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Properties of two variables Toeplitz type operators

Abstract. The investigation of properties of generalized Toeplitz operators with respect to the pairs of doubly commuting contractions (the abstract analogue of classical two variable Toeplitz operators) is proceeded. We especially concentrate on the condition of existence such a non-zero operator. There are also presented conditions of analyticity of such an operator.

1. Introduction

Let $L(H_1, H_2)$ denote the algebra of all bounded linear operators from H_1 into H_2, where H_1, H_2 are complex, separable Hilbert spaces. If $H_1 = H_2$ we will use the notation $L(H_1)$.

The classical Toeplitz operators on the Hardy space on the unit disc are well known and they are fully characterized by the relation $X = T_z^* X T_z$, where T_z is the shift operator – the multiplication operator by the independent variable on the Hardy space H^2 on the circle $\mathbb{T}$. This notion can be generalized when instead of the backward shift T_z^* in the equation above we will put arbitrary, possibly different, contractions. Namely, for given contractions $S \in L(H_1)$ and $T \in L(H_2)$, an operator $X \in L(H_2, H_1)$ is called *generalized Toeplitz operator* if $X = SXT^*$. These type of operators were studied in [1, 6, 11].

The classical Toeplitz operators are also considered on the Hardy space on the torus $H^2(\mathbb{T}^2)$. The space $H^2(\mathbb{T}^2)$ can be seen as a subspace of $L^2(\mathbb{T}^2) = L^2(\mathbb{T}^2, m \otimes m)$ (m denotes the normalized Lebesgue measure on $\mathbb{T}$) and $P_{H^2(\mathbb{T}^2)}$ is the appropriate projection. For any $\varphi \in L^\infty(\mathbb{T}^2) = L^\infty(\mathbb{T}^2, m \otimes m)$ we define the Toeplitz operator $T_\varphi \in L(H^2(\mathbb{T}^2))$ by $T_\varphi f = P_{H^2(\mathbb{T}^2)}(\varphi f)$ $(f \in H^2(\mathbb{T}^2))$.

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The function φ is called the symbol of the Toeplitz operator. The multiplication operators by the independent variables in this space we denote by T_{z_1}, T_{z_2}. As it was shown in [9, Proposition 3.3], the set of Toeplitz operators on $H^2(\mathbb{T}^2)$ can be characterized as a set of operators $X \in L(H^2(\mathbb{T}^2))$ such that $X = T_{z_1}^* X T_{z_1}$ and $X = T_{z_2}^* X T_{z_2}$. In [10], for given pairs of contractions $S_1, S_2 \in L(H_1)$ and $T_1, T_2 \in L(H_2)$, there were considered operators $X \in L(H_2, H_1)$ such that $X = S_1 X T_1^*$ and $X = S_2 X T_2^*$ and they were called *generalized Toeplitz operators with respect to the pairs* S_1, S_2 and T_1, T_2. A general assumption was that S_1, S_2 doubly commute (i.e. not only S_1, S_2 commute but also S_1^*, S_2 do) and T_1, T_2 also doubly commute. Observe that in the previous case operators $T_{z_1}^*, T_{z_2}^*$ doubly commute. One of the results, see [10] and Theorem 3.3, claims that for every generalized Toeplitz operator X there is an operator $Y \in L(K_2^+, K_1^+)$ with $X = P_{H_1} Y|_{H_2}$, $Y = W_1 Y V_1^*$ and $Y = W_2 Y V_2^*$, where pair W_1, W_2 and pair V_1, V_2 are minimal isometric dilations of the pairs of operators S_1, S_2 and T_1, T_2, respectively, defined on spaces K_1^+, K_2^+, respectively. Such an operator Y is called the *symbol* of X.

In this paper we continue the investigation of properties of generalized Toeplitz operator with respect to pairs of doubly commuting contractions. We especially concentrate on the condition of existing such a non zero operator (Section 3). There are also presented conditions of analyticity of such an operator (Section 4). The dilation theory of the pairs of contractions is the main tool. Hence, in Section 2, we recall results on dilations of pairs of doubly commuting contractions. In both sections 3 and 4 examples are given.

