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On nondegenerate umbilical affine hypersurfaces in recurrent affine manifolds

Dedicated to Professor Andrzej Zajtz
on the occasion of his 70th birthday

Abstract. Let $\widetilde{M}$ be a differentiable manifold of dimension $\geqslant 5$, which is endowed with a (torsion-free) affine connection $\widetilde{\nabla}$ of recurrent curvature. Let M be a nondegenerate umbilical affine hypersurface in $\widetilde{M}$, whose shape operator does not vanish at every point of M. Denote by ∇ and h, respectively, the affine connection and the affine metric induced on M from the ambient manifold. Under the additional assumption that the induced connection ∇ is related to the Levi-Civita connection ∇^* of h by the formula

$$\nabla_X Y = \nabla^*_X Y + \varphi(X)Y + \varphi(Y)X + h(X,Y)E,$$

φ being a 1-form and E a vector field on M, it is proved that the affine metric h is conformally flat. Relations to totally umbilical pseudo-Riemannian hypersurfaces are also discussed.

In this paper, certain ideas from my unpublished report [14] (cf. also [15]) are generalized.

1. Preliminaries ([11, 10])

Let $\widetilde{M}$ be an $(n+1)$-dimensional affine manifold, that is, a connected differentiable manifold endowed with an affine connection $\widetilde{\nabla}$ (only torsion-free affine connections will be considered).

Let M be an n-dimensional connected differentiable manifold immersed into $\widetilde{M}$ and assume that there exists a transversal vector field ξ along the submanifold M. If $\widetilde{X}$ is a vector field defined along the submanifold M (which is not tangent to M in general), by $\widetilde{X}^\top$ and $\widetilde{X}^\perp$ we indicate its tangential and transversal parts, respectively.

Denote by ∇ the affine connection induced on M by assuming $\nabla_X Y = (\widetilde{\nabla}_X Y)^\top$ for all vector fields X, Y tangent to M. In the sequel, M will be

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called an affine hypersurface of the affine manifold $\widetilde{M}$. Thus, we have the Gauss equation for M

$$\widetilde{\nabla}_X Y = \nabla_X Y + h(X,Y)\xi \tag{1}$$

for all vector fields X, Y tangent to M, where h is a symmetric $(0,2)$-tensor field, which is called the affine fundamental form of M or the affine metric corresponding to ξ.

The affine hypersurface M is said to be nondegenerate if the affine metric h is nondegenerate. In this case, h is a Riemannian or pseudo-Riemannian metric on M. It should be mentioned that there is no relation between the affine metric h and the induced connection ∇ in general.

For the affine hypersurface M, we also have the so-called Weingarten equation

$$\widetilde{\nabla}_X \xi = -AX + \tau(X)\xi, \tag{2}$$

where A is a (1,1)-tensor field and τ is a 1-form on M. A and τ are called, respectively, the shape operator and the transversal connection form of M.

Let $\widetilde{R}$ and R be the curvature tensor fields of the connection $\widetilde{\nabla}$ and the induced connection ∇. Thus,

$$\widetilde{R}\big(\widetilde{X},\widetilde{Y}\,\big) = [\widetilde{\nabla}_{\widetilde{X}},\widetilde{\nabla}_{\widetilde{Y}}] - \widetilde{\nabla}_{[\widetilde{X},\widetilde{Y}]} \qquad \text{for any vector fields } \widetilde{X},\ \widetilde{Y} \text{ on } \widetilde{M}$$

and

$$R(X,Y) = [\nabla_X,\nabla_Y] - \nabla_{[X,Y]} \qquad \text{for any vector fileds } X,\ Y \text{ on } M.$$

As the integrability conditions of (1) and (2), we have the so-called Gauss and Codazzi equations

$$\begin{aligned} \widetilde{R}(X,Y)Z = R(X,Y)Z &- h(Y,Z)AX + h(X,Z)AY \\ &+ \big((\nabla_X h)(Y,Z) + \tau(X)h(Y,Z) \\ &- (\nabla_Y h)(X,Z) - \tau(Y)h(X,Z)\big)\xi, \end{aligned} \tag{3}$$

$$\begin{aligned} \widetilde{R}(X,Y)\xi = &-(\nabla_X A)Y + \tau(X)AY + (\nabla_Y A)X - \tau(Y)AX \\ &+ \big(-h(X,AY) + h(Y,AX) + 2d\tau(X,Y)\big)\xi. \end{aligned} \tag{4}$$

In the above formulas and in the sequel, symbols X, Y, $Z, \ldots$ denote arbitrary vector fields tangent to M if it is not otherwise stated.

