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Strong ideals and horizontal ideals in pseudo-BCH-algebras

Abstract. In this paper we define strong ideals and horizontal ideals in pseudo-BCH-algebras and investigate the properties and characterizations of them.

1. Introduction

In 1966, Y. Imai and K. Iséki ([13, 14]) introduced BCK- and BCI-algebras. In 1983, Q.P. Hu and X. Li ([11]) introduced BCH-algebras. It is known that BCK- and BCI-algebras are contained in the class of BCH-algebras.

In 2001, G. Georgescu and A. Iorgulescu ([10]) introduced pseudo-BCK-algebras as an extension of BCK-algebras. In 2008, W.A. Dudek and Y.B. Jun ([2]) introduced pseudo-BCI-algebras as a natural generalization of BCI-algebras and of pseudo-BCK-algebras. These algebras have also connections with other algebras of logic such as pseudo-MV-algebras and pseudo-BL-algebras defined by G. Georgescu and A. Iorgulescu in [8] and [9], respectively. Those algebras were investigated by several authors in a number of papers (see for example [3, 5, 6, 7, 15, 17, 18, 19]). Recently, A. Walendziak ([20]) introduced pseudo-BCH-algebras as an extension of BCH-algebras and studied the set $\operatorname{Cen}\mathfrak{X}$ of all minimal elements of a pseudo-BCH-algebra $\mathfrak{X}$, the so-called centre of $\mathfrak{X}$. He also considered ideals in pseudo-BCH-algebras and established a relationship between the ideals of a pseudo-BCH-algebra and the ideals of its centre.

In this paper we define strong ideals and horizontal ideals in pseudo-BCH-algebras and investigate the properties and characterizations of them.

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2. Preliminaries

We recall that an algebra $\mathfrak{X} = (X; *, 0)$ of type $(2, 0)$ is called a *BCH-algebra* if it satisfies the following axioms:

(BCH-1) $x * x = 0$;

(BCH-2) $(x * y) * z = (x * z) * y$;

(BCH-3) $x * y = y * x = 0 \Longrightarrow x = y$.

A BCH-algebra $\mathfrak{X}$ is said to be a *BCI-algebra* if it satisfies the identity

(BCI) $((x * y) * (x * z)) * (z * y) = 0$.

A *BCK-algebra* is a BCI-algebra $\mathfrak{X}$ satisfying the law $0 * x = 0$.

Definition 2.1 ([2])
A *pseudo-BCI-algebra* is a structure $\mathfrak{X} = (X; \leq, *, \diamond, 0)$, where $\leq$ is a binary relation on the set X, $*$ and $\diamond$ are binary operations on X and 0 is an element of X, satisfying the axioms:

(pBCI-1) $(x * y) \diamond (x * z) \leq z * y, \quad (x \diamond y) * (x \diamond z) \leq z \diamond y$;

(pBCI-2) $x * (x \diamond y) \leq y, \quad x \diamond (x * y) \leq y$;

(pBCI-3) $x \leq x$;

(pBCI-4) $x \leq y,\ y \leq x \Longrightarrow x = y$;

(pBCI-5) $x \leq y \Longleftrightarrow x * y = 0 \Longleftrightarrow x \diamond y = 0$.

A pseudo-BCI-algebra $\mathfrak{X}$ is called a *pseudo-BCK-algebra* if it satisfies the identities

(pBCK) $0 * x = 0 \diamond x = 0$.

Definition 2.2 ([20])
A *pseudo-BCH-algebra* is an algebra $\mathfrak{X} = (X; *, \diamond, 0)$ of type $(2, 2, 0)$ satisfying the axioms:

(pBCH-1) $x * x = x \diamond x = 0$;

(pBCH-2) $(x * y) \diamond z = (x \diamond z) * y$;

(pBCH-3) $x * y = y \diamond x = 0 \Longrightarrow x = y$;

(pBCH-4) $x * y = 0 \Longleftrightarrow x \diamond y = 0$.