2. Preliminaries on dilations of pairs of operators

In what follows some properties of a minimal isometric dilation for a pair of contractions will be needed. For a pair $T_1, T_2 \in L(H)$ of commuting contractions, by Ando's theorem [8, Theorem I.6.1], there is a pair of commuting isometries $V_1, V_2 \in L(K^+)$, $H \subset K^+$, being a minimal isometric dilation of the given pair T_1, T_2, i.e. for all non-negative integers n, m the following holds

$$T_1^n T_2^m = P_H V_1^n V_2^m\big|_H \qquad \text{and} \qquad K^+ = \bigvee_{n,m \geqslant 0} V_1^n V_2^m H. \tag{1}$$

A minimal isometric dilations of a pair of commuting contractions is not unique but for each minimal isometric dilation we have (see [8, 10])

$$T_i P_H = P_H V_i, \quad V_i^* H \subset H \quad \text{and} \quad V_i^*\big|_H = T_i^*, \qquad i = 1, 2. \tag{2}$$

The aim of the paper is to consider a doubly commuting pairs of contractions $T_1, T_2 \in L(H)$, i.e. we assume that not only T_1, T_2 commute but also T_1^*, T_2 commute. The important observation, which was made in [12] (Lemma 1 and remarks afterwards), is that in this case the isometries V_1, V_2 can be also chosen doubly commuting. Then as it was noticed in [10, 12, 13] we have the specific properties. Let $\mathcal{R}$ be a maximal subspace of K^+, such that $R_1 = V_1|_{\mathcal{R}}$, $R_2 = V_2|_{\mathcal{R}}$, is a pair of unitary operators. As it was shown in [10] the projection $P_{\mathcal{R}}$ can be defined as follows

$$P_{\mathcal{R}} k = \lim_{n,m \to \infty} V_1^n V_2^m V_1^{*n} V_2^{*m} k \qquad \text{for } k \in K^+ \tag{3}$$

and that
$$V_iP_{\mathcal{R}} = P_{\mathcal{R}}V_i, \quad V_i^*P_{\mathcal{R}} = P_{\mathcal{R}}V_i^* \qquad \text{for } i = 1, 2. \tag{4}$$
The following will be used later.

Lemma 2.1
Let T_1, T_2 be a pair of doubly commuting contractions. Using the notations introduced above the doubly index sequence of closed subspaces $R_1^nR_2^mP_{\mathcal{R}}H$ is increasing according to the natural order in $\mathbb{N} \times \mathbb{N}$. Moreover,
$$\mathcal{R} = \bigvee_{n,m\geqslant 0} R_1^nR_2^mP_{\mathcal{R}}H. \tag{5}$$

Proof. By (2) and (3)
$$P_{\mathcal{R}}h = \lim_{n,m\to\infty} V_1^nV_2^mT_1^{*n}T_2^{*m}h \qquad \text{for } h \in H.$$
Hence
$$V_1P_{\mathcal{R}}T_1^*h = \lim_{n,m\to\infty} V_1^{n+1}V_2^mT_1^{*n+1}T_2^{*m}h = P_{\mathcal{R}}h$$
and
$$P_{\mathcal{R}}T_1^*h = R_1^*P_{\mathcal{R}}h \qquad \text{for } h \in H.$$
Similarly
$$P_{\mathcal{R}}T_2^*h = R_2^*P_{\mathcal{R}}h \qquad \text{for } h \in H,$$
which implies $R_i^*P_{\mathcal{R}}H \subset P_{\mathcal{R}}H$ thus $P_{\mathcal{R}}H \subset R_iP_{\mathcal{R}}H$ for $i = 1, 2$. In consequence
$$R_1^{n_1}R_2^{n_2}P_{\mathcal{R}}H \subset R_1^{m_1}R_2^{m_2}P_{\mathcal{R}}H \qquad \text{for } n_1 \leqslant m_1,\ n_2 \leqslant m_2.$$
Applying (1) and (4) we have
$$\mathcal{R} = P_{\mathcal{R}}K^+ = P_{\mathcal{R}} \bigvee_{n,m\geqslant 0} V_1^nV_2^mH = \bigvee_{n,m\geqslant 0} P_{\mathcal{R}}V_1^nV_2^mH = \bigvee_{n,m\geqslant 0} R_1^nR_2^mP_{\mathcal{R}}H.$$

