be

$$S(f,u,I_n,\xi)=\sum_{i=0}^{n-1}f(\xi_i)[u(x_{i+1})-u(x_i)]. \tag{7}$$

Then

$$\int_a^b f(t)\,du(t) = S(f,u,I_n,\xi) + R(f,u,I_n,\xi),$$

where $S(f,u,I_n,\xi)$ *is given in* (7) *and the remainder* $R(f,u,I_n,\xi)$ *satisfies the bound in* (6)*.*

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