We define a binary relation $\leqslant$ on X by

$$x \leqslant y \Longleftrightarrow x * y = 0 \Longleftrightarrow x \diamond y = 0.$$

Throughout this paper $\mathfrak{X}$ will denote a pseudo-BCH-algebra.

Remark
Observe that if $(X; *, 0)$ is a BCH-algebra, then letting $x \diamond y := x * y$, produces a pseudo-BCH-algebra $(X; *, \diamond, 0)$. Therefore, every BCH-algebra is a pseudo-BCH-algebra in a natural way. It is easy to see that if $(X; *, \diamond, 0)$ is a pseudo-BCH-algebra, then $(X; \diamond, *, 0)$ is also a pseudo-BCH-algebra. From Proposition 3.2 of [2] we conclude that if $(X; \leq, *, \diamond, 0)$ is a pseudo-BCI-algebra, then $(X; *, \diamond, 0)$ is a pseudo-BCH-algebra.

Example 2.3 ([21])
Let $(G;\cdot,e)$ be a group. Define binary operations $*$ and $\diamond$ on G by

$$a*b=ab^{-1}\quad\text{and}\quad a\diamond b=b^{-1}a$$

for all $a,b\in G$. Then $\mathfrak{G}=(G;*,\diamond,e)$ is a pseudo-BCH-algebra.

We say that a pseudo-BCH-algebra $\mathfrak{X}$ is *proper* if $*\neq\diamond$ and $\mathfrak{X}$ is not a pseudo-BCI-algebra.

Example 2.4
Consider the set $X=\{0,a,b,c,d,e,f,g,h\}$ with the operations $*$ and $\diamond$ defined by the following tables:

$*$	0	a	b	c	d	e	f	g	h
0	0	0	0	0	d	e	f	h	g
a	a	0	c	c	d	e	f	h	g
b	b	0	0	b	d	e	f	h	g
c	c	0	0	0	d	e	f	h	g
d	d	d	d	d	0	h	g	e	f
e	e	e	e	e	g	0	h	f	d
f	f	f	f	f	h	g	0	d	e
g	g	g	g	g	e	f	d	0	h
h	h	h	h	h	f	d	e	g	0

and

$\diamond$	0	a	b	c	d	e	f	g	h
0	0	0	0	0	d	e	f	h	g
a	a	0	c	c	d	e	f	h	g
b	b	0	0	b	d	e	f	h	g
c	c	0	0	0	d	e	f	h	g
d	d	d	d	d	0	h	g	f	e
e	e	e	e	e	g	0	h	d	f
f	f	f	f	f	h	g	0	e	d
g	g	g	g	g	e	f	d	0	g
h	h	h	h	h	f	d	e	h	0

Then $(X;*,\diamond,0)$ is a proper pseudo-BCH-algebra (see [21]).

From [20] it follows that in any pseudo-BCH-algebra $\mathfrak{X}$ for all $x,y\in X$ we have:

(a1) $x*(x\diamond y)\leqslant y$ and $x\diamond(x*y)\leqslant y$;
(a2) $x*0=x\diamond 0=x$;
(a3) $0*x=0\diamond x$;
(a4) $0*(0*(0*x))=0*x$;
(a5) $0*(x*y)=(0*x)\diamond(0*y)$;
(a6) $0*(x\diamond y)=(0*x)*(0*y)$.

Following the terminology of [20], the set $\{a\in X:\ a=0*(0*a)\}$ will be called the *centre* of $\mathfrak{X}$. W shall denote it by $\operatorname{Cen}\mathfrak{X}$. By Proposition 4.1 of [20], $\operatorname{Cen}\mathfrak{X}$ is the set of all minimal elements of $\mathfrak{X}$, that is, $\operatorname{Cen}\mathfrak{X} = \{a \in X : \forall_{x\in X}\ x \leqslant a \Longrightarrow x = a\}$.

Example 2.5
Let $\mathfrak{X} = (X; *, \diamond, 0)$ be the pseudo-BCH-algebra given in Example 2.4. It is easily seen that $\operatorname{Cen}\mathfrak{X} = \{0, d, e, f, g, h\}$.