3. Review on existence of a symbol

Let us consider two pairs of doubly commuting contractions
$$S_1, S_2 \in L(H_1), \qquad T_1, T_2 \in L(H_2).$$
Let the pairs $W_1, W_2 \in L(K_1^+)$, $V_1, V_2 \in L(K_2^+)$ be minimal isometric dilations of the pairs S_1, S_2 and T_1, T_2, respectively. Chose using [10, 12], as above, the pairs W_1, W_2 and V_1, V_2 doubly commuting. Let $\mathcal{R}_1, \mathcal{R}_2$ be maximal subspaces of K_1^+, K_2^+, respectively, such that both pairs $W_1|_{\mathcal{R}_1}, W_2|_{\mathcal{R}_1}$ and $V_1|_{\mathcal{R}_2}, V_2|_{\mathcal{R}_2}$ are unitary. An operator $Y \in L(K_2^+, K_1^+)$, following [10], is called a *symbol with respect to the pairs* S_1, S_2 and T_1, T_2 if $Y = W_1YV_1^*$ and $Y = W_2YV_2^*$.

Recall after [10] some basic fact about the symbols.

Remark 3.1 (Remark 3.1, [10])
If an operator $Y \in L(K_2^+, K_1^+)$ is a symbol with respect to the pairs $S_1, S_2 \in L(H_1)$ and $T_1, T_2 \in L(H_2)$, then the operator $X = P_{H_1} Y\big|_{H_2}$ is a generalized Toeplitz operator with respect to the pais S_1, S_2 and T_1, T_2.

Now we recall a characterization of the symbol.

Proposition 3.2 (Proposition 3.2, [10])
Let $Y \in L(K_2^+, K_1^+)$. Then the following are equivalent

(i) *Y is a symbol with respect to the pairs S_1, S_2 and T_1, T_2,*

(ii) *$YV_i = W_iY$, $i = 1, 2$ and $Y = YP_{\mathcal{R}_2}$,*

(iii) *$YV_i^* = W_i^*Y$, $i = 1, 2$ and $Y = P_{\mathcal{R}_1}Y$,*

(iv) *$Y = \lim_{n,m\to\infty} W_1^n W_2^m P_{H_1} Y P_{H_2} V_1^{*n} V_2^{*m}$ in SOT.*

Now let us recall the theorem about an existence of a symbol.

Theorem 3.3 (Theorem 3.5, [10])
Suppose $X \in L(H_2, H_1)$. Let $S_1, S_2 \in L(H_1)$ and $T_1, T_2 \in L(H_2)$ be pairs of doubly commuting contractions. Assume that $X = S_1XT_1^*$, $X = S_2XT_2^*$. Then there exists exactly one operator $Y \in L(K_2^+, K_1^+)$ such that

(i) *Y is a symbol with respect to the pairs S_1, S_2 and T_1, T_2,*

(ii) *$X = P_{H_1} Y|_{H_2}$,*

(iii) *$\|X\| = \|Y\|$.*

4. Existence of non-zero generalized Toeplitz operators

The next two theorems characterize when a non–zero generalized Toeplitz operator with respect to the pairs of doubly commuting contractions can exists.

Theorem 4.1
Let $T_1, T_2 \in L(H_2)$ be a pair of doubly commuting contractions, then the following are equivalent.

