DE G DE GRUYTER OPEN

Ann. Univ. Paedagog. Crac. Stud. Math. 16 (2017), 51-59
DOI: 10.1515/aupcsm-2017-0004

FOLIA 206

Annales Universitatis Paedagogicae Cracoviensis
Studia Mathematica XVI (2017)

Joanna Jureczko

Strong sequences and partition relations

Communicated by Tomasz Szemberg

Abstract. The first result in partition relations topic belongs to Ramsey (1930). Since that this topic has been still explored. Probably the most famous partition theorem is Erdös-Rado theorem (1956). On the other hand in 60's of the last century Efimov introduced strong sequences method, which was used for proving some famous theorems in dyadic spaces. The aim of this paper is to generalize theorem on strong sequences and to show that it is equivalent to generalized version of well-known Erdös-Rado theorem. It will be also shown that this equivalence holds for singulars. Some applications and conclusions will be presented too.

1. Introduction

Let α, β and λ be cardinals and $n < \omega$. The *arrow notation*

$$(\alpha) \to (\beta)^n_\lambda$$

denotes the following partition relation: if $[\alpha]^n = \bigcup_{i<\alpha} P_i$, then there are $A \subset \alpha$ and $i < \lambda$ such that $|A| = \beta$ and $[A]^n \subset P_i$, (see e.g. [2]).

Partition calculus has been considered by many researchers. Below there are presented some famous results in this topic of partition relations, (followed by [1]). The first result in this topic is given by Ramsey (1930) who proved that

$$U \to (U)^2_2$$

for non-principal ultrafiltres on ω. Such ultrafiltres were called "Ramsey" by Galvin around 1968. Other results belong to Rowbottom (1971) and Ketonen (1973).

AMS (2010) Subject Classification: 03E02, 03E05, 03E20, 54A25.
Keywords and phrases: Strong sequences, Erdös-Rado theorem, partition relations, inaccessible cardinals, singular cardinals.

There are results for ultrafiltres on uncountable measurable cardinals. Moreover, Sierpiński (1933) proved that

$$2^{\aleph_0} \not\to (\aleph_1)^2_2.$$

Probably the most famous theorem in partition calculus belongs to Erdös-Rado (1956) who proved that for every n - natural number and β - cardinal there is

$$\exp_n(\beta)^+ \to (\beta^+)^{n+1}_\beta.$$

The existence of different kinds of partition relations is still explored.

On the other hand a combinatorial method, so called strong sequences method, was introduced by Efimov in the 60s' of the last century in the subbase of the Cantor cube, which was used for proving some well known theorems in dyadic spaces, i.e., continuous images of the Cantor cube, see [3].

The problem of the existence and consequences of the existence of strong sequences in other spaces was explored in 90's of the last century, see [8] and [9] for some historical details.

The main goal of this paper is to show that the generalized versions on strong sequences, see [10], and Erdös-Rado theorem, see [5], are equivalent for regular and singular cardinals. There will be also shown some exceptions for which this equivalence does not hold.

2. Notations and definitions

Let (X, r) be a set with relation r and let κ be a cardinal. Let $a, b, c \in X$. We say that a and b have *a bound* iff there exists c such that

$$(a, c) \in r \quad \text{and} \quad (b, c) \in r.$$

We say that a set $A \subset X$ is a *κ-directed* iff every subset of A of cardinality less than κ has a bound.

Definition 2.1 ([10])
A sequence $(H_\phi)_{\phi<\alpha}$, $H_\phi \subset X$ is called a κ-strong sequence if

(1) H_ϕ is κ-directed for all $\phi < \alpha$,

(2) $H_\psi \cup H_\phi$ is not κ-directed for all $\phi < \psi < \alpha$.

Notice that in Definition 2.1 we do not really think about a sequence but rather about the family of sets with properties (1) and (2), so we can index $(H_\phi)_{\phi<\alpha}$ not by ordinal but by cardinal numbers.

