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Ann. Univ. Paedagog. Crac. Stud. Math. 14 (2015), 149-162
DOI: 10.1515/aupcsm-2015-0011

FOLIA 160

Annales Universitatis Paedagogicae Cracoviensis
Studia Mathematica XIV (2015)

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Stability of generalized quadratic functional equation on a set of measure zero

Abstract. In this paper we prove the Hyers-Ulam stability of the following $\mathcal{K}$-quadratic functional equation

$$\sum_{k\in\mathcal{K}} f(x+k\cdot y) = Lf(x) + Lf(y), \qquad x,y\in E,$$

where E is a real (or complex) vector space. This result was used to demonstrate the Hyers-Ulam stability on a set of Lebesgue measure zero for the same functional equation.

1. Introduction

The concept of the stability for functional equations was introduced for the first time by S.M. Ulam in 1940 [33]. Ulam started the stability by the following question

Given a group G, a metric group (G', d), a number $\delta > 0$ and a mapping $f\colon G\to G'$ which satisfies the inequality $d(f(xy), f(x)f(y)) < \delta$ for all $x, y \in G$, does there exist an homomorphism $h\colon G\to G'$ and a constant $\gamma > 0$, depending only on G and G' such that $d(f(x), h(x)) \le \gamma\delta$ for all x in G?

In 1941, Ulam's problem for the case of approximately additive mappings was solved by D.H. Hyers [16] on Banach spaces. In 1950 T. Aoki [3] provided a generalization of the Hyers theorem for additive mappings and in 1978 Th.M. Rassias [30] generalized the Hyers theorem for linear mappings by considering an unbounded

AMS (2010) Subject Classification: Primary 39B82; Secondary 39B52;
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Cauchy difference. For more information on the concept of the stability of functional equations see, for example [6, 12, 15, 17, 18, 19, 22, 24].

The first stability theorem for the quadratic and Cauchy functional equations was proved in 1941 by Hyers-Ulam [16] and in 1978 by Rassias [30] then by F. Skof in 1983 [32] in Banach spaces. In 1984, P.W. Cholewa [9] extended Skof's result to an abelian group. In 1992, S. Czerwik [13], in the spirit of Hyers-Ulam-Rassias generalized the Skof's theorem.

Recently, the stability problem of the quadratic Cauchy type functional equations has been investigated by a number of mathematicians, the interested reader can refer to, for example [1, 4, 7, 8, 19, 20, 21, 22, 23, 25, 30].

The stability problems of several functional equations on a restricted domain have been extensively investigated by a number of authors, for example [5, 11, 13, 21, 29, 31].

It is very natural to ask if the restricted domain $D = \{(x, y) \in \mathbb{E}^2 : \|x\| + \|y\| \geq d\}$ can be replaced by a much smaller subset $\Omega \subset D$, i.e. a subset of measure zero in a measurable space E.

In 2013, J. Chung in [10] answered to this question by considering the stability of the Cauchy functional equation

$$f(x + y) = f(x) + f(y)$$

in a set $\Omega \subset \{(x, y) \in \mathbb{R}^2 : \|x\| + \|y\| \geq d\}$, where $m(\Omega) = 0$ and $f : \mathbb{R} \to \mathbb{R}$.

In 2014, J. Chung and J.M. Rassias [11] proved the stability of the quadratic functional equation in a set of measure zero. In 2015, M. Almahalebi in [2], proved the Hyers-Ulam stability for the Drygas functional equation

$$f(x + y) + f(x - y) = 2f(x) + f(y) + f(-y)$$

for all $(x, y) \in \Omega$, where $\Omega \subset \mathbb{R}^2$ is of Lebesgue measure zero.

Throughout this paper, let E be a real (or complex) vector space and F be a real (or complex) Banach space.

Our aim is to prove the Hyers-Ulam stability on a set of Lebesgue measure zero for the $\mathcal{K}$-quadratic functional equation

$$\sum_{k \in \mathcal{K}} f(x + k \cdot y) = Lf(x) + Lf(y), \qquad x, y \in E, \tag{1}$$

where $f : E \to F$ are applications and $\mathcal{K}$ is a finite subgroup of the group of automorphisms of E and $\operatorname{card} \mathcal{K} = L$.

These results are applied to the study of an asymptotic behaviour of this functional equation.

2. Notations and preliminary results

In this section, we need to introduce some notions and notations.