(i) *The only operator $X \in L(H_2)$ satisfying $X = T_1XT_1^*$ and $X = T_2XT_2^*$ is the zero operator,*

(ii) *$\lim_{n,m\to\infty} T_1^{*n} T_2^{*m} h = 0$ for $h \in H_2$,*

(iii) *$P_{\mathcal{R}_2} H_2 = 0$,*

(iv) *$P_{H_2} \mathcal{R}_2 = 0$,*

(v) *$P_{H_2} P_{\mathcal{R}_2} P_{H_2} = 0$,*

(vi) *$\mathcal{R}_2 = 0$.*

Proof. Note firstly that projection $P_{\mathcal{R}_2}$ satisfies condition (ii) in Proposition 3.2, thus $P_{\mathcal{R}_2}$ is a symbol by condition (i) of this Proposition. Hence, by Remark 3.1, $X = P_{H_2} P_{\mathcal{R}_2|_{H_2}}$ is a generalized Toeplitz operator with respect to the pair T_1, T_2. If (i) is satisfied, then $X = 0$ and $P_{H_2} P_{\mathcal{R}_2} P_{H_2} = X = 0$ and (v) is fulfilled. Note that $0 = P_{H_2}P_{\mathcal{R}_2}P_{H_2} = P_{H_2}P_{\mathcal{R}_2}(P_{H_2}P_{\mathcal{R}_2})^*$ implies $P_{H_2}P_{\mathcal{R}_2} = 0$, i.e. (v) $\Rightarrow$ (iv). If (iv) is satisfied, then also $P_{\mathcal{R}_2}P_{H_2} = 0$ and we obtain (iii). Assuming (iii), by (5) we get (vi). When we assume (vi) and apply (5) we obtain $P_{\mathcal{R}_2}H_2 = 0$. Using (3) and isometric properties of V_1, V_2 we get (ii). The implication (ii) $\Rightarrow$ (i) is straightforward.

Example 4.2
Let us now consider $T^*_{z_1}, T^*_{z_2}$ the adjoints to the multiplication operators by the independent variables in the space $H^2(\mathbb{T}^2)$ as a pair of contractions. Note that the operators $T^*_{z_1}, T^*_{z_2}$ doubly commute. It was shown in [10, Example 3.7] that a minimal isometric dilation for the pair $T^*_{z_1}, T^*_{z_2}$ is the pair $M^*_{z_1}, M^*_{z_2}$ of multiplication operators by the conjugates of the independent variables in $L^2(\mathbb{T}^2)$. Hence $\mathcal{R}_2 = L^2(\mathbb{T}^2)$, so it is far from being zero. On the other hand, if $X \in L(H^2(\mathbb{T}^2))$ fulfils the equations $X = T^*_{z_i}XT_{z_i}$ for $i = 1, 2$, then, as it was shown in [10, Example 3.7], the symbol $Y \in L(L^2(\mathbb{T}^2))$ for X is represented by a function $\varphi \in L^\infty(\mathbb{T}^2)$ such that $Y = M_\varphi$, $(M_\varphi f)(z_1, z_2) = \varphi(z_1, z_2)f(z_1, z_2)$ for $f \in H^2(\mathbb{T}^2)$. Hence $X = P_{H^2(\mathbb{T}^2)}M_\varphi|_{H^2(\mathbb{T}^2)}$ (Theorem 3.3). Thus the set of X fulfilling the equations $X = T^*_{z_1}XT_{z_1}$, $X = T^*_{z_2}XT_{z_2}$ can be identify with $\varphi \in L^\infty(\mathbb{T}^2)$. Hence the set of generalized Toeplitz operators with respect to both pairs equal to $T^*_{z_1}, T^*_{z_2}$ is "rich". This is the case of the classical Toeplitz operator of two variables.

Example 4.3
Let T_{z_1}, T_{z_2} be the multiplication operators by the independent variables in the space $H^2(\mathbb{T}^2)$. Note that the operators T_{z_1}, T_{z_2} doubly commute. Since they are isometries, a minimal isometric dilation is the same pair T_{z_1}, T_{z_2} and $K^+ = H^2(\mathbb{T}^2)$. It is easy to see that $\lim_{n,m\to\infty} T^{*n}_{z_1}T^{*m}_{z_2}h = 0$ for $h \in H^2(\mathbb{T}^2)$ so that $\mathcal{R}_2 = 0$. Hence the only operator $X \in L(H^2(\mathbb{T}^2))$ satisfying $X = T_{z_1}XT^*_{z_1}$ and $X = T_{z_2}XT^*_{z_2}$ is the zero operator.

Now let us consider the general case.