Proposition 2.6 ([20])
Let $\mathfrak{X}$ be a pseudo-BCH-algebra, and let $a \in X$. Then the following conditions are equivalent:

(i) $a \in \operatorname{Cen}\mathfrak{X}$.

(ii) $a * x = 0 * (x * a)$ *for all* $x \in X$.

(iii) $a \diamond x = 0 * (x \diamond a)$ *for all* $x \in X$.

Proposition 2.7 ([20])
$\operatorname{Cen}\mathfrak{X}$ *is a subalgebra of* $\mathfrak{X}$.

Definition 2.8
A subset I of X is called an *ideal* of $\mathfrak{X}$ if it satisfies for all $x, y \in X$:

(I1) $0 \in I$;

(I2) if $x * y \in I$ and $y \in I$, then $x \in I$.

We will denote by $\operatorname{Id}(\mathfrak{X})$ the set of all ideals of $\mathfrak{X}$. Obviously, $\{0\}, X \in \operatorname{Id}(\mathfrak{X})$.

Proposition 2.9 ([20])
Let I be an ideal of $\mathfrak{X}$. For any $x, y \in X$, if $y \in I$ and $x \leqslant y$, then $x \in I$.

Proposition 2.10 ([20])
Let $\mathfrak{X}$ be a pseudo-BCH-algebra and I be a subset of X satisfying (I1). *Then I is an ideal of $\mathfrak{X}$ if and only if for all $x, y \in X$,*

(I2') *if $x \diamond y \in I$ and $y \in I$, then $x \in I$.*

Proposition 2.11
Let I be an ideal of $\mathfrak{X}$ and $x \in I$. Then $0 * (0 * x) \in I$.

Proof. Let $x \in X$. From (a3) and (a1) it follows that $0 * (0 * x) = 0 * (0 \diamond x) \leqslant x$. Since $x \in I$, by Proposition 2.9, $0 * (0 * x) \in I$.

Example 2.12
Consider the pseudo-BCH-algebra $\mathfrak{G}$, which is given in Example 2.3. Let a be an element of G. It is routine to verify that $\{a^n : n \in \mathbb{N} \cup \{0\}\}$ is an ideal of $\mathfrak{G}$.

Proposition 2.13 ([21])
Let $\mathfrak{X}$ be a pseudo-BCH-algebra and I be a subset of X containing 0. The following statements are equivalent:

(i) *I is an ideal of $\mathfrak{X}$.*

(ii) $x \in I$, $y \in X - I \Longrightarrow y * x \in X - I$.

(iii) $x \in I$, $y \in X - I \Longrightarrow y \diamond x \in X - I$.

For any pseudo-BCH-algebra $\mathfrak{X}$, we set

$$\mathrm{K}(\mathfrak{X}) = \{x \in X : \ 0 \leqslant x\}.$$

From ([20]) it follows that $\mathrm{K}(\mathfrak{X})$ is a subalgebra of $\mathfrak{X}$. Observe that

$$\operatorname{Cen}\mathfrak{X} \cap \mathrm{K}(\mathfrak{X}) = \{0\}. \tag{1}$$

Indeed, $0 \in \operatorname{Cen}\mathfrak{X} \cap \mathrm{K}(\mathfrak{X})$ and if $x \in \operatorname{Cen}\mathfrak{X} \cap \mathrm{K}(\mathfrak{X})$, then $x = 0*(0*x) = 0*0 = 0$.

3. Closed, strong, and horizontal ideals

An ideal I of $\mathfrak{X}$ is said to be *closed* if $0 * x \in I$ for every $x \in I$.

Proposition 3.1 ([20])
An ideal I of $\mathfrak{X}$ is closed if and only if I is a subalgebra of $\mathfrak{X}$.

Proposition 3.2 ([20])
Every ideal of a finite pseudo-BCH-algebra is closed.