Definition 2.2 ([13])
A pair (F, G) of functions, where Φ is an ordinal.

$$G\colon \Phi \to\ 2^X \quad \text{and} \quad F\colon X \to 2^\Phi$$

with the property that for any α, β if $\alpha < \beta$, then there exists $a \in G(\alpha)$ such that $\beta \in F(a)$ is called twin functions.

Definition 2.3 ([13])
A map $g\colon K \to X$, where $K \subset \Phi$ such that

(1) $g(\alpha) \in G(\alpha)$ for any $\alpha \in K$,

(2) for any $\alpha, \beta \in K$ if $\alpha < \beta$, then $\beta \in F(g(\beta))$

is called a selector of twin functions (F, G).

All other notations are standard in the field and can find in [2, 4, 7].

3. Theorems

In this part we present generalizations of three theorems: on strong sequences (compare [10]), on twin functions (compare [13]) and Erdös-Rado theorem (compare [5, 2]). Theorem 3.1 is proved in [10], Theorem 3.2 is a generalized version of Theorem on twin functions in [13]. Theorem 3.3 has been proved in [2], pp.8-11.

Let β and η be cardinals. By $\beta \ll \eta$ we denote: η is *β-strong inaccessible*, i.e., $\beta < \eta$ and $\alpha^\lambda < \eta$, whenever $\alpha < \beta$, $\lambda < \eta$.

Theorem 3.1 (on κ-strong sequences)
Let β, μ, η be cardinals such that $\omega \leq \beta \ll \eta$, $\mu < \beta$ and β, η be regular. Let X be a set of cardinality η. If there exists a κ-strong sequences $\{H_\alpha \subset X : \alpha < \eta\}$ with $|H_\alpha| \leq 2^\mu$ for all $\alpha < \eta$, then there exists a κ-strong sequence $\{T_\alpha \subset X : \alpha < \beta\}$ with $|T_\alpha| < \kappa$ for all $\alpha < \beta$ and $\kappa < \eta$.

Theorem 3.2 (on twin functions)
Let β, μ, η be cardinals such that $\omega \leq \beta \ll \eta$, $\mu < \beta$ and β, η be regular. Let X be a set of cardinality η. If there exists a pair (F, G) of twin functions $G\colon \eta \to 2^X$, $F\colon X \to 2^\eta$ such that $|G(\alpha)| \leq 2^\mu$ for all $\alpha < \eta$, then there exists a selector $g\colon K \to X$ of twin functions with $|K| = \beta$.

Theorem 3.3 (Erdös-Rado type)
Let β, μ, η be cardinals such that $\omega \leq \beta \ll \eta$, $\mu < \beta$ and β, η be regular. Let X be a set of cardinality η. Then

$$(\eta) \to (\beta)^2_\mu.$$

4. Proofs

In this part all proofs needed for showing equivalence of theorems from previous part will be presented. We introduce the following notation

SS means theorem on strong sequence,
TF means theorem on twin functions,
ER means Erdös-Rado type theorem.

The structure of this section is as follows

$$\begin{array}{ccc} SS & \Longrightarrow & TF \\ \Uparrow & & \Downarrow \\ TF & \Longleftarrow & ER \end{array}$$

Proof. $SS \Longrightarrow TF$
Take a pair (F, G) of twin functions such that $|G(\alpha)| \leq 2^\mu$. For any $\alpha < \eta$ consider

$$H_\alpha = G(\alpha) \setminus F^{-1}(\alpha).$$

Notice that $\{H_\alpha : \ \alpha < \eta\}$ is a κ-strong sequence for $\kappa < \eta$. If not, then by Definition 2.1 the set $H_\alpha \cup H_\beta$ would be an κ-directed set for $\alpha < \beta$. By Definition 2.2 there exists $a \in H_\alpha$ such that $\beta \in F(a)$. Hence $a \in F^{-1}(\beta)$, which contradicts to definition of H_β.

