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On the superstability of the cosine and sine type functional equations

Abstract. In this paper, we study the superstablity problem of the cosine and sine type functional equations:

$$f(x\sigma(y)a) + f(xya) = 2f(x)f(y)$$

and

$$f(x\sigma(y)a) - f(xya) = 2f(x)f(y),$$

where $f\colon S \to \mathbb{C}$ is a complex valued function; S is a semigroup; σ is an involution of S and a is a fixed element in the center of S.

1. Introduction

The stability problem of the functional equation was conjectured by Ulam [12] during the conference in the University of Wisconsin in 1940. In the next year, Hyers in [5] solved the problem of stability in the case of additive mapping. Since then it is called the Hyers-Ulam stability.

In 1979, Baker et al. in [4] introduced the following: if f satisfies the inequality $|E_1(f) - E_2(f)| \leq \epsilon$, then either f is bounded or $E_1(f) = E_2(f)$. The stability of this type is called the superstability.

The superstability of the cosine functional equation (also called the d'Alembert equation)

$$f(x + y) + f(x - y) = 2f(x)f(y),$$

was investigated by Baker [3]. Their results were improved by Badora [1] and Badora and Ger [2].

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The superstability of the sine functional equation

$$f(x+y)-f(x-y)=2f(x)f(y),$$

was investigated by Kim ([6, 7]).

The aim of this paper is to investigate the superstability problem of the cosine type functional equation

$$f(x\sigma(y)a)+f(xya)=2f(x)f(y) \tag{1}$$

and the sine type functional equation

$$f(x\sigma(y)a)-f(xya)=2f(x)f(y). \tag{2}$$

The form of solutions (1) (resp. (2)) are determined in [8, 11] (resp. [8, 10, 13]).

In this paper S is a semigroup, $\mathbb{C}$ stands for the field of complex numbers and a is a fixed element in the center of S. We may assume that f is a nonzero function, δ is a nonnegative real constant and σ is an involution of S, i.e. $\sigma(\sigma(x))=x$ and $\sigma(xy)=\sigma(y)\sigma(x)$ for all $x,y\in S$. If all the results of this article are given on the semigroup S, we will obtain identical results for a group G.

2. Stability of the equation (1)

In this section, we will investigated the superstability of the functional equation (1) related to the cosine functional equation. We start with the proof that f is an even function.

Lemma 2.1
Let $\delta\geq 0$. Let S be a semigroup and let f be a complex-valued function defined on S such that

$$|f(x\sigma(y)a)+f(xya)-2f(x)f(y)|\leq\delta,\qquad x,y\in S. \tag{3}$$

If f is unbounded, then it is even, i.e. $f(\sigma(x))=f(x)$ for all $x\in S$.

Proof. Assume that f is unbounded on S and satisfies inequality (3). So, for all $x,y\in G$ we have

$$|f(x\sigma(y)a)+f(xya)-2f(x)f(y)|\leq\delta,$$

replacing y by $\sigma(y)$ in (3) we obtain

$$|f(xya)+f(x\sigma(y)a)-2f(x)f(\sigma(y))|\leq\delta,$$

and by triangle inequality we find

$$|2f(x)||f(y)-f(\sigma(y))|\leq 2\delta\qquad\text{for all } x,y\in S.$$

Since f is assumed to be unbounded, then we get $f(y)=f(\sigma(y))$ for all $y\in S$.

Proposition 2.2
Suppose that $f\colon S \to \mathbb{C}$ satisfies the inequality (3), then

(*i*) *f is bounded and*

$$|f(x)| \leq \frac{1+\sqrt{1+2\delta}}{2}, \qquad x \in S.$$

Or

(*ii*) *f satisfies the functional equation*

$$f(x\sigma(y)\sigma(a)) + f(xy\sigma(a)) = 2f(x)f(y), \qquad x, y \in S. \tag{4}$$