Example 3.3
Let M be the set of all matrices of the form $A = \begin{bmatrix} x & y \\ 0 & 1 \end{bmatrix}$, where x and y are rational numbers such that $x > 0$. Evidently, $(M; \cdot, E)$, where $\cdot$ is the usual multiplication of matrices and $E = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$, is a group. We define the binary operations $*$ and $\diamond$ on M by

$$A * B = AB^{-1} \quad \text{and} \quad A \diamond B = B^{-1}A$$

for all $A, B \in M$. Then $\mathfrak{M} = (M; *, \diamond, E)$ is a pseudo-BCH-algebra (by Example 2.3). Let $C = \begin{bmatrix} 2 & 0 \\ 0 & 1 \end{bmatrix}$. The set $I = \{C^n : \ n \in \mathbb{N} \cup \{0\}\}$ is an ideal of $\mathfrak{M}$ (see Example 2.12). Observe that I is not closed. Indeed, $E * C = EC^{-1} = C^{-1} \notin I$.

Proposition 3.4 ([20])
$\mathrm{K}(\mathfrak{X})$ *is a closed ideal.*

Definition 3.5
An ideal I of a pseudo-BCH-algebra $\mathfrak{X}$ is called *strong* if, for all $x, y \in X$, $x \in I$ and $y \in X - I$ imply $x * y \in X - I$.

It is clear that X is a strong ideal of $\mathfrak{X}$. Note that in BCI-algebras such ideals were investigated in [1] (see also [12]).

Theorem 3.6
Let I be an ideal of $\mathfrak{X}$. Then the following statements are equivalent:

(i) *I is strong.*

(ii) *For any $x, y \in X$, $x * y \in I$ and $x \in I$ imply $y \in I$.*

(iii) *For every* $x \in X$ *both* x *and* $0 * x$ *belong or not belong to* I.

(iv) *For every* $x \in X$, $0 * x \in I$ *implies* $x \in I$.

Proof. (i) $\Longrightarrow$ (ii) Let I be a strong ideal. Let $x \in I$ and $x * y \in I$. Suppose that $y \notin I$. By the strongness of I, $x * y \in X - I$. This is a contradiction.

(ii) $\Longrightarrow$ (iii) Let $x \in I$. Then, by Proposition 2.11, $0 * (0 * x) \in I$. Since $0 \in I$, according to (ii), we deduce $0 * x \in I$. Thus, if $x \in I$, then $0 * x \in I$. Suppose now that $x \notin I$ and $0 * x \in I$. Applying (pBCH-2) and (pBCH-1) we have

$$[(0 * x) * x] \diamond (0 * x) = ((0 * x) \diamond (0 * x)) * x = 0 * x \in I,$$

and from the definition of ideal we conclude that $(0 * x) * x \in I$. By (ii), $x \in I$, which is a contradiction. Thus, if $x \notin I$, then $0 * x \notin I$.

(iii) $\Longrightarrow$ (iv) Obvious.

(iv) $\Longrightarrow$ (i) Any ideal I with the property that both x and $0 * x$ belong or not belong to I, is obviously closed. To prove that I is strong, let $x \in I$ and $y \in X - I$. On the contrary, assume that $x * y \in I$. Hence $0 * (x * y) \in I$, and by (a5) we obtain $(0 * x) \diamond (0 * y) \in I$. Also $0 * x \in I$. Since I is a subalgebra of $\mathfrak{X}$ (by Proposition 3.1) it follows that $((0 * x) \diamond (0 * y)) * (0 * x) \in I$. Then $0 * (0 * y) \in I$, because

$$\begin{aligned} 0 * (0 * y) &= 0 \diamond (0 * y) && \text{[by (a3)]} \\ &= ((0 * x) * (0 * x)) \diamond (0 * y) && \text{[by (pBCH-1)]} \\ &= ((0 * x) \diamond (0 * y)) * (0 * x). && \text{[by (pBCH-2)]} \end{aligned}$$

Using (iv) we conclude that $y \in I$, a contradiction.

From the proof of Theorem 3.6 we have the following corollaries.

