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An integro-differential inequality related to the smallest positive eigenvalue of $p(x)$-Laplacian Dirichlet problem

Abstract. We consider the eigenvalue problem for the $p(x)$-Laplace-Beltrami operator on the unit sphere. We prove same integro-differential inequalities related to the smallest positive eigenvalue of this problem.

1. Introduction

In recent years there has been an increasing interest in the study of various mathematical problems with variable exponent (see for example [1, 11, 12, 14, 17]). Differential equations and variational problems with $p(x)$-growth conditions arise from the study of elastic mechanics, oscillation problem, electrorheological fluids or image restoration ([5, 6, 16]). The basic properties of variable exponent function spaces were derived by O. Kováčik and J. Rákosník in [13] and (by different methods) by X.-L. Fan and D. Zhao in [10]. For a comprehensive survey concerning Lebesgue and Sobolev spaces with variable exponent we refer to [7].

One of the most interesting topics is the $p(x)$-Laplacian Dirichlet problem. Our interest is in the smallest eigenvalue of this problem. X. Fan, Q. Zhang and D. Zhao in [9] studied the following eigenvalue problem

$$\begin{cases} -\operatorname{div}(|\nabla u|^{p(x)-2}\nabla u) = \lambda |u|^{p(x)-2}u & \text{in } D, \\ u = 0 & \text{on } \partial D, \end{cases}$$

where D is a bounded domain in $\mathbb{R}^n$, $p\colon \overline{D} \to (1,\infty)$ is a continuous function and λ is a real number. Denoting by Λ the set of all nonnegative eigenvalues, they

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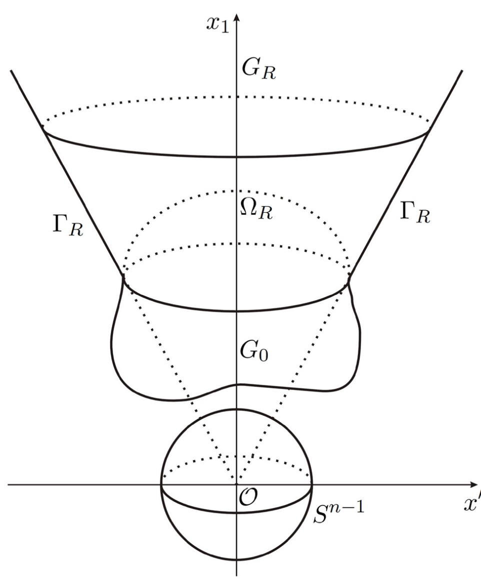
showed that $\sup\Lambda = +\infty$. They also pointed out that, in contrast with the case $p(x) = const$, only under special conditions we have that $\inf\Lambda > 0$. In the case of $p(x)$-Laplacian with Neumann boundary condition, unlike the p-Laplacian case, for very general variable exponent $p(x)$, the first eigenvalue is not isolated, that is the infimum of all positive eigenvalues of this problem is 0 (see [8]).

Our aim is to extend and develop the theory introduced in [3, 4] to the case of $p(x)$-Laplacian Dirichlet problem. In this article we prove some integro-differential inequalities related to the smallest positive eigenvalue of the eigenvalue problem for the $p(x)$-Laplace-Beltrami operator on the unit sphere. Such inequalities play very important role – they are necessary to investigate the behaviour of weak solutions of boundary value problems (Dirichlet, Neumann, Robin and mixed) for linear, weak quasilinear, and quasilinear elliptic divergence second order equations in cone-like domains (see [4, 18]) and domains with boundary singularities: angular, conic points or edges (see [2, 3]).