Remark

Note that for an immersion of a differentiable manifold M into an affine manifold $\widetilde{M}$, a choice of a transversal vector field ξ provides the induced connection ∇ on M in such a way that this immersion becomes an affine immersion of (M,∇) into $(\widetilde{M},\widetilde{\nabla})$ in the sense of [9].

2. Umbilical affine hypersurfaces

An affine hypersurface M is said to be umbilical ([5, 8, 10]) if its shape operator A is proportional to the identity tensor at every point of the hypersurface, that is, we have $A = \rho\,\mathrm{Id}$, where Id is the identity tensor field and ρ is a certain function on M. Consequently, for such a hypersurface, we also have $\nabla A = d\rho \otimes \mathrm{Id}$, where d indicates the exterior derivative.

For an umbilical affine hypersurface, the Gauss and Codazzi equations (3) and (4) take the forms

$$\begin{aligned}\widetilde{R}(X,Y)Z = R(X,Y)Z &- \rho h(Y,Z)X + \rho h(X,Z)Y \\ &+ \big((\nabla_X h)(Y,Z) + \tau(X)h(Y,Z) \\ &- (\nabla_Y h)(X,Z) - \tau(Y)h(X,Z)\big)\xi,\end{aligned} \tag{5}$$

$$\widetilde{R}(X,Y)\xi = (\rho\tau - d\rho)(X)Y - (\rho\tau - d\rho)(Y)X + 2d\tau(X,Y)\xi. \tag{6}$$

The following proposition can be found in my unpublished report [14], and we include its proof to the presented paper for completness only.

Proposition 1
For an umbilical affine hypersurface M in an affine manifold $\widetilde{M}$, we have

$$\begin{aligned}((\widetilde{\nabla}_Z\widetilde{R})(X,Y)\xi)^{\top} = \rho R(X,Y)Z& \\ &- 2\rho d\tau(X,Y)Z - \rho^2(h(Y,Z)X - h(X,Z)Y) \\ &- \big((\nabla_Z(\rho\tau - d\rho))(Y) - \tau(Z)(\rho\tau - d\rho)(Y)\big)X \\ &+ \big((\nabla_Z(\rho\tau - d\rho))(X) - \tau(Z)(\rho\tau - d\rho)(X)\big)Y \\ &+ h(Y,Z)(\widetilde{R}(\xi,X)\xi)^{\top} - h(X,Z)(\widetilde{R}(\xi,Y)\xi)^{\top}.\end{aligned} \tag{7}$$

Proof. Applying the equalities (1), (2) and $A = \rho\,\mathrm{Id}$ into the general formula

$$\begin{aligned}(\widetilde{\nabla}_Z\widetilde{R})(X,Y)\xi = \widetilde{\nabla}_Z\widetilde{R}(X,Y)\xi &- \widetilde{R}(\widetilde{\nabla}_Z X,Y)\xi \\ &- \widetilde{R}(X,\widetilde{\nabla}_Z Y)\xi - \widetilde{R}(X,Y)\widetilde{\nabla}_Z\xi,\end{aligned}$$

we find

$$\begin{aligned}(\widetilde{\nabla}_Z\widetilde{R})(X,Y)\xi = \widetilde{\nabla}_Z\widetilde{R}(X,Y)\xi &- \widetilde{R}(\nabla_Z X,Y)\xi - \widetilde{R}(X,\nabla_Z Y)\xi \\ &- h(Z,X)\widetilde{R}(\xi,Y)\xi + h(Z,Y)\widetilde{R}(\xi,X)\xi \\ &+ \rho\widetilde{R}(X,Y)Z - \tau(Z)\widetilde{R}(X,Y)\xi.\end{aligned} \tag{8}$$

On the other hand, with the help of (6), (1) and (2), we find

$$
\begin{aligned}
&(\widetilde{\nabla}_Z \widetilde{R}(X,Y)\xi - \widetilde{R}(\nabla_Z X, Y)\xi - \widetilde{R}(X, \nabla_Z Y)\xi)^\top \\
&\qquad = (\nabla_Z(\rho\tau - d\rho))(X)Y - (\nabla_Z(\rho\tau - d\rho))(Y)X \\
&\qquad\quad - 2\rho d\tau(X,Y)Z.
\end{aligned} \tag{9}
$$

Moreover, (5) and (6) imply

$$
(\widetilde{R}(X,Y)Z)^\top = R(X,Y)Z - \rho h(Y,Z)X + \rho h(X,Z)Y, \tag{10}
$$

$$
(\widetilde{R}(X,Y)\xi)^\top = (\rho\tau - d\rho)(X)Y - (\rho\tau - d\rho)(Y)X. \tag{11}
$$

Now, to obtain (7) it is sufficient to take the tangential parts of the both sides of (8) and use identities (9)-(11).