A function $A : E \to F$ between two vector spaces E and F is said to be additive if $A(x + y) = A(x) + A(y)$ for all $x, y \in E$. In this case, it is easily seen that $A(rx) = rA(x)$ for all $x \in E$ and all $r \in \mathbb{Q}$.

Let $k \in \mathbb{N}^* \setminus \{1\}$ and $A\colon E^k \to F$ be a function, then we say that A is k-additive if it is additive with respect to each variable. In addition, we say that A is symmetric if

$$A(x_{\sigma(1)}, x_{\sigma(2)}, \ldots, x_{\sigma(k)}) = A(x_1, x_2, \ldots, x_k),$$

whenever $x_1, x_2, \ldots, x_k \in E$ and σ is a permutation of $(1, 2, \ldots, k)$.

Let $k \in \mathbb{N}^* \setminus \{1\}$ and $A\colon E^k \to F$ be symmetric and k-additive and let $A_k(x) = A(x, x, \ldots, x)$ for $x \in E$, then $A_k(rx) = r^k A_k(x)$, whenever $x \in E$ and $r \in \mathbb{Q}$.

In this way, a function $A_k\colon E \to F$ which satisfies for all $\lambda \in \mathbb{Q}$ and $x \in E$ $A_k(\lambda x) = \lambda^k A_k$ will be called a rational-homogeneous form of degree k (assuming $A_k \not\equiv 0$).

A function $p\colon E \to F$ is called a generalized polynomial (GP) function of degree $m \in \mathbb{N}$ if there exist $a_0 \in E$ and a rational-homogeneous form $A_k\colon E \to F$ (for $1 \leq k \leq m$) of degree k, such that

$$p(x) = a_0 + \sum_{k=1}^{m} A_k(x)$$

for all $x \in E$.

Let F^E denote the vector space (over a field $\mathbb{K}$) consisting of all maps from E into F. For $h \in E$, define the linear difference operator Δ_h on F^E by

$$\Delta_h f(x) = f(x + h) - f(x)$$

for $f \in F^E$ and $x \in E$. Notice that these difference operators commute ($\Delta_{h_1}\Delta_{h_2} = \Delta_{h_2}\Delta_{h_1}$ for all $h_1, h_2 \in E$) and if $h \in E$ and $n \in \mathbb{N}$, then Δ_h^n the n-th iterate of Δ_h satisfies

$$\Delta_h^n f(x) = \sum_{k=0}^{n} (-1)^{n-k} \binom{n}{k} f(x + kh)$$

for all $f \in F^E$ and all $x, h \in E$.

3. Hyers-Ulam stability of (1)

The Hyers-Ulam stability of (1) was proved by M. Ait Sibaha, B. Bouikhalene and E. Elqorachi in [1], A. Charifi, B. Bouikhalene and E. Elqorachi in [7] and R. Łukasik in [26]. The purpose of this section is to establish this stability by another approach, the approach based on results by Mazur and Orlicz [27] and by Djoković [14].

Lemma 3.1

Let E be a vector space, F a Banach space, $\mathcal{K}$ a finite subgroup of the group of automorphisms of E and let $L = \operatorname{card} \mathcal{K}$. Let moreover $f\colon E \to F$ satisfy

$$\left\| \sum_{k \in \mathcal{K}} f(x + k \cdot y) - \sum_{k \in \mathcal{K}} f(k \cdot y) - Lf(x) \right\| \leq \delta, \qquad x, y \in E. \tag{2}$$

Then

$$\|\Delta_v^L f(u) - g(v)\| \leq \frac{2^L}{L}\delta, \qquad u, v \in E,$$

where $g(x) = -\sum_{i=0}^{L-1}(-1)^{L-i}\sum_{j=1}^{\binom{L}{i}} f(\sum_{k\in\mathcal{K}_{ij}} k\cdot x)$ *and* $\mathcal{K}_{ij} \subset \mathcal{K}$ *are pairwise different sets such that* $\operatorname{card}\mathcal{K}_{ij} = L - i$ *for* $j \in \{1, \ldots, \binom{L}{i}\}$, $i \in \{0, \ldots, L\}$.