By Theorem 3.1 there exists a κ-strong sequence $\{T_\alpha : \ \alpha < \beta\}$ such that $|T_\alpha| < \kappa$ for any $\alpha < \beta$. Consider a function $g\colon \beta \to X$ such that $g(\alpha) \in T_\alpha$ for $\alpha < \beta$. Obviously $g(\alpha) \in H_\alpha \subset G(\alpha)$, hence condition (1) of Definition 2.3 is fulfilled. Let $a \in g(\alpha) \in G(\alpha)$. By Definition 2.3 we have $\gamma \in F(a) = F(g(\alpha))$ for $\alpha < \gamma$. Hence condition (2) of Definition 2.3 of twin functions is fulfilled.

Proof. $TF \Longrightarrow ER$
Without the loss of generality we can assume that $X = \{x_\alpha : \ \alpha < \eta\}$. Let

$$f\colon [X]^2 \to \lambda$$

be a function which determines a partition

$$[X]^2 = \{A_\gamma : \ \gamma < \lambda\}.$$

We will show that for some A_γ of cardinality β there is $f|A_\gamma = const$. Define functions

$G\colon \eta \to 2^X$ such that

$$G(\alpha) = \{x_\alpha \in X : \ \{x_\alpha, x_\xi\} \in A_\gamma \text{ for some } \gamma < \lambda \text{ and any } \alpha < \xi\},$$

$F\colon X \to 2^\eta$ such that

$$F(x_\alpha) = \{\xi < \kappa : \ \{x_\alpha, x_\xi\} \in A_\gamma \text{ for some } \gamma < \lambda \text{ and } \alpha < \xi\}.$$

We will show that functions in a pair (F, G) are twin functions.

Take $\alpha < \xi$. Let $x_\alpha \in G(\alpha)$, then $\{x_\alpha, x_\xi\} \in A_\gamma$ for some $\alpha < \lambda$. Hence $\xi \in F(x_\alpha)$. By Theorem 3.2 there exists a selector of twin functions $g\colon K \to X$ with $K \subset \kappa$ such that $|K| = \beta$. This means that for each $\alpha, \xi \in K$ we have $\{x_\alpha, x_\xi\} \in A_\gamma$ for some $\alpha < \xi$. Suppose that $[K]^2 \not\subset A_\gamma$ for any $\gamma < \lambda$. Let

$$\{x_\alpha, x_\xi\} \in A_\rho \qquad \text{for } \alpha < \xi$$

and

$$\{x_\gamma, x_\delta\} \in A_\sigma \qquad \text{for } \gamma < \delta$$

and $\rho \neq \sigma$. By Definition 2.3 we have respectively

$$g(\alpha) = x_\alpha \quad \text{and hence} \quad \xi \in F(g(\alpha))$$

and

$$g(\gamma) = x_\gamma \quad \text{and hence} \quad \delta \in F(g(\gamma)).$$

If $\xi < \delta$, then $\xi \in F(g(\alpha))$ and then $\{x_\alpha, x_\xi\}, \{x_\gamma, x_\delta\} \in A_\rho$. If $\delta < \xi$, then $\xi \in F(g(\gamma))$ and then $\{x_\alpha, x_\xi\}, \{x_\gamma, x_\delta\} \in A_\sigma$. A contradiction. Hence $[K]^2 \subset A_\gamma$ for some $\gamma < \lambda$.

Proof. $ER \Longrightarrow TF$
Without the loss of generality we can assume that $X = \{x_\alpha : \ \alpha < \eta\}$. Assume that (F, G) is a pair $G \colon \eta \to 2^X$, $F \colon X \to 2^\eta$ of twin function. We can assume that for any $\beta < \eta$

$$G(\xi) = \{x_\gamma(\xi) : \ \gamma < \lambda\}.$$

Let $P \colon [X]^2 \to \lambda$ be a function such that

$$\text{if } P(\{x_\alpha, x_\xi\}) = \gamma, \qquad \text{then } \alpha \in F(x_\gamma(\xi)).$$

By Theorem 3.3 there exists a set $K \subset X$ of cardinality β such that $[K]^2 \subset P^{-1}(\gamma)$ for some $\gamma \leq \lambda$. By above information, $\alpha \in F(x_\gamma(\xi))$ for all $\alpha, \xi \in K$ with $\alpha < \xi$. Then $g(\xi) = x_\gamma(\xi)$ for $\xi \in K$ is a selector of twin functions.