Corollary 3.7
Every strong ideal of $\mathfrak{X}$ *is closed.*

Corollary 3.8
Let I *be an ideal of* $\mathfrak{X}$. *Then the following statements are equivalent:*

(i) I *is strong.*

(ii) *For any* $x, y \in X$, $x \in I$ *and* $y \in X - I$ *imply* $x \diamond y \in X - I$.

(iii) *For any* $x, y \in X$, $x \diamond y \in I$ *and* $x \in I$ *imply* $y \in I$.

Combining Proposition 2.13 and Corollary 3.8 we get

Theorem 3.9
Let $\mathfrak{X}$ *be a pseudo-BCH-algebra and* I *be a subset of* X *containing* 0. *The following statements are equivalent:*

(i) I *is a strong ideal of* $\mathfrak{X}$.

(ii) *For any* $x, y \in X$, $x \in I$ *and* $y \in X - I$ *imply* $x * y$, $y * x \in X - I$.

(iii) *For any* $x, y \in X$, $x \in I$ *and* $y \in X - I$ *imply* $x \diamond y$, $y \diamond x \in X - I$.

Theorem 3.10
Let I be a closed ideal of $\mathfrak{X}$. Then the following statements are equivalent:

(i) *I is strong.*

(ii) *For all $x, y \in X$, $x \leqslant y$ and $x \in I$ imply $y \in I$.*

(iii) *For every $x \in X$, $0 * (0 * x) \in I$ implies $x \in I$.*

Proof. (i) $\Longrightarrow$ (ii) Let I be a strong ideal. Let $x \leqslant y$ and $x \in I$. Then $x * y = 0 \in I$. As I satisfies (ii) of Theorem 3.6, we get $y \in I$.

(ii) $\Longrightarrow$ (iii) Let $0 * (0 * x) \in I$. Since $0 * (0 * x) \leqslant x$, applying (ii) we see that $x \in I$.

(iii) $\Longrightarrow$ (i) Let $0 * x \in I$. Then $0 * (0 * x) \in I$, because I is closed. From (iii) it follows that $x \in I$. Thus condition (iv) of Theorem 3.6 holds. Consequently, I is a strong ideal.

As a consequence of Proposition 3.2 and Theorem 3.10 we get the following

Proposition 3.11
Let I be an ideal of a finite pseudo-BCH-algebra $\mathfrak{X}$. Then the following statements are equivalent:

(i) *I is strong.*

(ii) *For all $x, y \in X$, $x \leqslant y$ and $x \in I$ imply $y \in I$.*

(iii) *For every $x \in X$, $0 * (0 * x) \in I$ implies $x \in I$.*

Proposition 3.12
$\mathrm{K}(\mathfrak{X})$ *is a strong ideal.*

Proof. By Proposition 3.4, $\mathrm{K}(\mathfrak{X})$ is closed. Let $0 * (0 * x) \in \mathrm{K}(\mathfrak{X})$. Then $0 * (0 * (0 * x)) = 0$. Since $0 * (0 * (0 * x)) = 0 * x$ (see (a4)), we have $0 * x = 0$. Hence $x \in \mathrm{K}(\mathfrak{X})$, and thus $\mathrm{K}(\mathfrak{X})$ satisfies condition (iii) of Theorem 3.10. Therefore $\mathrm{K}(\mathfrak{X})$ is strong.

Proposition 3.13
Let $I \in \mathrm{Id}(\mathfrak{X})$. If $I \subset \mathrm{K}(\mathfrak{X})$, then I is not a strong ideal.

Proof. Let $a \in \mathrm{K}(\mathfrak{X}) - I$. Then $0 * (0 * a) = 0 * 0 = 0 \in I$ but $a \notin I$.

Example 3.14
Let $\mathfrak{X} = (X; *, \diamond, 0)$ be the pseudo-BCH-algebra given in Example 2.4. $\mathfrak{X}$ has six strong ideals, namely: $I = \{0, a, b, c\}$, $I \cup \{d\}$, $I \cup \{e\}$, $I \cup \{f\}$, $I \cup \{g, h\}$, X. In $\mathfrak{X}$, $\{0\}$ is not a strong ideal by Proposition 3.13.