Proof. (*i*) Assume that f satisfies the inequality (3). If f is bounded, let $M = \sup|f|$, then we get for all $x \in S$ that

$$|2f(x)f(x)| \leq \delta + 2M,$$

from which we obtain that $2M^2 - 2M - \delta \leq 0$ such that

$$M \leq \frac{1+\sqrt{1+2\delta}}{2}.$$

(*ii*) Assume that f is unbounded and satisfies the inequality (3). For all $x, y, z \in S$, we have

$$\begin{aligned}
&2|f(z)||f(x\sigma(y)\sigma(a)) + f(xy\sigma(a)) - 2f(x)f(y)| \\
&\quad = |2f(x\sigma(y)\sigma(a))f(z) + 2f(xy\sigma(a))f(z) - 4f(x)f(y)f(z)| \\
&\quad \leq |f(x\sigma(y)\sigma(a)\sigma(z)a) + f(x\sigma(y)\sigma(a)za) - 2f(x\sigma(y)\sigma(a))f(z)| \\
&\qquad + |f(xy\sigma(a)\sigma(z)a) + f(xy\sigma(a)za) - 2f(xy\sigma(a))f(z)| \\
&\qquad + |f(x\sigma(y)\sigma(a)\sigma(z)a) + f(xzaya) - 2f(x)f(zay)| \\
&\qquad + |f(x\sigma(y)\sigma(a)za) + f(x\sigma(z)aya) - 2f(x)f(\sigma(z)ay)| \\
&\qquad + |f(xy\sigma(a)\sigma(z)a) + f(xza\sigma(y)a) - 2f(x)f(za\sigma(y))| \\
&\qquad + |f(xy\sigma(a)za) + f(x\sigma(z)a\sigma(y)a) - 2f(x)f(\sigma(z)a\sigma(y))| \\
&\qquad + |f(x\sigma(z)a\sigma(y)a) + f(x\sigma(z)aya) - 2f(x\sigma(z)a)f(y)| \\
&\qquad + |f(xza\sigma(y)a) + f(xzaya) - 2f(xza)f(y)| \\
&\qquad + |2f(x\sigma(z)a)f(y) + 2f(xza)f(y) - 4f(x)f(z)f(y)| \\
&\qquad + |2f(\sigma(z)a\sigma(y))f(x) + 2f(\sigma(z)ay)f(x) - 4f(x)f(\sigma(z))f(y)| \\
&\qquad + |2f(za\sigma(y))f(x) + 2f(zay)f(x) - 4f(x)f(z)f(y)| \\
&\qquad + |4f(x)f(z)f(y) - 4f(x)f(\sigma(z))f(y)|.
\end{aligned}$$

By virtue of inequality (3) and using Lemma 2.1, we have

$$2|f(z)||f(x\sigma(y)\sigma(a)) + f(xy\sigma(a)) - 2f(x)f(y)|$$

$$\begin{aligned}&\le 8\delta + 2|f(y)|\delta\\&\quad + |2f(\sigma(z)a\sigma(y))f(x) + 2f(\sigma(z)ay)f(x) - 4f(x)f(\sigma(z))f(y)|\\&\quad + |2f(za\sigma(y))f(x) + 2f(zay)f(x) - 4f(x)f(z)f(y)|.\end{aligned}$$

Using that a is an element in the center of S and inequality (3) we find that

$$2|f(z)||f(x\sigma(y)\sigma(a)) + f(xy\sigma(a)) - 2f(x)f(y)| \le 8\delta + 2(|f(y)| + 2|f(x)|)\delta.$$

Since f is unbounded, from the last inequality, we conclude that f is a solution of the equation (4).

Theorem 2.3
Suppose that $f\colon S \to \mathbb{C}$ satisfies (3). Then
(i) f is bounded and

$$|f(x)| \le \frac{1+\sqrt{1+2\delta}}{2}, \qquad x \in S.$$

Or

(ii) f satisfies the functional equation

$$f(x\sigma(y)a) + f(xya) = 2f(x)f(y), \qquad x, y \in S. \tag{5}$$

Proof. (i) Assume that f is bounded and using Proposition 2.2, (i) we get that $|f(x)| \le \frac{1+\sqrt{1+2\delta}}{2}$, $x \in S$.