Proof. $TF \Longrightarrow SS$
Assume that there exists a κ-strong sequence $\{H_\alpha \subset X : \ \alpha < \eta\}$ for $\kappa < \eta$ such that $|H_\alpha| \leq 2^\mu$. Consider the following set

$$A_\alpha = \{T \subset H_\alpha : \ |T_\alpha| < \omega \text{ and } T \cup H_\xi \text{ is not } \kappa\text{-directed for any } \xi > \alpha\}.$$

By Definition 2.1 the set A_α is non-empty for any $\alpha < \eta$. Let

$$\mathcal{X} = \{T : \ T \in A_\alpha \text{ for some } \alpha < \eta\}.$$

Define a pair of functions (F, G):

$$G \colon \eta \to 2^{\mathcal{X}} \text{ such that } G(\alpha) = A_\alpha,$$
$$F \colon \mathcal{X} \to 2^\eta \text{ such that } F(T) = \{\xi < \eta : \ T \cup H_\xi \text{ is not } \kappa\text{-directed}\}.$$

We will show that (F, G) are twin functions. Let $\alpha < \xi$. Then for some $T \in G(\alpha)$ the set $T \cup H_\xi$ is not κ-directed. Hence $\xi \in F(T)$. By Theorem 3.2 there exists a selector of twin functions $g \colon K \to \mathcal{X}$, where $|K| = \beta$. By (1) in Definition 2.3 the sets $g(\alpha)$ are κ-directed for any $\alpha \in K$, by (2) in Definition 2.3 the sets $g(\alpha) \cup H_\xi$ are not κ-directed whenever $\alpha < \xi$. Hence $\{g(\alpha) : \ \alpha \in K\}$ is a required κ-strong sequence.

5. Some counterexamples

We show that such strong assumptions of cardinals in Theorem 3.1 are necessary because if we omit them we can construct the following counterexamples.

Example 5.1
Let $\mu = \aleph_1$, $\eta = \aleph_2$ and $X = \omega_2$. Let $X = \bigcup\{H_\alpha : \ \alpha < \omega_2\}$ with $|H_\alpha| = \aleph_1$ be a partition of X. Let G be a graph on X such that for each $\beta < \alpha < \omega_2$ there is $e_{\beta,\alpha}$ between H_β and H_α such that the endvertices of $\{e_{\beta,\alpha} : \ \beta < \alpha\}$ in H_α are distinct. Define the relation r: for finite $S \subset X$ there is b with $(a, b) \in r$, $a \in S$ iff S is bounded in Z. This gives a system $\{H_\alpha : \ \alpha < \omega_2\}$ which is an ω-strong sequence assumed in the theorem. By Theorem 3.1 there exists an ω-strong sequence $\{T_\alpha : \ \alpha < \omega_1\}$ consisting of finite sets. Define the set with using the following function $f \colon \omega_1 \to P(\omega_1)$ given by the formula $\eta \in f(\xi)$ iff there are $x \in T_\xi$, $y \in T_\eta$, $y < x$ with $\{y, x\} \in Z$. Then each $f(\xi)$ is finite and for $\xi < \eta < \omega_1$ either $\xi \in f(\eta)$ or $\eta \in f(\xi)$ holds, but this contradicts Hajnal Set Mapping Theorem, (see [6]).