In [16], K.H. Kim and E.H. Roh introduced the notion of H-ideal in BCH-algebras. Similarly, we define horizontal ideals in pseudo-BCH-algebras.

Let $I \in \mathrm{Id}(\mathfrak{X})$. We say that I is a *horizontal ideal* of $\mathfrak{X}$ if $I \cap \mathrm{K}(\mathfrak{X}) = \{0\}$. Obviously, $\{0\}$ is a horizontal ideal of $\mathfrak{X}$.

Remark
In pseudo-BCI-algebras, horizontal ideals were considered by G. Dymek in [4].

Example 3.15
Let $\mathfrak{M}$ and I be given as in Example 3.3. It is not difficult to verify that I is a horizontal ideal of $\mathfrak{M}$.

Proposition 3.16
If $\mathfrak{X}$ is a pseudo-BCH-algebra, then $\mathrm{K}(\mathfrak{X}) = \{0\}$ *if and only if every ideal of $\mathfrak{X}$ is horizontal.*

Proof. The proof is straightforward.

Theorem 3.17
Let I be a closed ideal of $\mathfrak{X}$. Then I is horizontal if and only if $I \subseteq \mathrm{Cen}\,\mathfrak{X}$.

Proof. Let I be a closed horizontal ideal and $x \in I$. By Proposition 2.11, $0 * (0 * x) \in I$. Since I is a closed ideal, from Proposition 3.1 it follows that I is a subalgebra of $\mathfrak{X}$. Then

$$x * (0 * (0 * x)) \in I. \tag{2}$$

Observe that $x * (0 * (0 * x)) \in \mathrm{K}(\mathfrak{X})$. By (a5) and (a4), $0 * [x * (0 * (0 * x))] = (0 * x) \diamond (0 * (0 * (0 * x))) = (0 * x) \diamond (0 * x) = 0$, and hence

$$x * (0 * (0 * x)) \in \mathrm{K}(\mathfrak{X}). \tag{3}$$

From (2) and (3) it follows that $x * (0 * (0 * x)) \in I \cap \mathrm{K}(\mathfrak{X}) = \{0\}$. Therefore $x * (0 * (0 * x)) = 0$, that is, $x \leqslant 0 * (0 * x)$. By (a3) and (a1) we have $0 * (0 * x) = 0 * (0 \diamond x) \leqslant x$. Thus $x = 0 * (0 * x)$. Consequently, $x \in \mathrm{Cen}\,\mathfrak{X}$.

Conversely, let $I \subseteq \mathrm{Cen}\,\mathfrak{X}$. Then $I \cap \mathrm{K}(\mathfrak{X}) \subseteq \mathrm{Cen}\,\mathfrak{X} \cap \mathrm{K}(\mathfrak{X}) = \{0\}$ (see (1)). From this $I \cap \mathrm{K}(\mathfrak{X}) = \{0\}$, so I is a horizontal ideal.

Corollary 3.18
If $\mathrm{Cen}\,\mathfrak{X}$ *is an ideal of $\mathfrak{X}$, then it is horizontal.*

Proof. Let $\mathrm{Cen}\,\mathfrak{X}$ be an ideal of $\mathfrak{X}$. Since $\mathrm{Cen}\,\mathfrak{X}$ is a subalgebra of $\mathfrak{X}$ (see Proposition 2.7), $\mathrm{Cen}\,\mathfrak{X}$ is closed by Proposition 3.1. From Theorem 3.17 we deduce that $\mathrm{Cen}\,\mathfrak{X}$ is horizontal.

Theorem 3.19
Let I be a closed ideal of $\mathfrak{X}$. Then the following statements are equivalent:

(i) *I is horizontal.*

(ii) $x = (x * a) * (0 * a)$ *for* $x \in X$, $a \in I$.

(iii) *For all* $x \in X$, $a \in I$, $x * a = 0 * a$ *implies* $x = 0$.