Theorem 4.4
Let $S_1, S_2 \in L(H_1)$ and $T_1, T_2 \in L(H_2)$ be pairs of doubly commuting contractions. Then the following are equivalent.

(i) *The only operator $X \in L(H_2, H_1)$ satisfying $X = S_1XT_1^*$ and $X = S_2XT_2^*$ is the zero operator.*

(ii) *One of the subspaces $\mathcal{R}_1$, $\mathcal{R}_2$ is trivial or the pairs of operators $W_1|_{\mathcal{R}_1}$, $W_2|_{\mathcal{R}_1}$ and $V_1|_{\mathcal{R}_2}, V_2|_{\mathcal{R}_2}$ are relatively singular.*

Proof. Let $X \in L(H_2, H_1)$ satisfying $X = S_1XT_1^*$, $X = S_2XT_2^*$. Then there exists its symbol $Y \in L(K_2^+, K_1^+)$ such that

$$Y = W_1YV_1^*, \qquad Y = W_2YV_2^*.$$

Let $Z = Y|_{\mathcal{R}_2}$. By definition of $\mathcal{R}_1$, $\mathcal{R}_2$ we have

$$ZV_1\big|_{\mathcal{R}_2} = W_1\big|_{\mathcal{R}_1}Z, \qquad ZV_2\big|_{\mathcal{R}_2} = W_2\big|_{\mathcal{R}_1}Z.$$

Let $Z = AU$ be a polar decomposition of Z. By [1, Lemma 4.1] $\ker Z^{\perp}$ reduces $V_1|_{\mathcal{R}_2}$ and $V_2|_{\mathcal{R}_2}$ and the subspace $\overline{\operatorname{Ran} Z}$ reduces $W_1|_{\mathcal{R}_1}$, $W_2|_{\mathcal{R}_1}$. Moreover, operators $V_1|_{\ker Z^{\perp}}$, $W_1|_{\overline{\operatorname{Ran} Z}}$ are unitarily equivalent taking $U|_{\ker Z^{\perp}} \colon \ker Z^{\perp} \to \overline{\operatorname{Ran} Z}$ and $V_2|_{\ker Z^{\perp}}$, $W_2|_{\overline{\operatorname{Ran} Z}}$ are unitarily equivalent taking the same unitary operator $U|_{\ker Z^{\perp}}$. Thus pairs $V_1|_{\ker Z^{\perp}}$, $V_2|_{\ker Z^{\perp}}$ and $W_1|_{\overline{\operatorname{Ran} Z}}$, $W_2|_{\overline{\operatorname{Ran} Z}}$ are unitarily equivalent taking $U_{\ker Z^{\perp}}$.

By (i), X have to be the zero operator. Hence Y and Z have to be zero operators. If $Z = Y|_{\mathcal{R}_2}$ is zero operator, then $\mathcal{R}_1 = 0$ or $\mathcal{R}_2 = 0$.

For the proof of the converse implication we assume that $\mathcal{R}_1 \neq 0$, $\mathcal{R}_2 \neq 0$. Let $A(\mathbb{D}^2)$ be the algebra of all holomorphic functions on $\mathbb{D}^2$ and continuous in $\overline{\mathbb{D}^2}$ ($\mathbb{D}$ is the unit disc). It is a standard technique (see [4, 5]) that the pair $W_1|_{\mathcal{R}_1}$, $W_2|_{\mathcal{R}_1}$ generate the representation $\Phi_W \colon A(\mathbb{D}^2) \to L(\mathcal{R}_1)$, i.e. Φ_W is linear, $\Phi_W(uv) = \Phi_W(u)$, $\Phi_W(v)$ and $\|\Phi_W(u)\| \leqslant \|u\|_\infty$ for $u, v \in A(\mathbb{D}^2)$. For any polynomial p of two variables the representation Φ_W is defined as $\Phi_W(p) := p(W_1|_{\mathcal{R}_1}, W_2|_{\mathcal{R}_1})$. Next Φ_W is uniquely extended to $A(\mathbb{D}^2)$. Then, for any $x \in \mathcal{R}_1$, there exists a positive regular Borel measure μ_x on $\mathbb{T}^2$ such that