Example 5.2
Let $\mu = \aleph_1$ and $\eta = 2^{\aleph_0}$. Let $D^{\aleph_2}$, where $D = \{0, 1\}$ denote the generalized Cantor discontinuum. For each $\alpha < \aleph_2$ consider the projection $\pi_\alpha \colon D^{\aleph_2} \to \{0, 1\}$ and let $B = \{\pi_\alpha^{-1} : \ \alpha < \aleph_2, \ i \in \{0, 1\}\}$. The family B forms a subbase in $D^{\aleph_2}$. Take the base generated by B as a set X and consider inclusion as a relation on X. The weight and the character of $D^{\aleph_2}$ are equal $\aleph_2$. Let $\lambda \colon 2^{\aleph_2} \to D^{\aleph_2}$ be a bijection. Consider a family $H_\alpha = \{x \in X : \ \lambda(\alpha) \in x\}$. Hence the family is centered because $\bigcap H_\alpha = \lambda(\alpha)$ and of cardinality $\aleph_2$ and obviously $\{H_\alpha : \ \alpha < 2^{\aleph_2}\}$ forms an ω-strong sequence. By Theorem 3.1 there exists an ω-strong sequence $\{T_\alpha \colon \alpha < \aleph_1\}$ consisting of finite sets. Then the family $\{\bigcap T_\alpha : \ \alpha < \aleph_1\}$ is consisting of open pairwise disjoint sets. It means that cellularity of $D^{\aleph_2}$ is not less than $\aleph_1$. But from Marczewski Theorem on cellularity of generalized Cantor discontinua, (see [11]), we have that the cellularity is $\aleph_0$. A contradiction.

6. Applications and conclusions for regulars

In this section some applications and conclusions of theorems from Section 3 will be presented. The next theorem is the well known theorem proved by Erdös-Rado in [5], (see also [2]).

Corollary 6.1 (Erdös-Rado theorem)
For any infinite cardinal λ the following statement is true

$$(2^\lambda)^+ \to (\lambda^+)^2_\lambda.$$

Proof. We have $(2^\lambda)^\lambda = 2^\lambda$, thus $\lambda^+ \ll (2^\lambda)^+$ for all $\lambda \geq \omega$.

The next two corollaries were proved with using Theorem 3.1 on strong sequences in [12].

Corollary 6.2
If X is a regular topological space, then

$$d(X) \leq \chi(X)^{c(X)}.$$

Corollary 6.3
If X is a regular topological space, then

$$w(X) \leq \chi(X)^{c(X)}.$$

For completeness this paper we prove two corollaries using Theorem 3.1 on strong sequences.

Corollary 6.4
If X, Y are topological spaces. Then

$$c(X \times Y) \leq 2^{c(X)+c(Y)}.$$

Proof. Let

$$\mathcal{X} = \{U_\alpha \times V_\alpha : \ \alpha(2^\lambda)^+\}$$

be a family of open non-empty sets and let $c(X \times Y) \leq 2^\lambda$. We will show that $c(X) + c(Y) = \lambda$.

Let $\mathcal{M}_0 \subset \mathcal{X}$ be a maximal family of pairwise disjoint sets. Let $\mathcal{M}_0$ be the first element of an ω-strong sequence. Suppose that for $\beta < \alpha < (2^\lambda)^+$ the ω-strong sequence $\{\mathcal{M}_\beta : \ \beta < \alpha\}$ such that $\mathcal{M}_\beta \subset \mathcal{X} \setminus \bigcup_{\gamma<\beta} \mathcal{M}_\gamma$ is a maximal family of pairwise disjoint sets has been defined. Obviously $|\mathcal{M}_\beta| \leq 2^\lambda$ for all $\beta < \alpha$. Hence $\mathcal{X} \setminus \bigcup_{\beta<\alpha} \mathcal{M}_\beta$ is non-empty. Let $\mathcal{M}_\alpha \subset \mathcal{X} \setminus \bigcup_{\beta<\alpha} \mathcal{M}_\beta$ be a maximal family of pairwise disjoint sets. Thus we have obtained the ω-strong sequence $\{\mathcal{M}_\alpha : \ \alpha < (2^\lambda)^+\}$. By Theorem 3.1 applying for $\eta = (2^\lambda)^+, \beta = \lambda^+$, $\mu = \lambda$ and $\kappa = \omega$ there exists an ω-strong sequence $\{\mathcal{T}_\alpha : \ \alpha < \lambda^+\}$ such that $|\mathcal{T}_\alpha| < \omega$ for all $\alpha < \lambda^+$. Hence $c(X) = \lambda^+$ or $c(Y) = \lambda^+$. A contradiction.