Proof. We have

$$\sum_{j=1}^{\binom{L}{i}}\sum_{\mu\in\mathcal{K}} f\Big(\sum_{k\in\mathcal{K}_{ij}} \mu k\cdot x\Big) = L\sum_{j=1}^{\binom{L}{i}} f\Big(\sum_{k\in\mathcal{K}_{ij}} k\cdot x\Big), \qquad x\in E. \tag{3}$$

Since for all $\beta\in\mathcal{K}$,

$$\beta\mathcal{K}_{ij} = \mathcal{K}_{ik}, \qquad i\in\{0,\ldots,L\},\ j,k\in\Big\{1,\ldots,\binom{L}{i}\Big\}.$$

Now, fix $u, v\in E$ and let

$$x_i = u + iv,\ y_{ij} = \sum_{k\in\mathcal{K}_{ij}} k\cdot v, \qquad i\in\{0,\ldots,L\},\ j\in\Big\{1,\ldots,\binom{L}{i}\Big\}.$$

For all $\beta\in\mathcal{K}$, $i\in\{0,\ldots,L\}$, $j\in\big\{1,\ldots,\binom{L}{i}\big\}$ we study two different cases.

Case 1: $\beta^{-1}\in\mathcal{K}_{ij}$.

In this case, we have $i\neq L$. So let $k\in\{1,\ldots,\binom{L}{i+1}\}$ be such that $\mathcal{K}_{ij} = \mathcal{K}_{(i+1)k}\cup\{\beta^{-1}\}$. Thus, we have

$$x_i + \beta y_{ij} = u + iv + \sum_{l\in\mathcal{K}_{ij}}\beta l\cdot v = u + (i+1)v + \sum_{l\in\mathcal{K}_{(i+1)k}}\beta l\cdot v = x_{i+1} + \beta y_{(i+1)k}.$$

Case 2: $\beta^{-1}\notin\mathcal{K}_{ij}$.

Since $i\neq 0$, let $k\in\{1,\ldots,\binom{L}{i-1}\}$ be such that $\mathcal{K}_{(i-1)k} = \mathcal{K}_{ij}\cup\{\beta^{-1}\}$. By a similar calculation to the previous case, we obtain

$$x_i + \beta y_{ij} = x_{i-1} + \beta y_{(i-1)k}.$$

Consequently, we get

$$\sum_{i=0}^{L-1}(-1)^{L-i}\sum_{j=1}^{\binom{L}{i}}\sum_{\mu\in\mathcal{K}} f(x_i + \mu\cdot y_{ij}) = 0. \tag{4}$$

Now, in view of (2), (3) and (4) we have

$$\begin{aligned}&\|L\Delta_v^L f(u) - Lg(v)\|\\ &\quad = \Big\|L\sum_{i=0}^{L}(-1)^{L-i}\binom{L}{i} f(u+iv) + L\sum_{i=0}^{L-1}\sum_{j=1}^{\binom{L}{i}}(-1)^{L-i} f\Big(\sum_{k\in\mathcal{K}_{ij}} k\cdot v\Big)\Big\|\end{aligned}$$

$$= \left\| L \sum_{i=0}^{L} (-1)^{L-i} \binom{L}{i} f(u+iv) + \sum_{0}^{L-1} \sum_{j=1}^{\binom{L}{i}} \sum_{\mu\in\mathcal{K}} (-1)^{L-i} f\Big(\sum_{k\in\mathcal{K}_{ij}} \mu k\cdot v \Big) \right\|$$
$$= \left\| \sum_{i=0}^{L} (-1)^{L-i} \sum_{j=1}^{\binom{L}{i}} \Big[\sum_{\mu\in\mathcal{K}} f(x_i+\mu\cdot y_{ij}) - Lf(x_i) - \sum_{\mu\in\mathcal{K}} f(\mu\cdot y_{ij}) \Big] \right\|$$
$$\le 2^L\delta$$

which ends the proof.

Theorem 3.2
Let E be a vector space, F a Banach space, $\mathcal{K}$ a finite subgroup of the group of automorphisms of E and let $L = \operatorname{card}\mathcal{K}$. Let moreover $f\colon E\to F$ satisfy

$$\left\| \sum_{k\in\mathcal{K}} f(x+k\cdot y) - \sum_{k\in\mathcal{K}} f(k\cdot y) - Lf(x) \right\| \le \delta, \qquad x,y\in E. \tag{5}$$