$$\langle \Phi_W(u)x, x\rangle = \int u \, d\mu_x \qquad \text{for } x \in \mathcal{R}_1,\ u \in A(\mathbb{D}^2)$$

and $\|\mu_x\| \leqslant \|x\|^2$. Let $\mathcal{M}_\mu$ be a band of measures generated (for definition see [4]) by $\{\mu_x\}_{x \in \mathcal{R}_1}$. Similarly the pair $V_1|_{\mathcal{R}_2}$, $V_2|_{\mathcal{R}_2}$ generate the representation Φ_V and there are measures ν_y, $y \in \mathcal{R}_2$, such that

$$\langle \Phi_V(u)y, y\rangle = \int u \, d\nu_y \qquad \text{for } y \in \mathcal{R}_2,\ u \in A(\mathbb{D}^2).$$

Let $\mathcal{M}_\nu$ be a band of measures generated by ν_y, $y \in \mathcal{R}_2$. If the pairs $W_1|_{\mathcal{R}_1}$, $W_2|_{\mathcal{R}_1}$ and $V_1|_{\mathcal{R}_2}$, $V_2|_{\mathcal{R}_2}$ are not singular, there is a measure $\eta \in \mathcal{M}_\mu \cap \mathcal{M}_\nu$. By [4, Proposition 1.4] there is $x \in \mathcal{R}_1$ such that $\eta \ll \mu_x$. By the theory of spectral multiplicity (see [2]), mainly by [2, §65, Theorem 3], there are vectors $x_0 \in \mathcal{R}_1$, $y_0 \in \mathcal{R}_2$ and a unitary operator $U \colon \mathcal{Z}(y_0) \to \mathcal{Z}(x_0)$, where $\mathcal{Z}(x_0)$ is the smallest closed subspace containing x_0 and reducing for $W_1|_{\mathcal{R}_1}$ and $W_2|_{\mathcal{R}_1}$ and $\mathcal{Z}(y_0)$ is the smallest closed subspace containing y_0 and reducing for $V_1|_{\mathcal{R}_1}$ and $V_2|_{\mathcal{R}_1}$. Moreover, $UV_i|_{\mathcal{Z}(y_0)} = W_i|_{\mathcal{Z}(x_0)}U$ for $i = 1, 2$. Let us define nonzero operator $Y \in L(K_2^+, K_1^+)$ as $Y = U$ on $\mathcal{Z}(y_0)$ and $Y = 0$ on $K_+^2 \ominus \mathcal{Z}(y_0)$. Clearly $Y = W_i Y V_i^*$ for $i = 1, 2$, i.e. Y is a symbol with respect to the pairs S_1, S_2 and T_1, T_2. By Theorem 3.3 and Remark 3.1 the operator $X = P_{H_1} Y|_{H_2}$ fulfils equalities $X = S_1 X T_1^*$ and $X = S_2 X T_2^*$. Moreover, $X \not\equiv 0$, since $\|X\| = \|Y\|$.

Example 4.5

Let the first pair of contractions be the pair M_{z_1}, M_{z_2} of multiplication operators by the independent variables in the space $L^2(\mathbb{T}^2, \mu \otimes \mu)$, where μ is a non-atomic normalized measure concentrated on the Cantor set on the unit circle $\mathbb{T}$ of the Lebesgue measure zero. The operators M_{z_1}, M_{z_2} on $L^2(\mathbb{T}^2, \mu \otimes \mu)$ doubly commute as unitary. The pair M_{z_1}, M_{z_2} is its own isometric (and unitary) dilation. Hence the space $\mathcal{R}_1 = L^2(\mathbb{T}^2, \mu \otimes \mu)$ is non-zero. Let the second pair of contractions be as in Example 4.2, i.e. $T_{z_1}^*, T_{z_2}^*$ in the space $H^2(\mathbb{T}^2)$. As we have noticed above $\mathcal{R}_2 =$

$L^2(\mathbb{T}^2, m \otimes m)$, so it is also non-zero. There is no non-zero generalized Toeplitz operator with respect to this two pairs since the pair M_{z_1}, M_{z_2} on $L^2(\mathbb{T}^2, \mu \otimes \mu)$ and the pair $M_{z_1}^*, M_{z_2}^*$ on $L^2(\mathbb{T}^2, m \otimes m)$ are relatively singular ($\mu \otimes \mu$ and $m \otimes m$ are singular measures).