2. Preliminaries

Let $B_1(\mathcal{O})$ be the unit ball in $\mathbb{R}^n$, $n \geq 2$, with center at the origin $\mathcal{O}$ and $G \subset \mathbb{R}^n \setminus B_1(\mathcal{O})$ be an unbounded domain with the smooth boundary ∂G. We assume that $G = G_0 \cup G_R$, where G_0 is a bounded domain in $\mathbb{R}^n$, $G_R = \{x = (r,\omega) \in \mathbb{R}^n |\ r \in (R,\infty),\ \omega \in \Omega \subset S^{n-1},\ n \geq 2\}$, $R \gg 1$, S^{n-1} is the unit sphere. We also define a domain $\Omega_R = G \cap \{|x| = R\}$, $R \gg 1$.



an unbounded cone-like domain

Let us recall some well known formulae related to spherical coordinates:

$$dx = r^{n-1}drd\Omega, \qquad d\Omega_\varrho = \varrho^{n-1}d\Omega, \qquad |\nabla u|^2 = u_r^2 + \frac{1}{r^2}|\nabla_\omega u|^2; \tag{1}$$

$$\nabla_\omega u = \Big\{ \frac{1}{\sqrt{q_1}} \frac{\partial u}{\partial \omega_1}, \dots, \frac{1}{\sqrt{q_{n-1}}} \frac{\partial u}{\partial \omega_{n-1}} \Big\},$$
$$q_1 = 1,\ q_i = (\sin\omega_1 \cdot \ldots \cdot \sin\omega_{i-1})^2,\ i \geq 2;$$

$$\operatorname{div}_\omega u = \frac{1}{J(\omega)} \sum_{i=1}^{n-1} \frac{\partial}{\partial \omega_i} \Big(\frac{J(\omega)}{\sqrt{q_i}} u_i \Big), \quad J(\omega) = \sin^{n-2}\omega_1 \sin^{n-3}\omega_2 \cdot \ldots \cdot \sin\omega_{n-2}.$$

We define the variable exponent Lebesgue space $L^{p(x)}(G)$ as the set of measurable functions $u\colon G \to R$ such that $\int_G |u(x)|^{p(x)}\, dx < \infty$ with the Luxemburg norm

$$\|u\|_{L^{p(x)}(G)} = \inf \Big\{ \sigma > 0 \,|\, \int\limits_G \Big| \frac{u(x)}{\sigma} \Big|^{p(x)} dx \leq 1 \Big\}.$$

The variable exponent Sobolev space $W^{k,p(x)}(G)$ is defined as the set of functions $u \in L^{p(x)}(G)$ such that $D^\alpha u \in L^{p(x)}(G)$ for every multiindex α, $|\alpha| \leq k$, with the norm

$$\|u\|_{W^{k,p(x)}(G)} = \sum_{|\alpha| \leq k} \|D^\alpha u\|_{L^{p(x)}(G)}.$$

The space $W_0^{1,p(x)}(G)$ is defined by the closure of C_0^∞ in $W^{1,p(x)}(G)$.

2.1. The eigenvalue problem

We consider the eigenvalue problem for the $p(x)$-Laplace-Beltrami operator on the unit sphere

$$\begin{cases} \operatorname{div}_\omega(|\nabla_\omega \psi|^{p(\omega)-2} \nabla_\omega \psi) + \vartheta |\psi|^{p(\omega)-2}\psi = 0, & \omega \in \Omega; \\ \psi(\omega) = 0, & \omega \in \partial\Omega, \end{cases} \tag{$NEVP$}$$

which consists of the determination of all values ϑ (eigenvalues) for which $(NEVP)$ has weak solutions $\psi(\omega) \neq 0$ (eigenfunctions). Here $p(\omega) > 1$ and $p(\omega) \in C^0(\overline{\Omega})$.

Definition 2.1
A function ψ is said to be a weak solution of problem $(NEVP)$ provided that $\psi \in {W_0}^{1,p(\omega)}(\Omega)$ and satisfies the integral identity

$$\int\limits_\Omega \Big(|\nabla_\omega \psi|^{p(\omega)-2} \frac{1}{q_i} \frac{\partial \psi}{\partial \omega_i} \frac{\partial \eta}{\partial \omega_i} - \vartheta |\psi|^{p(\omega)-2} \psi \eta \Big)\, d\Omega = 0$$

for all $\eta(\omega) \in W_0^{1,p(\omega)}(\Omega)$.