(ii) Assume that f satisfies the inequality (3). For all $x, y, z \in S$, we have

$$\begin{aligned}&2|f(z)||f(x\sigma(y)a) + f(xya) - 2f(x)f(y)|\\&\quad = |2f(x\sigma(y)a)f(z) + 2f(xya)f(z) - 4f(x)f(y)f(z)|\\&\quad \le |f(x\sigma(y)a\sigma(z)a) + f(x\sigma(y)aza) - 2f(x\sigma(y)a)f(z)|\\&\qquad + |f(xya\sigma(z)a) + f(xyaza) - 2f(xya)f(z)|\\&\qquad + |f(x\sigma(y)a\sigma(z)a) + f(xz\sigma(a)ya) - 2f(x)f(\sigma(y)a\sigma(z))|\\&\qquad + |f(x\sigma(y)aza) + f(x\sigma(z)\sigma(a)ya) - 2f(x)f(\sigma(y)az)|\\&\qquad + |f(xya\sigma(z)a) + f(xz\sigma(a)\sigma(y)a) - 2f(x)f(ya\sigma(z))|\\&\qquad + |f(xyaza) + f(x\sigma(z)\sigma(a)\sigma(y)a) - 2f(x)f(yaz)|\\&\qquad + |f(x\sigma(z)a\sigma(y)\sigma(a)) + f(x\sigma(z)ay\sigma(a)) - 2f(x\sigma(z)a)f(y)|\\&\qquad + |f(xza\sigma(y)\sigma(a)) + f(xzay\sigma(a)) - 2f(xza)f(y)|\\&\qquad + 2|f(y)||f(x\sigma(z)a) + f(xza) - 2f(x)f(z)|\\&\qquad + |2f(x)||f(ya\sigma(z)) + f(yaz) - 2f(y)f(z)|\\&\qquad + |2f(x)||f(\sigma(y)a\sigma(z)) + f(\sigma(y)az) - 2f(\sigma(y))f(z)|\\&\qquad + 4|f(x)f(z)f(y) - f(x)f(\sigma(y))f(z)|.\end{aligned}$$

By virtue of inequality (3) and according to Lemma 2.1, we have

$$\begin{aligned}
2|f(z)||f(x\sigma(y)a)+f(xya)-2f(x)f(y)|&\\
\le 6\delta+2|f(y)|\delta&\\
+|f(x\sigma(z)a\sigma(y)\sigma(a))+f(x\sigma(z)ay\sigma(a))-2f(x\sigma(z)a)f(y)|&\\
+|f(xza\sigma(y)\sigma(a))+f(xzay\sigma(a))-2f(xza)f(y)|&\\
+|2f(x)||f(ya\sigma(z))+f(yaz)-2f(y)f(z)|&\\
+|2f(x)||f(\sigma(y)a\sigma(z))+f(\sigma(y)az)-2f(\sigma(y))f(z)|.&
\end{aligned}$$

Since a is an element in the center of S and f is unbounded then, according to Proposition 2.2 (ii), f is a solution of the equation (5).

As an immediate consequence of Theorem 2.3, we have the following result which has been the subject of [9, Corollary 1] in the case where $a = e$.

Corollary 2.4 (11, Corollary 1)
Suppose that $f\colon S \to \mathbb{C}$ satisfies (3). Then

(i) *f is bounded and*

$$|f(x)| \le \frac{1+\sqrt{1+2\delta}}{2}, \qquad x \in S.$$

Or

(ii) *f satisfies the functional equation*

$$f(x\sigma(y)) + f(xy) = 2f(x)f(y), \qquad x, y \in S.$$

3. Stability of equation (2)

In this section, we will investigated the superstability of the functional equation (2) related to the sine functional equation.

Lemma 3.1
Let $\delta \ge 0$. Let S be a semigroup and let f be a complex-valued function defined on S such that

$$|f(x\sigma(y)a) - f(xya) - 2f(x)f(y)| \le \delta, \qquad x, y \in S. \tag{6}$$

If f is unbounded, then it is odd, i.e. $f(\sigma(x)) = -f(x)$ for all $x \in S$.