Then there exists a unique (GP) function $p\colon E\to F$ of degree at most L such that p is a solution of (1) *and*

$$\|f(x)-f(0)-p(x)\| \le \frac{2^{L+1}}{L}\delta.$$

Proof. According to (5), we have

$$\|\Delta_v^L f(u) - g(v)\| \le \frac{2^L}{L}\delta, \qquad u,v\in E. \tag{6}$$

Replacing u by $u+v$, we get

$$\|\Delta_v^L f(u+v) - g(v)\| \le \frac{2^L}{L}\delta. \tag{7}$$

Using (6) and (7), we obtain

$$\|\Delta_v^{L+1} f(u)\| \le \frac{2^{L+1}}{L}\delta.$$

Then by [19, Theorem II] there exists a (GP) function $q\colon E\to F$ of degree at most L such that

$$\|f(x)-q(x)\| \le \frac{2^{L+1}}{L}\delta. \tag{8}$$

For $0\le k\le L$, there is a rational-homogeneous form $A_k\colon E\to F$ of degree k such that

$$q(x) = f(0) + \sum_{k=1}^{m} A_k(x). \tag{9}$$

By (5) and (8), we get

$$\begin{aligned}\Big\| \sum_{k\in\mathcal{K}} q(x+k\cdot y) - Lq(x) - Lq(y)\Big\| \\ &\le \Big\| \sum_{k\in\mathcal{K}} (q(x+k\cdot y) - f(x+k\cdot y))\Big\| + \Big\| \sum_{k\in\mathcal{K}} (q(k\cdot y) - f(k\cdot y))\Big\| \\ &\quad + \|L(q(x)-f(x))\| + \Big\| \sum_{k\in\mathcal{K}} f(x+k\cdot y) - \sum_{k\in\mathcal{K}} f(k\cdot y) - Lf(x)\Big\| \\ &\le 2^{L+1}\cdot\delta + 2^{L+1}\cdot\delta + 2^{L+1}\cdot\delta + \delta \\ &\le (3\cdot 2^{L+1}+1)\delta\end{aligned} \tag{10}$$

for all $x, y \in E$. Now (9) says, in light of (10), that for all $x, y \in E$,

$$\Big\| - Lf(0) + \sum_{j=1}^{L} \sum_{k\in\mathcal{K}} \Big(A_j(x+k\cdot y) - Aj(k\cdot y) - \sum_{j=1}^{L} A_j(x)\Big)\Big\| \le (3\cdot 2^{L+1}+1)\delta. \tag{11}$$

Replacing x by rx and y by ry in (11), where $r \in \mathbb{Q}$, we conclude that

$$\begin{aligned}\Big\| - Lf(0) + \sum_{j=1}^{L} r^j \sum_{k\in\mathcal{K}} \Big(A_j(x+k\cdot y) - \sum_{j=1}^{L} r^j LA_j(x) \\ - \sum_{k\in\mathcal{K}} \sum_{j=1}^{L} r^j A_j(k\cdot y)\Big)\Big\| \\ \le (3\cdot 2^{L+1}+1)\delta\end{aligned} \tag{12}$$

for all $x, y \in E$. By continuity, (12) holds for all real r and all $x, y \in E$. Now suppose that $\phi\colon F \to \mathbb{R}$ is a continuous linear functional. Then by (12), we get

$$\begin{aligned}\Big\| - \phi(Lf(0)) + \sum_{j=1}^{L} r^j \phi\Big(\sum_{k\in\mathcal{K}} \Big(A_j(x+k\cdot y) \\ - \sum_{j=1}^{L} LA_j(x) - \sum_{k\in\mathcal{K}} \sum_{j=1}^{L} A_j(y)\Big)\Big)\Big\| \\ \le (3\cdot 2^{L+1}+1)\delta\|\phi\|\end{aligned}$$

for all $x, y \in E$ and all $r \in \mathbb{R}$.

Since a real polynomial function is bounded if and only if it is constant, from the last inequality we surmise that, for $1 \le j \le L$,

$$\phi\Big(\sum_{k\in\mathcal{K}} \Big(A_j(x+k\cdot y) - LA_j(x) - \sum_{k\in\mathcal{K}} A_j(k\cdot y)\Big)\Big) = 0$$

for all $x, y \in E$. As $\phi\colon F \to \mathbb{R}$ is arbitrary continuous linear functional, by the Hahn-Banach theorem,

$$\sum_{k\in\mathcal{K}} \left(A_j(x + k\cdot y) - LA_j(x) - \sum_{k\in\mathcal{K}} A_j(k\cdot y) \right) = 0, \qquad x, y \in E,\ 1 \le j \le L. \tag{13}$$

Let $p(x) = q(x) - q(0)$, then p is a (GP) function of degree at most L and by (13) it is a solution of equation (1)

$$\sum_{k\in\mathcal{K}} \left(p(x + k\cdot y) - Lp(x) - \sum_{k\in\mathcal{K}} p(k\cdot y) \right) = 0, \qquad x, y \in E. \tag{14}$$

Finally, by using (8) and (14), we get

$$\|f(x) - f(0) - p(x)\| < \frac{2^{L+1}}{L}\delta, \qquad x \in E.$$

Let p' be another (GP) function solution of (1) of degree at most L such that

$$\|f(x) - f(0) - p'(x)\| < \frac{2^{L+1}}{L}\delta, \qquad x \in E.$$

Then, we get $\|p(x) - p'(x)\| < \frac{2^{L+2}}{L}\delta$, $x \in E$. A similar proof to that in [19, Theorem III] yields $p = p'$.