Throughout the paper we need only the smallest positive eigenvalue

$$\vartheta_* := \inf_{\psi \in W_0^{1,p(\omega)} \setminus \{0\}} \frac{\int_\Omega |\nabla_\omega \psi|^{p(\omega)}\, d\Omega}{\int_\Omega |\psi|^{p(\omega)}\, d\Omega}.$$

By [9, 15] ϑ_* exists only under some special conditions:

Theorem 2.2 (see Theorem 3.2 [9])
If $n = 2$, then $\vartheta_* > 0$ if and only if the function $p(\omega)$ is monotonic.

Theorem 2.3 (see Theorem 3.3 [9])
Suppose $n > 2$. If there is a vector $l \in R^{n-1} \setminus \{0\}$, such that for any $\omega \in \Omega$, $f(t) = p(\omega + tl)$ is monotonic for $t \in I_\omega = \{t \mid \omega + tl \in \Omega\}$, then $\vartheta_* > 0$.

Theorem 2.4 (see Remark 1 [15])
Let $\vec{a}\colon \Omega \to R^{n-1}$. Suppose that there exists a constant $a_0 > 0$, such that

$$\operatorname{div}_\omega \vec{a}(\omega) \geq a_0 > 0, \quad \forall \omega \in \overline{\Omega}.$$

Let $p\colon \overline{\Omega} \to (1, n-1)$ be a function of class C^1 satisfying

$$\vec{a}(\omega) \cdot \nabla_\omega p(\omega) = 0, \quad \forall \omega \in \overline{\Omega}.$$

Then $\vartheta_* > 0$.

From the definition of ϑ_* we obtain the following Friedrichs-Wirtinger type inequality:

Theorem 2.5
Let assumptions either of theorems 2.2–2.4 be satisfied, $\psi \in W_0^{1,p(\omega)}(\Omega)$, $\Omega \subset S^{n-1}$. Then

$$\int\limits_\Omega |\psi|^{p(\omega)}\, d\Omega \leq \frac{1}{\vartheta_*} \int\limits_\Omega |\nabla_\omega \psi|^{p(\omega)}\, d\Omega \tag{$(W)_{p(\omega)}$}$$

with the sharp constant $\frac{1}{\vartheta_*}$.

2.2. Some algebraic inequalities

Let us recall some elementary inequalities which will be used in the next chapter.

Lemma 2.6 (Cauchy's inequality)
For any $a, b \in \mathbb{R}$ and $\varepsilon > 0$, we have

$$ab \leq \frac{\varepsilon}{2} a^2 + \frac{1}{2\varepsilon} b^2.$$

Lemma 2.7 (Young's inequality)
For any $a, b \geq 0$, $\varepsilon > 0$ and $q, q' > 1$ with $\frac{1}{q} + \frac{1}{q'} = 1$, we have

$$ab \leq \frac{1}{q} \varepsilon a^q + \frac{1}{q'} b^{q'} \varepsilon^{-\frac{q'}{q}}.$$

Lemma 2.8 (Jensen's inequality)
Let a_i, $i = 1, \ldots, n$, be any nonnegative real numbers and $q > 0$. Then

$$\min(1, n^{q-1}) \cdot \sum_{i=1}^n a_i^q \leq \left(\sum_{i=1}^n a_i \right)^q \leq \max(1, n^{q-1}) \cdot \sum_{i=1}^n a_i^q.$$