In the final section, we will study the case when the ambient affine manifold $\widetilde{M}$ is a recurrent affine manifold, that is, the curvature tensor field $\widetilde{R}$ of $\widetilde{M}$ is non-zero and its covariant derivative $\widetilde{\nabla}\widetilde{R}$ satisfies the conditon ([19, 20, 6])

$$
\widetilde{\nabla}\widetilde{R} = \psi \otimes \widetilde{R} \tag{12}
$$

for a certain 1-form ψ.

We will need the following result:

Proposition 2

Let M be an umbilical affine hypersurface in a recurrent affine manifold $\widetilde{M}$. Then the curvature tensor R of the induced connection ∇ is given by

$$
\begin{aligned}
&\rho R(X,Y)Z \\
&\quad = 2\rho d\tau(X,Y)Z + \rho^2(h(Y,Z)X - h(X,Z)Y) \\
&\qquad + \big((\nabla_Z(\rho\tau - d\rho))(Y) - (\tau+\psi)(Z)(\rho\tau - d\rho)(Y)\big)X \\
&\qquad - \big((\nabla_Z(\rho\tau - d\rho))(X) - (\tau+\psi)(Z)(\rho\tau - d\rho)(X)\big)Y \\
&\qquad - h(Y,Z)(\widetilde{R}(\xi,X)\xi)^\top + h(X,Z)(\widetilde{R}(\xi,Y)\xi)^\top
\end{aligned} \tag{13}
$$

Proof. At first, note that (12) and (6) enable us to find

$$
(\widetilde{\nabla}_Z\widetilde{R})(X,Y)\xi = \psi(Z)\big((\rho\tau - d\rho)(X)Y - (\rho\tau - d\rho)(Y)X + 2d\tau(X,Y)\xi\big).
$$

Then, applying the above into (7), we obtain (13).

3. A special class of affine connections

In the next section, a geometric situation occurs in which a pseudo-Riemannian manifold (M,g) admits an affine connection ∇ which is related to the Levi-Civita connection ∇^* of the metric g by the formula

$$
\nabla_X Y = \nabla^*_X Y + \varphi(X)Y + \varphi(Y)X + g(X,Y)E, \tag{14}
$$

where φ is a 1-form and E a vector field on a M.

The following proposition is of our special interest in the next section.

PROPOSITION 3
Let ∇ be an affine connection on a pseudo-Riemannian manifold (M, g), which is related to the Levi-Civita connection ∇^* of g by the formula (14). Then for the curvature tensor fields R and R^* of ∇ and ∇^*, respectively, it holds

$$\begin{aligned} R^*(X,Y)Z \\ &= R(X,Y)Z - 2d\varphi(X,Y)Z - \varphi(E)(g(Y,Z)X - g(X,Z)Y) \\ &\quad + ((\nabla^*_Y\varphi)(Z) - \varphi(Y)\varphi(Z))X - ((\nabla^*_X\varphi)(Z) - \varphi(X)\varphi(Z))Y \\ &\quad - g(Y,Z)(\nabla^*_X E + g(X,E)E) + g(X,Z)(\nabla^*_Y E + g(Y,E)E). \end{aligned} \tag{15}$$

Proof. Let ∇^2 and ∇^{*2} denote the second covariant derivatives with respect to ∇ and ∇^*, respectively,

$$\nabla^2_{XY}Z = \nabla_X\nabla_Y Z - \nabla_{\nabla_X Y}Z, \qquad \nabla^{*2}_{XY}Z = \nabla^*_X\nabla^*_Y Z - \nabla^*_{\nabla^*_X Y}Z.$$

Then obviously

$$R(X,Y) = \nabla^2_{XY} - \nabla^2_{YX}, \qquad R^*(X,Y) = \nabla^{*2}_{XY} - \nabla^{*2}_{YX}\,. \tag{16}$$

At first, using (14), we find the following relation for the second covariant derivatives

$$\begin{aligned} \nabla^{*2}_{XY}Z &= \nabla^2_{XY}Z - (\nabla^*_X\varphi)(Y)Z - \varphi(E)g(Y,Z)E - (\nabla^*_X\varphi)(Z)Y \\ &\quad - \varphi(Y)\varphi(Z)X - g(Y,Z)(\nabla^*_X E + g(X,E)E) \\ &\quad + \mathrm{SP}(X,Y)Z, \end{aligned} \tag{17}$$

where $\mathrm{SP}(X,Y)Z$ indicates an expression which is symmetric with respect to X and Y. Next, we find (15), by applying (17), (16) and the following expression for the exterior derivative

$$d\varphi(X,Y) = \frac{1}{2}((\nabla^*_X\varphi)(Y) - (\nabla^*_Y\varphi)(X)).$$

Below, we discuss two typical geometric circumstances leading to (14).