5. Analytic generalized Toeplitz operators

Let us above $S_1, S_2 \in L(H_1)$ and $T_1, T_2 \in L(H_2)$ be two pairs of doubly commuting contractions. Let $Y \in L(K_2^+, K_1^+)$ be a symbol with respect to this pairs. We call a symbol Y *analytic* if $YH_2 \subset H_1$. The following theorem characterizes the analyticity of the symbol.

Theorem 5.1
Let $S_1, S_2 \in L(H_1)$ and $T_1, T_2 \in L(H_2)$ be pairs of doubly commuting contractions. Assume that $X \in L(H_2, H_1)$ such that

$$S_1^* X = X T_1^* \qquad \text{and} \qquad S_2^* X = X T_2^*. \tag{6}$$

Then the operator $P_{H_1} P_{\mathcal{R}_1} X$ is a generalized Toeplitz operator with respect to the pairs S_1, S_2 and T_1, T_2 and the following are equivalent

(i) *X is a generalized Toeplitz operator with respect to the pairs S_1, S_2 and T_1, T_2,*
(ii) *$X = P_{H_1} P_{\mathcal{R}_1} X$,*
(iii) *$X = P_{H_1 \cap \mathcal{R}_1} X$,*
(iv) *$X(H_2) \subset H_1 \cap \mathcal{R}_1$.*

Additionally, if the operator X satisfies (6) *and one of the above conditions is fulfilled then X is a generalized Toeplitz operator whose symbol is analytic.*

Adversely, if Y is an analytic symbol with respect to the pairs S_1, S_2 and T_1, T_2, then the related Toeplitz $X = P_{H_1} Y|_{H_2}$ operator satisfies (6) *and conditions* (ii), (iii)*,* (iv).

Proof. Let $X \in L(H_2, H_1)$ be such that $S_1^* X = X T_1^*$ and $S_2^* X = X T_2^*$. Then, by (2), we have

$$\begin{aligned} S_1 P_{H_1} P_{\mathcal{R}_1} X T_1^* &= S_1 P_{H_1} P_{\mathcal{R}_1} S_1^* X = P_{H_1} W_1 P_{\mathcal{R}_1} S_1^* X \\ &= P_{H_1} W_1 P_{\mathcal{R}_1} W_1^* X = P_{H_1} W_1 W_1^* P_{\mathcal{R}_1} X \\ &= P_{H_1} P_{\mathcal{R}_1} X, \end{aligned}$$

since $\mathcal{R}_1$ is reducing for W_1. Similarly we prove that $S_2 P_{H_1} P_{\mathcal{R}_1} X T_2^* = P_{H_1} P_{\mathcal{R}_1} X$. Hence the operator $P_{H_1} P_{\mathcal{R}_1} X$ is a generalized Toeplitz operator with respect to the pairs S_1, S_2 and T_1, T_2.

Let now (i) be fulfilled. For nonnegative integers n, m and $h_2 \in H_2$ we have

$$\begin{aligned} X h_2 &= S_1^n S_2^m X T_1^{*n} T_2^{*m} h_2 = S_1^n S_2^m S_1^{*n} X T_2^{*m} h_2 = S_1^n S_2^m S_1^{*n} S_2^{*m} X h_2 \\ &= P_{H_1} W_1^n W_2^m S_1^{*n} S_2^{*m} X h_2 = P_{H_1} W_1^n W_2^m W_1^{*n} W_2^{*m} X h_2, \end{aligned}$$

by (1) and (2). Taking the limit when $n, m \to \infty$, by (3), we obtain

$$P_{H_1} W_1^n W_2^m W_1^{*n} W_2^{*m} X h_2 \longrightarrow P_{H_1} P_{\mathcal{R}_1} X h_2 \tag{7}$$

and (ii) is satisfied. Assume that (ii) holds, then

$$X = (P_{H_1} P_{\mathcal{R}_1})^n P_{H_1} X \longrightarrow P_{H_1 \cap \mathcal{R}_1} X,$$

by [7, p.192]. The remaining implications in the equivalence (i)–(iv) are straightforward.