Theorem 3.3
Let E be a vector space, F a Banach space, $\mathcal{K}$ a finite subgroup of the group of automorphisms of E and let $L = \operatorname{card}\mathcal{K}$. Let $f\colon E \to F$ be a function satisfying

$$\left\| \sum_{k\in\mathcal{K}} f(x + k\cdot y) - Lf(x) - Lf(y) \right\| \le \delta, \qquad x, y \in E. \tag{15}$$

Then there exists a unique (GP) function $p\colon E \to F$ of degree at most L such that p is a solution of (1) and

$$\|f(x) - f(0) - p(x)\| \le \frac{2^{L+3}}{L}\delta, \qquad x \in E.$$

Proof. By posing $f' = f - f(0)$, it is easy to show that f' satisfies

$$\left\| \sum_{k\in\mathcal{K}} f'(x + k\cdot y) - Lf'(x) - Lf'(y) \right\| \le 2\delta, \qquad x, y \in E. \tag{16}$$

First, we observe that

$$\left\| Lf'(y) - \sum_{k\in\mathcal{K}} f'(k\cdot y) \right\| \le 2\delta. \tag{17}$$

From the above inequalities (16) and (17), we have

$$\left\| \sum_{k\in\mathcal{K}} f'(x + k\cdot y) - \sum_{k\in\mathcal{K}} f'(k\cdot y) - Lf'(x) \right\| \le 4\delta.$$

By Theorem 3.2, the result follows.

4. Stability of equations (1) on a set of measure zero

Let E be a vector space and F be a real (or complex) Banach space. For given $x, y, t \in E$ and a finite subgroup $\mathcal{K}$ of the group of automorphisms of E, we define

$$P_{x,y,t} = \{(x, t), (x + k' \cdot t, y), (x + k \cdot y, t);\ \forall k, k' \in \mathcal{K}\}.$$

Let $\Omega \subset E^2$. Throughout this section we assume that Ω satisfies the condition: For given $x, y \in E$ there exists $t \in E$ such that

$$P_{x,y,t} \subset \Omega. \tag{C}$$

In the following, we prove the Hyers-Ulam stability theorem for the generalized quadratic functional equation (15) in Ω.

Theorem 4.1
Let $\delta \geq 0$ and suppose that E is a vector space and F is real (or complex) Banach space. If $f\colon E \to F$ satisfies

$$\Big\| \sum_{k\in\mathcal{K}} f(x + k \cdot y) - Lf(x) - Lf(y) \Big\| \leq \delta$$

for all $(x, y) \in \Omega$, then there exists a unique generalized polynomial (GP) $q\colon E \to F$ of degree at most L such that

$$\|f(x) - q(x)\| \leq\ 3 \cdot \frac{2^{L+3}}{L}\delta, \qquad x \in E. \tag{18}$$

Proof. Let

$$D(x, y) = \sum_{k\in\mathcal{K}} f(x + k \cdot y) - Lf(x) - Lf(y).$$

Since Ω satisfies (C), for given $x, y \in E$, there exists $t \in E$ such that

$$\|D(x + k \cdot y, t)\| \leq \delta, \quad \|D(x + k' \cdot t, y)\| \leq \delta \quad \text{and} \quad \|D(x, t)\| \leq \delta.$$

Thus, we have

$$\begin{aligned} L\Big[\sum_{k\in\mathcal{K}} & f(x+k \cdot y) - Lf(x) - Lf(y)\Big] \\ &= \Big[\sum_{k\in\mathcal{K}} \Big(Lf(x + k \cdot y) + Lf(t) - \sum_{k'\in\mathcal{K}} f(x + k \cdot y + k' \cdot t)\Big)\Big] \\ &\quad + \Big[\sum_{k'\in\mathcal{K}} \Big(- Lf(x + k' \cdot t) - Lf(y) + \sum_{k\in\mathcal{K}} f(x + k' \cdot t + k \cdot y)\Big)\Big] \\ &\quad + L\Big[\sum_{k'\in\mathcal{K}} f(x + k' \cdot t) - Lf(x) - Lf(t)\Big] \\ &= - \sum_{k\in\mathcal{K}} D(x + k \cdot y, t) + \sum_{k'\in\mathcal{K}} D(x + k' \cdot t, y) + LD(x, t). \end{aligned}$$