A. Weyl connections ([2, 4, 11]). A Weyl structure on a differentiable manifold M is a conformal class of pseudo-Riemannian metrics $\mathfrak{C}$ together with a mapping $F\colon \mathfrak{C} \longrightarrow \Lambda^1(M)$ such that

$$F(e^\lambda g) = F(g) - d\lambda$$

for any $\lambda\colon M \longrightarrow \mathbb{R}$ and $g \in \mathfrak{C}$, $\Lambda^1(M)$ being the space of 1-forms on M. We say that an affine connection ∇ is compatible with the given Weyl structure $\mathfrak{C}$ on M if

$$\nabla g + F(g) \otimes g = 0 \qquad \text{for all } g \in \mathfrak{C}.$$

Given a Weyl structure $\mathfrak{C}$ on M, there exists a unique connection compatible with this structure, and this connection can be described in the following way

$$\nabla = \nabla^* + \varphi \otimes \mathrm{Id} + \mathrm{Id} \otimes \varphi - g \otimes \varphi^\sharp,$$

where g is a (pseudo-)Riemannian metric belonging to the conformal class, ∇^* is the Levi-Civita connection of g, $\varphi = F(g)/2$ and $\varphi^\sharp$ is the vector field related to the 1-form φ by $g(\cdot, \varphi^\sharp) = \varphi(\cdot)$.

Given a pseudo-Riemannian metric g, an affine connection ∇ and a 1-form φ satisfying the condition

$$\nabla g + 2\varphi \otimes g = 0 \tag{18}$$

on a manifold M, there is a Weyl structure on M for which ∇ is compatible. Namely it is sufficient to suppose $\mathfrak{C} = [g]$ ($\mathfrak{C}$ is the equivalence class of pseudo-Riemannian metrics conformal to g) and define $F\colon \mathfrak{C} \longrightarrow \Lambda^1(M)$ by $F(e^\lambda g) = 2\varphi - d\lambda$.

To be consistent with a certain geometrical tradition, an affine connection ∇ is called a Weyl connection for a pseudo-Riemannian metric g if there exists a 1-form φ such that the relation (18) is fulfilled. Of course, then ∇ is related to the Levi-Civita connection ∇^* of g by

$$\nabla_X Y = \nabla^*_X Y + \varphi(X)Y + \varphi(Y)X - g(X,Y)\varphi^\sharp,$$

so that we have (14) with $E = -\varphi^\sharp$.

B. Projectively related connections ([2, 10, 18], cf. also [16]). Let M be a differentiable manifold endowed with an affine connection ∇. A curve γ in M is called a ∇-pregeodesic (or a path with respect to ∇) if $\nabla_t \dot\gamma(t) = \sigma(t)\dot\gamma(t)$ for a function σ of the parameter t. Geometrically, this condition means that the tangent line field is parallel along γ. A ∇-pregeodesic γ can always be reparametrized so that $\nabla_s \dot\gamma(s) = 0$ with respect to the new parameter s. Two affine connections ∇ and ∇^* on M have the same paths if and only if there is a 1-form φ such that

$$\nabla_X Y = \nabla^*_X Y + \varphi(X)Y + \varphi(Y)X.$$

Clearly, if ∇^* is taken to be the Levi-Civita connection of a pseudo-Riemannian metric g on M, then we get (14) with $E = 0$.

4. Main result

THEOREM 4
Let $\widetilde{M}$ be a recurrent affine manifold with $\dim \widetilde{M} \geqslant 5$. Let M be a nondegenerate umbilical affine hypersurface in $\widetilde{M}$, whose shape operator A does not vanish at every point of M. Moreover, assume that the induced connection ∇ is related to the Levi-Civita connection ∇^* of h by the formula

$$\nabla_X Y = \nabla^*_X Y + \varphi(X)Y + \varphi(Y)X + h(X,Y)E, \tag{19}$$

where φ is a 1-form and E a vector field on M. Then the induced affine metric h is conformally flat.