3. The main result

Theorem 3.1
Let G_R be an unbounded conical domain, $v(\varrho,\omega) \in W_0^{1,p(r,\omega)}(\Omega)$ for almost all $\varrho > R \gg 1$ and

$$V(\varrho) = \int\limits_{G_\varrho} |\nabla v|^{p(x)}\, dx < \infty,$$

where $1 < \inf_{x\in G} p(x) = p_- \le p(x) \le p_+ = \sup_{x\in G} p(x) < \infty$. Let ϑ_* be the smallest positive eigenvalue of problem $(NEVP)$. Then for almost all $\varrho > R \gg 1$

$$\int\limits_{\Omega_\varrho} v\frac{\partial v}{\partial r}|\nabla v|^{p(r,\omega)-2}\, d\Omega_\varrho \ge -\Xi(\vartheta_*,\varrho)V'(\varrho), \tag{2}$$

where

$$\Xi(\vartheta_*,\varrho) = \begin{cases} \varrho^{p_+-1}\cdot \begin{cases} \left(\frac{p_+}{p_++2}\right)^{\frac{p_++2}{2p_+}} \vartheta_*^{-\frac{1}{p_+}} & \text{if } \vartheta_* \ge \frac{p_+}{p_++2}, \\ \frac{1}{2\sqrt{\vartheta_*}} + \frac{p_+-2}{2p_+}\sqrt{\vartheta_*} & \text{if } 0 < \vartheta_* \le \frac{p_+}{p_++2} \end{cases} & \text{if } 2 \le p(x) \le p_+, \\ \varrho^{3-p_-}\cdot \frac{2^{\frac{2-p_-}{2}}}{p_-}\cdot \begin{cases} \vartheta_*^{-\frac{1}{p_-}} & \text{if } \vartheta_* \le 1, \\ \vartheta_*^{-\frac{1}{2}} & \text{if} \vartheta_* \ge 1 \end{cases} & \text{if } p_- \le p(x) \le 2. \end{cases}$$

Proof. First of all we observe that writing the function $V(\varrho)$ in spherical coordinates

$$V(\varrho) = \int\limits_{\varrho}^{\infty} r^{n-1}\left(\int\limits_{\Omega} |\nabla v(r,\omega)|^{p(r,\omega)}\, d\Omega\right) dr$$

and differentiating it with respect to ϱ we obtain

$$V'(\varrho) = -\varrho^{n-1}\int\limits_{\Omega} |\nabla v(\varrho,\omega)|^{p(\varrho,\omega)}\, d\Omega. \tag{3}$$

We need to consider two possible cases.

Case 1: $2 < p(x) \le p_+$.
Using the Cauchy and next the Young inequality with $q = \frac{p(r,\omega)}{2}$ and $q' = \frac{p(r,\omega)}{p(r,\omega)-2}$ we obtain for all $\varepsilon, \delta > 0$

$$\begin{aligned} &\int\limits_{\Omega_\varrho} v\frac{\partial v}{\partial r}|\nabla v|^{p(r,\omega)-2} d\,\Omega_\varrho \\ &\quad = \varrho^n \int\limits_{\Omega} \frac{v}{\varrho}\cdot\frac{\partial v}{\partial r}|\nabla v|^{p(r,\omega)-2}\Big|_{r=\varrho} d\Omega \\ &\quad \ge -\varrho^n \int\limits_{\Omega} \left\{\frac{\varepsilon}{2}\left(\frac{v}{\varrho}\right)^2 + \frac{1}{2\varepsilon}\left(\frac{\partial v}{\partial r}\right)^2\right\}|\nabla v|^{p(r,\omega)-2}\Big|_{r=\varrho} d\Omega \end{aligned}$$

$$\geq -\varrho^n \int\limits_{\Omega} \Big\{ \frac{\varepsilon\delta}{p(r,\omega)} \Big|\frac{v}{\varrho}\Big|^{p(r,\omega)} + \frac{p(r,\omega)-2}{p(r,\omega)} \cdot \frac{\varepsilon}{2} \delta^{\frac{2}{2-p(r,\omega)}} |\nabla v|^{p(r,\omega)} + \frac{1}{2\varepsilon} \Big|\frac{\partial v}{\partial r}\Big|^2 |\nabla v|^{p(r,\omega)-2} \Big\}\Big|_{r=\varrho} d\Omega.$$