Using the triangle inequality, we get

$$L\Big\|\sum_{k\in\mathcal{K}} f(x+k\cdot y)-Lf(x)-Lf(y)\Big\|\le 3L\delta,\qquad x,y\in E.$$

This implies that

$$\Big\|\sum_{k\in\mathcal{K}} f(x+k\cdot y)-Lf(x)-Lf(y)\Big\|\le 3\delta,\qquad x,y\in E.$$

Next, according to Theorem 3.3, there exists a unique generalized polynomial (GP) $q\colon E\to F$ of degree at most L such that

$$\|f(x)-q(x)\|\le 3\cdot\frac{2^{L+3}}{L}\delta,\qquad x\in E.$$

This completes the proof.

Corollary 4.2
Suppose that $f\colon E\to F$ satisfies the functional equation

$$\sum_{k\in\mathcal{K}} f(x+k\cdot y)=Lf(x)+Lf(y)$$

for all $(x,y)\in\Omega$. Then (18) holds for all $x,y\in E$.

Corollary 4.3
Let $\varepsilon\ge 0$ be fixed. Suppose that $f\colon E\to F$ satisfies the functional inequality

$$\|f(x+y)+f(x+\sigma(y))-2f(x)-2f(y)\|\le\varepsilon,\qquad x,y\in\Omega,$$

where $\sigma\colon E\to F$ is an involution. Then there is a unique quadratic $Q\colon E\to F$, and an additive $A\colon E\to F$ such that $A(\sigma(x))=-A(x)$, $x\in E$ and

$$\|f(x)-f(0)-A(x)-Q(x)\|\le 16\varepsilon,\qquad x\in E.$$

Proof. By taking $L=2$ and $\mathcal{K}=\{I,\sigma\}$ in Theorem 4.1, there exists a unique generalized polynomial (GP) $p(x)=f(0)-A(x)-Q(x)$ of degree at most 2 which is a solution of the following functional equation

$$f(x+y)+f(x+\sigma(y))-2f(x)-2f(y)=0,\qquad x,y\in E.$$

We use [Theorem 6, [25]] to complete the proof.

5. Applications

In this section, we construct a set of measure zero satisfying the condition (C) for $E=\mathbb{R}$. From now on, we identify $\mathbb{R}^2$ with $\mathbb{C}$. Using $\mathcal{K}=\{I\}$, respectively $\mathcal{K}=\{I,-I\}$ for $\mathbb{R}$. The following lemma is a crucial key of our construction [28, Theorem 1.6].

Lemma 5.1
The set $\mathbb{R}$ of real numbers can be partitioned as $\mathbb{R} = F \cup K$, where F is of the first Baire category, i.e. F is a countable union of nowhere dense subsets of $\mathbb{R}$, and K is of Lebesgue measure zero.

The following lemma was proved by J. Chung and J.M. Rassias in [10] and [11].

Lemma 5.2
Let K be a subset of $\mathbb{R}$ of measure zero such that $\complement K = \mathbb{R}\backslash\{K\}$ is of first Baire category. Then, for any countable subsets $U \subset \mathbb{R}$, $V \subset \mathbb{R}\backslash\{0\}$ and $M > 0$, there exists $t \geq M$ such that

$$U + tV = \{u + tv : \; u \in U, \, v \in V\} \subset K. \tag{19}$$

In the following theorem, we give the construction of a set Ω of Lebesgue measure zero.

Theorem 5.3
Let $\Omega = \exp(\frac{-\pi}{6} i)(K \times K)$ be the rotation of $K \times K$ by $\frac{\pi}{6}$, i.e.

$$\Omega = \left\{ (p, q) \in \mathbb{R}^2 : \; \frac{\sqrt{3}}{2} p - \frac{1}{2} q \in K, \, \frac{1}{2} p + \frac{\sqrt{3}}{2} q \in K \right\},$$

where K is a subset of $\mathbb{R}$ of measure zero such that $\complement K = \mathbb{R}\backslash\{K\}$ is of the first Baire category. Then Ω satisfies the condition (C) *and is of two-dimensional Lebesgue measure zero.*