Proof. Note that (19) is just of the form (14) with $g = h$, so we can apply Proposition 3. Using (13) and (15) with $g = h$, we conclude the following

$$\begin{aligned}\rho h(R^*(X,Y)Z,W) = \omega_0(X,Y)h(Z,W) &\\ + \alpha(h(Y,Z)h(X,W) - h(X,Z)h(Y,W)) &\\ + h(Y,Z)\omega_1(X,W) - h(X,Z)\omega_1(Y,W) &\\ + \omega_2(Y,Z)h(X,W) - \omega_2(X,Z)h(Y,W), &\end{aligned} \tag{20}$$

where α is the scalar function and ω_i's are the (0,2)-tensor fields defined by

$$\begin{aligned}\alpha &= \rho^2 - \rho\varphi(E),\\ \omega_0(X,Y) &= 2\rho(d\tau - d\varphi)(X,Y),\\ \omega_1(X,Y) &= -h\big(\rho h(X,E)E + \rho\nabla^*_X E + (\widetilde{R}(\xi,X)\xi)^\top, Y\big),\\ \omega_2(X,Y) &= \rho(\nabla^*_X\varphi)(Y) - \rho\varphi(X)\varphi(Y) + (\nabla_Y(\rho\tau - d\rho))(X)\\ &\quad - (\tau+\psi)(Y)(\rho\tau - d\rho)(X).\end{aligned}$$

The antisymmetrization of (20) with respect to Z and W gives

$$\begin{aligned}\rho h(R^*(X,Y)Z,W) = \alpha(h(Y,Z)h(X,W) - h(X,Z)h(Y,W)) &\\ + h(Y,Z)\omega(X,W) - h(X,Z)\omega(Y,W) &\\ + \omega(Y,Z)h(X,W) - \omega(X,Z)h(Y,W), &\end{aligned} \tag{21}$$

where

$$\omega = \frac{1}{2}(\omega_1 + \omega_2).$$

From (21), for the Ricci tensor S^* and the scalar curvature r^* of ∇^*, we find

$$\begin{aligned}\rho S^*(Y,Z) &= (n-2)\omega(Y,Z) + ((n-1)\alpha + \operatorname{Tr}_h(\omega))h(Y,Z),\\ \rho r^* &= 2(n-1)\operatorname{Tr}_h(\omega) + n(n-1)\alpha,\end{aligned}$$

where $\operatorname{Tr}_h(\omega)$ indicates the trace of the tensor ω with respect to the metric h. Next, from the last two equalities, one gets

$$\omega(Y,Z) = \frac{1}{n-2}\rho S^*(Y,Z) - \frac{1}{2}\left(\frac{1}{(n-1)(n-2)}\rho r^* + \alpha\right)h(Y,Z).$$

This applied to (21), gives

$$\rho\Big(h(R^*(X,Y)Z,W) - \frac{1}{n-2}\big(S^*(Y,Z)h(X,W) - S^*(X,Z)h(Y,W) + h(Y,Z)S^*(X,W) - h(X,Z)S^*(Y,W)\big) + \frac{r^*}{(n-1)(n-2)}\big(h(Y,Z)h(X,W) - h(X,Z)h(Y,W)\big)\Big) = 0,$$

that is, $\rho C^* = 0$, where C^* is the Weyl conformal curvature tensor of the metric h. This implies the assertion since $n = \dim M \geqslant 4$ and ρ is non-zero everywhere on M.

5. The case of pseudo-Riemannian hypersurfaces

Let $\widetilde{M}$ be a connected differentiable manifold, which is endowed with a pseudo-Riemannian metric $\widetilde{g}$. Denote by $\widetilde{\nabla}$ the Levi-Civita connection of the metric $\widetilde{g}$. Let us assume that M is a pseudo-Riemannian hypersurface of $\widetilde{M}$, that is, M is a submanifold of codimension 1 in $\widetilde{M}$, on which a pseudo-Riemannian metric g is induced by $g(X,Y) = \widetilde{g}(X,Y)$ for any vector fields X, Y on M. Then the induced connection ∇ on M is just the Levi-Civita connection of g.

As it follows from [12, Theorem and Corollary 3], if $\dim \widetilde{M} \geqslant 5$, $(\widetilde{M}, \widetilde{g})$ is of recurrent curvature (more generally, of recurrent Weyl conformal curvature) and M is totally umbilical and not-totally geodesic ($g = \rho h$, $\rho \neq 0$, h being the second fundamental form), then (M, g) must be conformally flat. It is obvious that in this case, the second fundamental form h must be conformally flat too (h becomes the affine metric when we treat the pseudo-Riemannian submanifold as the affine hypersurface).

Thus, we claim that our Theorem 4 is an extension of the above result to the case of umbilical affine hypersurfaces.

Another theorems about totally umbilical hypersurfaces in pseudo-Riemannian manifolds of recurrent curvature are presented in [3, 7, 17], and of Riemannian or pseudo-Riemannian (locally) symmetric spaces in [1, 13] and in many others papers.

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