Now, because $\varrho \gg 1$, in our case we get that $-\frac{\varrho^{-p(r,\omega)}}{p(r,\omega)} \geq -\frac{\varrho^{-2}}{2}$. Hence, applying the Friedrichs-Wirtinger type inequality $(W)_{p(\omega)}$ for the function v, we see that

$$\begin{aligned} &\int\limits_{\Omega_\varrho} v \frac{\partial v}{\partial r} |\nabla v|^{p(r,\omega)-2} \, d\Omega_\varrho \\ &\quad \geq -\frac{1}{2} \varrho^n \int\limits_{\Omega} \Big\{ \frac{\varepsilon\delta}{\vartheta_*} \varrho^{-2} |\nabla_\omega v|^{p(r,\omega)} + \frac{p(r,\omega)-2}{p(r,\omega)} \cdot \varepsilon \delta^{\frac{2}{2-p(r,\omega)}} |\nabla v|^{p(r,\omega)} \\ &\qquad + \frac{1}{\varepsilon} \Big|\frac{\partial v}{\partial r}\Big|^2 |\nabla v|^{p(r,\omega)-2} \Big\}\Big|_{r=\varrho} d\Omega. \end{aligned} \tag{4}$$

In virtue of (1) we have $|\nabla_\omega v| \leq \varrho |\nabla v|$. Hence we get

$$\varrho^{-2} |\nabla_\omega v|^{p(r,\omega)} = \Big|\frac{\nabla_\omega v}{\varrho}\Big|^2 \cdot |\nabla_\omega v|^{p(r,\omega)-2} \leq \Big|\frac{\nabla_\omega v}{\varrho}\Big|^2 \varrho^{p(r,\omega)-2} |\nabla v|^{p(r,\omega)-2}.$$

Therefore, by the formula (1) and because $2 < p(r,\omega) \leq p_+$,

$$\begin{aligned} &-\int\limits_{\Omega} \Big\{ \frac{\varepsilon\delta}{\vartheta_*} \varrho^{-2} |\nabla_\omega v|^{p(r,\omega)} + \frac{1}{\varepsilon} \Big|\frac{\partial v}{\partial r}\Big|^2 |\nabla v|^{p(r,\omega)-2} \Big\}\Big|_{r=\varrho} d\Omega \\ &\qquad \geq -\frac{1}{\varepsilon} \int\limits_{\Omega} |\nabla v|^{p(r,\omega)-2} \Big\{ \varrho^{p_+-2} \Big|\frac{\nabla_\omega v}{\varrho}\Big|^2 + v_r^2 \Big\}\Big|_{r=\varrho} d\Omega \\ &\qquad \geq -\frac{1}{\varepsilon} \varrho^{p_+-2} \int\limits_{\Omega} |\nabla v|^{p(r,\omega)} \Big|_{r=\varrho} d\Omega, \end{aligned}$$

choosing $\varepsilon > 0$ from the equality

$$\frac{\varepsilon\delta}{\vartheta_*} = \frac{1}{\varepsilon}. \tag{5}$$

Therefore from (4) it follows that

$$\begin{aligned} &\int\limits_{\Omega_\varrho} v \frac{\partial v}{\partial r} |\nabla v|^{p(r,\omega)-2} \, d\Omega_\varrho \\ &\qquad \geq -\frac{1}{2} \varrho^{n+p_+-2} \int\limits_{\Omega} \Big(\frac{1}{\varepsilon} + \frac{p(r,\omega)-2}{p(r,\omega)} \varepsilon \delta^{\frac{2}{2-p(r,\omega)}} \Big) |\nabla v|^{p(r,\omega)} \Big|_{r=\varrho} d\Omega. \end{aligned} \tag{6}$$