Proof. Let f be a complex-valued function defined on S which satisfies the inequality (6), then for all $x, y \in S$ we have

$$\begin{aligned}
2|f(x)||f(y)+f(\sigma(y))|&\\
=|2f(x)f(y)+2f(x)f(\sigma(y))|&
\end{aligned}$$

$$\begin{aligned}&= |f(x\sigma(y)a) - f(xya) - 2f(x)f(y) - f(x\sigma(y)a)\\&\quad + f(x\sigma(\sigma(y))a) - 2f(x)f(\sigma(y))|\\&\le |f(x\sigma(y)a) - f(xya) - 2f(x)f(y)|\\&\quad + |f(x\sigma(\sigma(y))a) - f(x\sigma(y)a) - 2f(x)f(\sigma(y))|\\&\le 2\delta.\end{aligned}$$

Since f is unbounded it follows that $f(\sigma(y)) = -f(y)$ for all $y \in S$.

Proposition 3.2
Suppose that $f\colon S \to \mathbb{C}$ satisfies the inequality (6), then one of the assertions is satisfied

(i) *f is bounded and*

$$|f(x)| \le \frac{1+\sqrt{1+2\delta}}{2}, \qquad x \in S.$$

Or

(ii) *f satisfies the functional equation*

$$f(x\sigma(y)\sigma(a)) - f(xy\sigma(a)) = 2f(x)f(y), \qquad x, y \in S. \tag{7}$$

Proof. (i) Using the same method of proof as in Proposition 2.2 (i), i.e. let $M = \sup|f|$, then for all $x \in S$ we have

$$|2f(x)f(x)| \le \delta + 2M,$$

from which we obtain that $2M^2 - 2M - \delta \le 0$, hence

$$M \le \frac{1+\sqrt{1+2\delta}}{2}.$$

(ii) Assume that f is unbounded and satisfies the inequality (6). For all $x, y, z \in S$, we have

$$\begin{aligned}&2|f(z)||f(x\sigma(y)\sigma(a)) - f(xy\sigma(a)) - 2f(x)f(y)|\\&\quad= |2f(x\sigma(y)\sigma(a))f(z) - 2f(xy\sigma(a))f(z) - 4f(x)f(y)f(z)|\\&\quad\le |f(x\sigma(y)\sigma(a)\sigma(z)a) - f(x\sigma(y)\sigma(a)za) - 2f(x\sigma(y)\sigma(a))f(z)|\\&\qquad + |f(xy\sigma(a)\sigma(z)a) - f(xy\sigma(a)za) - 2f(xy\sigma(a))f(z)|\\&\qquad + |f(x\sigma(y)\sigma(a)\sigma(z)a) - f(xzaya) - 2f(x)f(zay)|\\&\qquad + |f(x\sigma(y)\sigma(a)za) - f(x\sigma(z)aya) - 2f(x)f(\sigma(z)ay)|\\&\qquad + |f(xy\sigma(a)\sigma(z)a) - f(xza\sigma(y)a) - 2f(x)f(za\sigma(y))|\\&\qquad + |f(xy\sigma(a)za) - f(x\sigma(z)a\sigma(y)a) - 2f(x)f(\sigma(z)a\sigma(y))|\\&\qquad + |f(x\sigma(z)a\sigma(y)a) - f(x\sigma(z)aya) - 2f(x\sigma(z)a)f(y)|\end{aligned}$$

$$\begin{aligned}&+|f(xza\sigma(y)a)-f(xzaya)-2f(xza)f(y)|\\&+|2f(y)||f(x\sigma(z)a)-f(xza)-2f(x)f(z)|\\&+|2f(x)||f(\sigma(z)a\sigma(y))-f(\sigma(z)ay)-2f(\sigma(z))f(y)|\\&+|2f(x)||f(za\sigma(y))-f(zay)-2f(z)f(y)|\\&+|4f(x)f(z)f(y)+4f(x)f(\sigma(z))f(y)|.\end{aligned}$$

In virtue of inequality (6) and that a belongs to the center of S and using Lemma 3.1, we have

$$2|f(z)||f(x\sigma(y)\sigma(a))-f(xy\sigma(a))-2f(x)f(y)|\leq 8\delta+2(|f(y)|+2|f(x)|)\delta.$$

Since f is unbounded, from the last inequality, we conclude that f is a solution of the equation (7).