Proof. By the construction of Ω, the condition (C) is equivalent to the fact that for every $x, y \in \mathbb{R}$, there exists $t \in \mathbb{R}$ such that

$$\exp\left(\frac{-\pi}{6} i\right) P_{x,y,t} \subset K \times K.$$

The inclusion (19) is equivalent to

$$S_{x,y,t} := \left\{ \frac{\sqrt{3}}{2} u - \frac{1}{2} v, \, \frac{1}{2} u + \frac{\sqrt{3}}{2} v : \; (u, v) \in P_{x,y,t} \right\} \subset K.$$

It is easy to check that the set $S_{x,y,t}$ is contained in a set of form $U + tV$. We consider two cases

(i) $\mathcal{K} = \{I\}$. Then

$$U = \left\{ \frac{\sqrt{3}}{2} x; \frac{1}{2} x; \frac{\sqrt{3}}{2} x - \frac{1}{2} y; \frac{1}{2} x + \frac{\sqrt{3}}{2} y; \frac{\sqrt{3}}{2}(x + y); \frac{1}{2}(x + y) \right\},$$
$$V = \left\{ \pm \frac{1}{2}; \frac{\sqrt{3}}{2} \right\}.$$

In this case we find the functional equation of Cauchy

$$f(x + y) = f(x) + f(y).$$

(ii) $\mathcal{K} = \{I, -I\}$. Then

$$U = \left\{ \frac{\sqrt{3}}{2}x; \frac{1}{2}x; \frac{\sqrt{3}}{2}x - \frac{1}{2}y; \frac{1}{2}x + \frac{\sqrt{3}}{2}y; \frac{\sqrt{3}}{2}(x \pm y); \frac{1}{2}(x \pm y) \right\},$$
$$V = \left\{ \pm\frac{1}{2}; \pm\frac{\sqrt{3}}{2} \right\}.$$

In this case we find the quadratic functional equation

$$f(x+y) + f(x-y) = 2f(x) + 2f(y).$$

By Lemma 5.2, for given $x, y \in \mathbb{R}$ and $M > 0$ there exists $\alpha \geq M$ such that

$$S_{x,y,t} \subset U + tV \subset K.$$

Thus, Ω satisfies (C). This completes the proof.

Corollary 5.4
Let $d > 0$ and $\Omega_d := \{(x, y) \in \Omega : \ |x| + |y| \geq d\}$. Then, Ω_d satisfies (C).

As a consequence of Theorem 4.1 and corollary, we obtain the asymptotic behaviour of f satisfying the asymptotic condition,then there exists a sequence δ_n monotonically decreasing to 0 such that

$$\left\| \sum_{k\in\mathcal{K}} f(x + k \cdot y) - Lf(x) - Lf(y) \right\| \to 0, \quad \text{as } \|x\| + \|y\| \to \infty. \tag{20}$$

Corollary 5.5
Suppose that $f : \mathbb{R} \to F$ satisfies the condition (20)*. Then, f is a generalized quadratic functional.*

Proof. The condition (20) implies that, for each $n \in \mathbb{N}$, there exists $d_n > 0$ such that

$$\left\| \sum_{k\in\mathcal{K}} f(x + k \cdot y) - Lf(x) - Lf(y) \right\| \leq \delta_n \tag{21}$$

for all $(x, y) \in \Omega_{d_n}$. From previous corollary, $\Omega_{d_n} := \{(x, y) \in \Omega : \ |x| + |y| \geq d_n\}$ satisfies (C). Thus, by Theorem 4.1, there exists a unique generalized polynomial $q_n : \mathbb{R} \to F$

$$\|f(x) - q_n(x)\| \leq 3 \cdot \frac{2^{L+3}}{L}\delta_n \tag{22}$$

for all $x \in \mathbb{R}$. By replacing $n \in \mathbb{N}$ by $m \in \mathbb{N}$ in (22) and using the triangle inequality, we have

$$\|q_n(x) - q_{n_1}(x)\| \leq 3 \cdot \frac{2^{L+3}}{L}\delta_n + 3 \cdot \frac{2^{L+3}}{L}\delta_m \leq 6 \cdot \frac{2^{L+3}}{L} \tag{23}$$

for all $x \in \mathbb{R}$. For all $n_1, n \in \mathbb{N}$ and $x \in \ \mathbb{R}$, we have necessarily $q_n = q_{n_1} + q_n(0) - q_{n_1}(0)$. Since $q_n(0) = q_{n_1}(0) = 0$, we have in (23) $q_n = q_{n_1}$. Now, letting $n \to \infty$ in (22), we get the result. This completes the proof.