Corollary 6.5
If X is a Hausdorff space then

$$|X| \leq 2^{\chi(X)+c(X)}.$$

Proof. Let $\lambda = \chi(X)+c(X)$. Assume that $|X| > 2^\lambda$. Without the loss of generality we can assume that $X = \{x_\alpha : \ \alpha < (2^\lambda)^+\}$. Let $x_0 \in X$ be an arbitrary element, $\mathcal{B}_0 \subset \mathcal{X}$ be a maximal local base in point x_0 and let $\mathcal{B}_0$ be the first element in a strong sequence. Suppose that for $\beta < \alpha < (2^\lambda)^+$ the ω-strong sequence $\{\mathcal{B}_\beta : \ \beta < \alpha\}$ such that $\mathcal{B}_\beta \subset \mathcal{X} \setminus \bigcup_{\gamma<\beta} \mathcal{B}_\gamma$ is a maximal local base in point $x_\beta \in X \setminus \bigcup_{\gamma<\beta} \mathcal{B}_\gamma$ has been defined. Obviously $|\mathcal{B}_\beta| \leq 2^\lambda$ for all $\beta < \alpha$. Hence $\mathcal{X} \setminus \bigcup_{\beta<\alpha} \mathcal{B}_\beta$ is non-empty. Let $x_\alpha \in X \setminus \bigcup_{\beta<\alpha} \mathcal{B}_\beta$ be an arbitrary element. Let $\mathcal{B}_\alpha \subset \mathcal{X} \setminus \bigcup_{\beta<\alpha} \mathcal{B}_\beta$ be a maximal local base in point x_α. Thus we have obtained the ω-strong sequence $\{\mathcal{B}_\alpha : \ \alpha < (2^\lambda)^+\}$.

By Theorem 3.1 applying for $\eta = (2^\lambda)^+$, $\beta = \lambda^+$, $\mu = \lambda$ and $\kappa = \omega$ there exists an ω-strong sequence $\{\mathcal{T}_\alpha : \ \alpha < \lambda^+\}$ such that $|\mathcal{T}_\alpha| < \omega$ for all $\alpha < \lambda^+$. By Definition 2.1 there exists a family of open non-empty pairwise disjoint sets $\{U_\alpha : \ \alpha < \lambda^+\}$ such that $U_\alpha \in \mathcal{T}_\alpha$. Hence $c(X) = \lambda^+$. A contradiction.

7. Results for singulars

In [2] p.8, the following result for singulars was proved. Since the proof of Theorem 7.1 is short we cite it here, (see also [2]).

Theorem 7.1
Let β and η be cardinals such that $\omega \leq \beta \ll \eta$ with η-regular and β-singular. Then

$$(\kappa) \to (\beta^+)^2_\beta.$$

Proof. For β-singular we have $\beta^+ \ll \eta$. By Theorem 3.3 (with β and μ replaced by β^+ and β respectively) we obtain our claim.

Using the similar arguments as in Section 4 we easy obtain that Theorem 7.1 is also equivalent to both following results.

Theorem 7.2
Let β and η be cardinals such that $\omega \leq \beta \ll \eta$ with η-regular and β-singular. Let X be a set of cardinality η. If there exists a κ-strong sequences $\{H_\alpha \subset X : \alpha < \eta\}$ with $|H_\alpha| \leq 2^\beta$ for all $\alpha < \eta$, then there exists a κ-strong sequence $\{T_\alpha \subset X : \alpha < \beta^+\}$ with $|T_\alpha| < \kappa$ for all $\alpha < \beta^+$ and $\kappa < \eta$.

Theorem 7.3
Let β and η be cardinals such that $\omega \leq \beta \ll \eta$ with η-regular and β-singular. Let X be a set of cardinality η. If there exists a pair (F, G) of twin functions $G\colon \eta \to 2^X$, $F\colon X \to 2^\eta$ such that $|G(\alpha)| \leq 2^\beta$ for all $\alpha < \eta$, then there exists a selector $g\colon K \to X$ of twin functions with $|K| = \beta^+$.