Let us choose $0 < \varepsilon \leq \sqrt{\vartheta_*}$. Then, by (5)

$$\delta^{\frac{2}{2-p(r,\omega)}} = \Big(\frac{\varepsilon^2}{\vartheta_*}\Big)^{\frac{2}{p(r,\omega)-2}} \leq \Big(\frac{\varepsilon^2}{\vartheta_*}\Big)^{\frac{2}{p_+-2}}.$$

Hence, we can rewrite (6) in the following form

$$\int\limits_{\Omega_\varrho} v\frac{\partial v}{\partial r}|\nabla v|^{p(r,\omega)-2}\,d\Omega_\varrho \geq -\frac{1}{2}\varrho^{n+p_+-2}w(\varepsilon)\int\limits_{\Omega}|\nabla v(\varrho,\omega)|^{p(\varrho,\omega)}\,d\Omega, \tag{7}$$

where $w(\varepsilon)=\frac{1}{\varepsilon}+(1-\frac{2}{p_+})\varepsilon(\frac{\varepsilon^2}{\vartheta_*})^{\frac{2}{p_+-2}}$. Now our aim is to obtain the best estimate for the integral on the left hand side of the above inequality. In order to do that we find $\min_{\varepsilon\in(0,\sqrt{\vartheta_*})} w(\varepsilon)$. Direct calculations give

$$w'(\varepsilon)=-\frac{1}{\varepsilon^2}+\frac{p_++2}{p_+}\Big(\frac{\varepsilon^2}{\vartheta_*}\Big)^{\frac{2}{p_+-2}},\qquad w''(\varepsilon)=\frac{2}{\varepsilon^3}+\frac{4(p_++2)}{p_+(p_+-2)}\frac{\varepsilon}{\vartheta_*}\Big(\frac{\varepsilon^2}{\vartheta_*}\Big)^{\frac{4-p_+}{p_+-2}}.$$

Hence, $w'(\varepsilon)=0$ only for $\varepsilon_0=(\frac{p_+}{p_++2})^{\frac{p_+-2}{2p_+}}\vartheta_*^{\frac{1}{p_+}}$ and $w''(\varepsilon_0)>0$.

If $\varepsilon_0\leq\sqrt{\vartheta_*}$, then $\vartheta_*\geq\frac{p_+}{p_++2}$. In this case

$$\min_{\varepsilon\in(0,\sqrt{\vartheta_*})} w(\varepsilon)=w(\varepsilon_0)=2\Big(\frac{p_+}{p_++2}\Big)^{\frac{p_++2}{2p_+}}\vartheta_*^{-\frac{1}{p_+}}.$$

On the other hand, we find that $w'(\varepsilon)<0$ for $0<\varepsilon<\varepsilon_0$.

If $\sqrt{\vartheta_*}\leq\varepsilon_0$, i.e. $0<\vartheta_*<\frac{p_+}{p_++2}$, then

$$\min_{\varepsilon\in(0,\sqrt{\vartheta_*})} w(\varepsilon)=w\big(\sqrt{\vartheta_*}\big)=\frac{1}{\sqrt{\vartheta_*}}+\frac{p_+-2}{p_+}\sqrt{\vartheta_*}.$$

Thus, from (7), above arguments and (3), we derive the required estimation (2) for $2<p(x)\leq p_+$.

Case 2: *$1<p_-\leq p(x)\leq 2$.*
We conclude from $|\nabla v|\geq|v_r|$ that $-|\nabla v|^{p(\varrho,\omega)-2}\geq-|v_r|^{p(\varrho,\omega)-2}$. Therefore, using the Young inequality with $q=p(\varrho,\omega)$ and $q'=\frac{p(\varrho,\omega)}{p(\varrho,\omega)-1}$, we have