Theorem 3.3
Suppose that $f\colon S\to\mathbb{C}$ satisfies (6). Then
(i) *f is bounded and*

$$|f(x)|\leq\frac{1+\sqrt{1+2\delta}}{2},\qquad x\in S.$$

Or

(ii) *f satisfies the functional equation*

$$f(x\sigma(y)a)-f(xya)=2f(x)f(y),\qquad x,y\in S. \tag{8}$$

Proof. (ii) Assume that f satisfies the inequality (6). By using Lemma 3.1 and Proposition 3.2 (ii). For all $x,y,z\in S$, we have

$$\begin{aligned}&2|f(z)||f(x\sigma(y)a)-f(xya)-2f(x)f(y)|\\&\quad=|2f(x\sigma(y)a)f(z)-2f(xya)f(z)-4f(x)f(y)f(z)|\\&\quad\leq|f(x\sigma(y)a\sigma(z)a)-f(x\sigma(y)aza)-2f(x\sigma(y)a)f(z)|\\&\quad\quad+|f(xya\sigma(z)a)-f(xyaza)-2f(xya)f(z)|\\&\quad\quad+|f(xz\sigma(a)ya)-f(x\sigma(y)a\sigma(z)a)-2f(x)f(\sigma(y)a\sigma(z))|\\&\quad\quad+|f(x\sigma(z)\sigma(a)ya)-f(x\sigma(y)aza)-2f(x)f(\sigma(y)az)|\\&\quad\quad+|f(xz\sigma(a)\sigma(y)a)-f(xya\sigma(z)a)-2f(x)f(ya\sigma(z))|\\&\quad\quad+|f(x\sigma(z)\sigma(a)\sigma(y)a)-f(xyaza)-2f(x)f(yaz)|\\&\quad\quad+|f(x\sigma(z)a\sigma(y)\sigma(a))-f(x\sigma(z)ay\sigma(a))-2f(x\sigma(z)a)f(y)|\\&\quad\quad+|f(xz\sigma(a)\sigma(y)a)-f(xz\sigma(a)ya)-2f(xza)f(y)|\\&\quad\quad+|2f(y)||f(x\sigma(z)a)-f(xza)-2f(x)f(z)|\\&\quad\quad+|2f(x)||f(\sigma(y)a\sigma(z))-f(\sigma(y)az)-2f(\sigma(y))f(z)|\\&\quad\quad+|2f(x)||f(ya\sigma(z))-f(yaz)-2f(y)f(z)|\\&\quad\quad+|4f(x)f(z)f(y)+4f(x)f(\sigma(y))f(z)|.\end{aligned}$$

Therefore

$$\begin{aligned} 2|f(z)||f(x\sigma(y)a) + f(xya) - 2f(x)f(y)| \\ \leq 6\delta + 2|f(y)|\delta \\ + |2f(x)||f(\sigma(y)a\sigma(z)) - f(\sigma(y)az) - 2f(\sigma(y))f(z)| \\ + |2f(x)||f(ya\sigma(z)) - f(yaz) - 2f(y)f(z)|. \end{aligned}$$

Since a is an element in the center of S and f is unbounded, then f satisfies the equation (8), which finished the proof of Theorem 3.3.

The following corollary is a particular case of Theorem 3.3.

Corollary 3.4 ([14])
Suppose that $f\colon S \to \mathbb{C}$ *satisfies* (6) *and* $a = e$. *Then*

(*i*) f *is bounded and*

$$|f(x)| \leq \frac{1+\sqrt{1+2\delta}}{2}, \qquad x \in S.$$

Or

(*ii*) f *satisfies the functional equation*

$$f(x\sigma(y)) - f(xy) = 2f(x)f(y), \qquad x, y \in S.$$

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