Notice that the assumption of regularity of η cannot be omitted. The next theorem says about it.

Theorem 7.4
Let β and η be cardinals such that $\omega \leq \beta \ll \eta$ and $\mu < \beta$ be cardinals with η-singular and β-regular. Then

$$\eta \not\to (\beta)^2_\mu.$$

Proof. Suppose that the theorem is not true. Let $f\colon [\eta]^2 \to \mu$ be a function which determines a partition

$$[\eta]^2 = \{A_\xi : \xi < \mu\}.$$

Notice that at least one A_ξ has cardinality greater than β. If not, then

$$\Big| \bigcup_{\xi<\mu} A_\xi \Big| = \sum_{\xi<\mu} A_\xi < \mu \cdot \beta < \eta.$$

A contradiction.

Consider a function $g\colon \mu \to \beta$ such that $g(\alpha) = |f^{-1}(\alpha)|$ for any $\alpha < \mu$. Since at least one A_ξ has cardinality greater than β, then $\sup\{g(\alpha)\colon \alpha < \mu\} > \beta$, but β is regular. A contradiction.

Corollary 7.5
If λ is an infinite cardinal, then

$$2^\lambda \not\to (\lambda^+)^2_2.$$

References

[1] Baumgartner, James E., and Alan D. Taylor, and Stanley Wagon. *Structural properties of ideals*. Vol 197 of Dissertationes Math. (Rozprawy Mat.). Warszawa: PWN–Polish Scientific Publishers, 1982. Cited on 51.

[2] Comfort, W. Wistar, and Stylianos A. Negrepontis. *Chain conditions in topology.* Vol 79 of Cambridge Tracts in Mathematics. Cambridge-New York: Cambridge University Press, 1982. Cited on 51, 53, 56 and 58.

[3] Efimov, B. A. "Dyadic bicompacta." (Russian) *Trudy Moskov. Mat. Obšč.* 14 (1965): 211–247. Cited on 52.

[4] Engelking, Ryszard. *General topology.* Vol. 60 of Monografie Matematyczne. Warszawa: PWN–Polish Scientific Publishers, 1977. Cited on 53.

[5] Erdös, Paul, and Richard Rado. "A partition calculus in set theory." *Bull. Amer. Math. Soc.* 62 (1956): 427–489. Cited on 52, 53 and 56.

[6] Hajnal, András. "Proof of a conjecture of S. Ruziewicz." *Fund. Math.* 50 (1961/1962): 123–128. Cited on 56.

[7] Jech, Thomas. *Set theory.* New York-London: Academic Press, 1978. Cited on 53.

[8] Jureczko, Joanna. "On inequalities among some cardinal invariants." *Math. Aeterna* 6, no. 1 (2016): 87–98 Cited on 52.

[9] Jureczko, Joanna. "Strong sequences and independent sets." *Math. Aeterna* 6, no. 2 (2016): 141–152. Cited on 52.

[10] Jureczko, Joanna. "κ-strong sequences and the existence of generalized independent families." *Open Mathematics* (2017) (to appear). Cited on 52 and 53.

[11] Szpilrajn, Edward. "Remarque sur les produits cartésiens d'espaces topologiques." *C. R. (Doklady) Acad. Sci. URSS (N. S.)* 31 (1941): 525–527. Cited on 56.

[12] Turzański, Marian. "Strong sequences and the weight of regular spaces." *Comment. Math. Univ. Carolin.* 33, no. 3 (1992): 557–561. Cited on 56.

[13] Turzański, Marian. "On the selector of twin functions." *Comment. Math. Univ. Carolin.* 39, no. 2 (1998): 303–307. Cited on 52 and 53.

Wrocław University of Science and Technology
Wrocław
Poland
E-mail: joanna.jureczko@pwr.edu.pl

Received: February 28, 2016; final version: August 26, 2017;
available online: October 2, 2017.