$$\begin{aligned}
&\int\limits_{\Omega_\varrho} v\frac{\partial v}{\partial r}|\nabla v|^{p(\varrho,\omega)-2}\,d\Omega_\varrho\\
&\quad\geq-\varrho^n\int\limits_{\Omega}\Big|\frac{v}{\varrho}\Big||v_r|^{p(\varrho,\omega)-1}\Big|_{r=\varrho}d\Omega\\
&\quad\geq-\varrho^n\int\limits_{\Omega}\Big\{\frac{\varepsilon}{p(r,\omega)}\Big|\frac{v}{\varrho}\Big|^{p(\varrho,\omega)}+\frac{p(\varrho,\omega)-1}{p(\varrho,\omega)}\varepsilon^{-\frac{1}{p(\varrho,\omega)-1}}|v_r|^{p(\varrho,\omega)}\Big\}\Big|_{r=\varrho}d\Omega,\quad\forall\varepsilon>0.
\end{aligned}$$

Next, because $\varrho\gg 1$, we have $-\frac{\varrho^{-p(\varrho,\omega)}}{p(\varrho,\omega)}\geq-\frac{\varrho^{-p_-}}{p_-}$. Therefore, applying the Friedrichs-Wirtinger type inequality $(W)_{p(\omega)}$ for the function v and noting that $-\frac{p(\varrho,\omega)-1}{p(\varrho,\omega)}\geq-\frac{1}{p(\varrho,\omega)}\geq\frac{1}{p_-}$, we obtain

$$\int\limits_{\Omega_\varrho} v\frac{\partial v}{\partial r}|\nabla v|^{p(\varrho,\omega)-2}\,d\Omega_\varrho$$

$$\geq -\frac{\varrho^n}{p_-}\int\limits_\Omega \left\{\frac{\varepsilon}{\vartheta_*}\varrho^{-p_-}|\nabla_\omega v|^{p(\varrho,\omega)} + \varepsilon^{-\frac{1}{p(\varrho,\omega)-1}}|v_r|^{p(\varrho,\omega)}\right\}\Big|_{r=\varrho} d\Omega.$$

Now we can observe that

$$\varrho^{-p_-}|\nabla_\omega v|^{p(\varrho,\omega)} = \varrho^{p(\varrho,\omega)-p_-}\left|\frac{\nabla_\omega v}{\varrho}\right|^{p(\varrho,\omega)} \leq \varrho^{2-p_-}\left|\frac{\nabla_\omega v}{\varrho}\right|^{p(\varrho,\omega)}.$$

Hence we obtain

$$\begin{aligned}\int\limits_{\Omega_\varrho} v\frac{\partial v}{\partial r}|\nabla v|^{p(\varrho,\omega)-2}\, d\Omega_\varrho \\ \geq -\frac{\varrho^{n-p_-+2}}{p_-}\int\limits_\Omega \left[\frac{\varepsilon}{\vartheta_*}\left|\frac{\nabla_\omega v}{\varrho}\right|^{p(\varrho,\omega)} + \varepsilon^{-\frac{1}{p(\varrho,\omega)-1}}|v_\varrho|^{p(\varrho,\omega)}\right] d\Omega.\end{aligned} \tag{8}$$

Now we choose $\frac{\varepsilon}{\vartheta_*} = \varepsilon^{-\frac{1}{p(\varrho,\omega)-1}}$ and therefore inequality (8) gives

$$\begin{aligned}&\int\limits_{\Omega_\varrho} v\frac{\partial v}{\partial r}|\nabla v|^{p(\varrho,\omega)-2}\, d\Omega_\varrho \\ &\geq -\frac{\varrho^{n-p_-+2}}{p_-}\int\limits_\Omega \vartheta_*^{-\frac{1}{p(\varrho,\omega)}}\left(\left|\frac{\nabla_\omega v}{\varrho}\right|^{p(\varrho,\omega)} + |v_\varrho|^{p(\varrho,\omega)}\right) d\Omega \\ &\geq -\frac{\varrho^{n-p_-+2}}{p_-}\int\limits_\Omega \left(\left|\frac{\nabla_\omega v}{\varrho}\right|^{p(\varrho,\omega)} + |v_\varrho|^{p(\varrho,\omega)}\right) d\Omega \cdot \begin{cases}\vartheta_*^{-\frac{1}{p_-}} & \text{if } \vartheta_* \leq 1,\\ \vartheta_*^{-\frac{1}{2}} & \text{if } \vartheta_* \geq 1.\end{cases}\end{aligned}$$