Assume that (iv), (6) hold and Y is a symbol for X. Using Proposition 3.2, (1), Theorem 3.3 we have the following

$$\begin{aligned} Y h_2 &= \lim_{n,m\to\infty} W_1^n W_2^m P_{H_1} Y P_{H_2} V_1^{*n} V_2^{*m} h_2 = \lim_{n,m\to\infty} W_1^n W_2^m P_{H_1} Y T_1^{*n} T_2^{*m} h_2 \\ &= \lim_{n,m\to\infty} W_1^n W_2^m X T_1^{*n} T_2^{*m} h_2 = \lim_{n,m\to\infty} W_1^n W_2^m S_1^{*n} X T_2^{*m} h_2 \\ &= \lim_{n,m\to\infty} W_1^n W_2^m S_1^{*n} S_2^{*m} X h_2 = \lim_{n,m\to\infty} W_1^n W_2^m V_1^{*n} V_2^{*m} X h_2 = P_{\mathcal{R}_1} X h_2 \\ &= X h_2 \end{aligned}$$

by (3) and (iv). Consequently Y is an analytic symbol.

For the proof of the converse implication we assume that Y is an analytic symbol, $X = Y|_{H_2}$ and $X(H_2) = Y(H_2) \subset H_1 \cap \mathcal{R}_1$ by Proposition 3.2 [10, Proposition 3.2,(3)]. Moreover, for $i = 1, 2$ and $h_2 \in H_2$, we have

$$X T_i^* h_2 = Y T_i^* h_2 = Y V_i^* h_2 = W_i^* Y h_2 = W_i^* X h_2 = S_i^* X h_2,$$

which finishes the proof of the theorem.

Example 5.2

Let, as in Example 4.2, $T_{z_1}^*, T_{z_2}^*$ be the adjoints to multiplication operators by the independent variables in the space $H^2(\mathbb{T}^2)$ as both pairs of contractions. Looking from one point of view, if an operator X fulfills (6), then $T_{z_1} X = X T_{z_1}$ and $T_{z_2} X = X T_{z_2}$, which means that $X \in \{T_{z_1}, T_{z_2}\}'$. Hence, by [3, Theorem 11], the operator X have to be equal to a Toeplitz operator $X = T_\varphi$ with φ being a bounded holomorphic function on $\mathbb{D}^2$. On the other hand the symbol $Y \in L(L^2(\mathbb{T}^2))$ for X is represented by a function $\varphi \in L^\infty(\mathbb{T}^2)$ such that $Y = M_\varphi$. The analyticity of the symbol means that $Y(H_2(\mathbb{T}^2)) = M_\varphi(H_2(\mathbb{T}^2)) \subset H_2(\mathbb{T}^2)$. It forces φ to be holomorphic.

Example 5.3

Let M_{z_1}, M_{z_2} be the multiplication operators by the independent variables in the space $L^2(\mathbb{T}^2)$. Note that the operators M_{z_1}, M_{z_2} doubly commute. Since they are unitary operators a minimal isometric dilation is the same pair M_{z_1}, M_{z_2} and $K^+ = L^2(\mathbb{T}^2)$. Hence the operator $X \in L(L^2(\mathbb{T}^2))$ satisfies $X = M_{z_1} X M_{z_1}^*$ and $X = M_{z_2} X M_{z_2}^*$ if and only if X belongs to the commutant $\{M_{z_1}, M_{z_2}\}'$. The commutant equals to the set of all multiplication operators M_φ with $\varphi \in L^\infty(\mathbb{T}^2)$. Thus each generalized Toeplitz operator X with respect to both pairs being M_{z_1}, M_{z_2} equals to its symbol Y and there is a function $\varphi \in L^\infty(\mathbb{T}^2)$ such that $X = Y = M_\varphi$. Moreover, the symbol Y is analytic in our sense, since $Y L^2(\mathbb{T}^2) \subset L^2(\mathbb{T}^2)$.

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