(iv) *For all* $x \in \mathrm{K}(\mathfrak{X})$, $a \in I$, $x * a = 0 * a$ *implies* $x = 0$.

Proof. (i) $\Longrightarrow$ (ii) Let I be a horizontal ideal of $\mathfrak{X}$. From Theorem 3.17 it follows that $I \subseteq \mathrm{Cen}\,\mathfrak{X}$. Let $x \in X$ and $a \in I$. By (pBCH-2) and (pBCH-1),

$$((x*a)*(0*a)) \diamond x = ((x*a) \diamond x)*(0*a) = ((x \diamond x)*a))*(0*a) = (0*a)*(0*a) = 0,$$

and hence

$$(x * a) * (0 * a) \leqslant x. \tag{4}$$

Using (pBCH-2) and (a1), we obtain

$$(x \diamond ((x * a) * (0 * a))) * a = (x * a) \diamond ((x * a) * (0 * a)) \leqslant 0 * a. \tag{5}$$

Since $a \in I$ and $0 * a \in I$, from (5) we see that

$$x \diamond ((x * a) * (0 * a)) \in I. \tag{6}$$

Applying (a5) and Proposition 2.6 we get

$$0 * ((x * a) * (0 * a)) = (0 * (x * a)) \diamond (0 * (0 * a)) = (a * x) \diamond a = (a \diamond a) * x = 0 * x.$$

Then by (a6), $0 * (x \diamond ((x * a) * (0 * a))) = (0 * x) * (0 * x) = 0$, and hence $x \diamond ((x * a) * (0 * a)) \in \mathrm{K}(\mathfrak{X})$. From this and (6) we have $x \diamond ((x * a) * (0 * a)) \in I \cap \mathrm{K}(\mathfrak{X}) = \{0\}$, that is, $x \diamond ((x * a) * (0 * a)) = 0$. Therefore

$$x \leqslant (x * a) * (0 * a). \tag{7}$$

Using (4), (7) and (pBCH-3) we obtain $x = (x * a) * (0 * a)$.

(ii) $\Longrightarrow$ (iii) Let $x \in X$, $a \in I$, and $x * a = 0 * a$. Then $x = (x * a) * (0 * a) = (x * a) * (x * a) = 0$.

(iii) $\Longrightarrow$ (iv) is obvious.

(iv) $\Longrightarrow$ (i) Let $x \in I \cap \mathrm{K}(\mathfrak{X})$. Hence $x * x = 0 = 0 * x$, and by (iv) we obtain $x = 0$. So $I \cap \mathrm{K}(\mathfrak{X}) = \{0\}$, and consequently, I is a horizontal ideal of $\mathfrak{X}$.

We also have theorem analogus to Theorem 3.19.

Theorem 3.20
Let I be a closed ideal of $\mathfrak{X}$. Then the following statements are equivalent:

(i) *I is horizontal.*
(ii) *$x = (x \diamond a) \diamond (0 \diamond a)$ for $x \in X$, $a \in I$.*
(iii) *For all $x \in X$, $a \in I$, $x \diamond a = 0 \diamond a$ implies $x = 0$.*
(iv) *For all $x \in \mathrm{K}(\mathfrak{X})$, $a \in I$, $x \diamond a = 0 \diamond a$ implies $x = 0$.*

Proposition 3.21
Let $\mathfrak{X}$ be a pseudo-BCH-algebra. Then:

(i) *If $\mathfrak{X}$ satisfies the condition* (pBCK)*, then the only $\{0\}$ is a horizontal ideal of $\mathfrak{X}$ and the only X is a strong ideal of $\mathfrak{X}$.*
(ii) *If $0 * x = x$ for all $x \in X$, then every ideal of $\mathfrak{X}$ is both strong and horizontal.*

Proof. The proof is straightforward.

Corollary 3.22
If $\mathfrak{X}$ is a pseudo-BCK-algebra, then the only $\{0\}$ is a horizontal ideal of $\mathfrak{X}$ and the only X is a strong ideal of $\mathfrak{X}$.

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