But, by (1), using the Jensen inequality, we can conclude that

$$\left|\frac{\nabla_\omega v}{\varrho}\right|^{p(\varrho,\omega)} + |v_\varrho|^{p(\varrho,\omega)} = \left(\left|\frac{\nabla_\omega v}{\varrho}\right|^2\right)^{\frac{p(\varrho,\omega)}{2}} + (|v_\varrho|^2)^{\frac{p(\varrho,\omega)}{2}} \leq 2^{\frac{2-p_-}{2}}|\nabla v(\varrho,\omega)|^{p(\varrho,\omega)}.$$

Hence, regarding (3) we obtain the desired estimate.

Remark 3.2
In [4] we can found an analogous inequality which is related to the smallest positive eigenvalue of eigenvalue problem for the m-Laplace-Beltrami operator on the unit sphere, mainly

$$\begin{cases}\operatorname{div}_\omega(|\nabla_\omega\psi|^{m-2}\nabla_\omega\psi) + \vartheta|\psi|^{m-2}\psi = 0, & \omega\in\Omega;\\ \alpha(\omega)|\nabla_\omega\psi|^{m-2}\frac{\partial\psi}{\partial\vec{\nu}} + \gamma(\omega)|\psi|^{m-2}\psi(\omega) = 0, & \omega\in\partial\Omega,\end{cases} \tag{EVP}$$

where

$$\alpha(\omega) = \begin{cases}0, & \text{if } \omega\in\partial_{\mathcal{D}}\Omega;\\ 1, & \text{if } \omega\in\partial\Omega\setminus\partial_{\mathcal{D}}\Omega,\end{cases}$$

$\partial_{\mathcal{D}}\Omega \subseteq \partial\Omega$ is the part of the boundary $\partial\Omega$ for which we consider the Dirichlet boundary condition; $\gamma(\omega)$ is a positive bounded piecewise smooth function on $\partial\Omega$ such that $\gamma(\omega) \geq \gamma_0 > 0$ and $m > 1$.

There was obtained the following theorem.

Theorem 3.3
Let G_R be an unbounded conical domain, $\Gamma_R = \{(r,\omega): \ r > R, \omega \in \partial\Omega\} \cap \partial G$. Let $m > 1$, $v(\varrho,\cdot) \in W^{1,m}(\Omega)$ for almost all $\varrho > R \gg 1$ and

$$V(\varrho) = \int\limits_{G_\varrho} |\nabla v|^m \, dx + \int\limits_{\Gamma_\varrho} \alpha(x) \frac{\gamma(\omega)}{r^{m-1}} |v|^m \, ds < \infty.$$

Let $\vartheta_*(m)$ be the smallest positive eigenvalue of problem (EVP). Then for almost all $\varrho > R \gg 1$

$$\int\limits_{\Omega_\varrho} v \frac{\partial v}{\partial r} |\nabla v|^{m-2} \, d\Omega_\varrho \geq \Xi(m) \cdot \frac{\varrho}{m\vartheta_*^{\frac{1}{m}}} V'(\varrho), \tag{9}$$

where

$$\Xi(m) = \begin{cases} m\sqrt{\frac{m}{2}} \cdot \left(\frac{2}{m+2}\right)^{\frac{m+2}{2m}}, & m \geq 2; \\ (m-1)^{\frac{m-1}{m}} \cdot 2^{\frac{2-m}{2}}, & 1 < m \leq 2. \end{cases}$$

One can easily see that taking in (EVP) $\alpha(\omega) = 0$ and $m = 2$ we obtain $(NEVP)$ problem. Then inequality (2) for special case $p(x) = p_- = p_+ = 2$ coincide with inequality (9).

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