If we define $\Omega \subset \mathbb{R}^{2n}$ as an appropriate rotation of $2n$-product K^{2n} of K, then Ω has $2n$-dimensional measure zero and satisfies (C). Consequently, we obtain the following.

Corollary 5.6
Let F be a Banach space. Suppose that $f\colon \mathbb{R}^n \to F$ satisfies the functional inequality

$$\Big\| \sum_{k\in\mathcal{K}} f(x+k\cdot y) - Lf(x) - Lf(y) \Big\| \le \varepsilon$$

for all $(x,y) \in \Omega$. Then there exists a unique quadratic mapping $q\colon \mathbb{R}^n \to F$ such that

$$\|f(x) - q(x)\| \le 3 \cdot \frac{2^{L+3}}{L}\varepsilon$$

for all $x \in \mathbb{R}^n$.

References

[1] M. Ait Sibaha, B. Bouikhalene, E. Elqorachi, *Hyers-Ulam-Rassias stability of the K-quadratic functional equation*, JIPAM. J. Inequal. Pure Appl. Math. **8** (2007), no. 3, Article 89, 13pp. Cited on 150 and 151.

[2] M. Almahalebi, *Approximate Drygas mappings on a set of measure zero*, (submitted). Cited on 150.

[3] T. Aoki, *On the stability of the linear transformation in Banach spaces*, J. Math. Soc. Japan **2** (1950), 64–66. Cited on 149.

[4] N. Brillouët-Belluot, J. Brzdęk, K. Ciepliński, *On some recent developments in Ulam's type stability*, Abstr. Appl. Anal. 2012, Art. ID 716936, 41pp. Cited on 150.

[5] J. Brzdęk, *On a method of proving the Hyers-Ulam stability of functional equations on restricted domains*, Austr. J. Math. Anal. Appl. **6** (2009), 1–10. Cited on 150.

[6] J. Brzdęk, *Hyperstability of the Cauchy equation on restricted domains*, Acta Math. Hungar. **141** (2013), no. 1-2, 58–67. Cited on 150.

[7] A. Charifi, B. Bouikhalene, E. Elqorachi, *Hyers-Ulam-Rassias stability of a generalized Pexider functional equation*, Banach J. Math. Anal. **1** (2007), no. 2, 176–185. Cited on 150 and 151.

[8] A.B. Chahbi, A. Charifi, B. Bouikhalene, S. Kabbaj, *Operatorial approach to the non-Archimedean stability of a Pexider K-quadratic functional equation*, Arab J. Math. Sci. **21** (2015), no. 1, 67–83. Cited on 150.

[9] P.W. Cholewa, *Remarks on the stability of functional equations*, Aequationes Math. **27** (1984), no. 1-2, 76–86. Cited on 150.

[10] Jaeyoung Chung, *Stability of a conditional Cauchy equation on a set of measure zero*, Aequationes Math. **87** (2014), no. 3, 391–400. Cited on 150 and 158.

[11] Jaeyoung Chung, J.M. Rassias, *Quadratic functional equations in a set of Lebesgue measure zero*, J. Math. Anal. Appl. **419** (2014), no. 2, 1065–1075. Cited on 150 and 158.

[12] K. Ciepliński, *Applications of fixed point theorems to the Hyers-Ulam stability of functional equations – a survey*, Ann. Funct. Anal. **3** (2012), no. 1, 151–164. Cited on 150.

[13] S. Czerwik, *On the stability of the quadratic mapping in normed spaces*, Abh. Math. Sem. Univ. Hamburg **62** (1992), 59–64. Cited on 150.

[14] D.Ž. Djoković, *A representation theorem for* $(X_1 - 1)(X_2 - 1) \cdots (X_n - 1)$ *and its applications*, Ann. Polon. Math. **22** (1969/1970) 189–198. Cited on 151.

[15] P. Găvruţa, *A generalization of the Hyers-Ulam-Rassias stability of approximately additive mappings*, J. Math. Anal. Appl. **184** (1994), 431–436. Cited on 150.

[16] D.H. Hyers, *On the stability of the linear functional equation*, Proc. Nat. Acad. Sci. U.S.A. **27** (1941), 222–224. Cited on 149 and 150.

[17] D.H. Hyers, Th.M. Rassias, *Approximate homomorphisms*, Aequationes Math. **44** (1992), no. 2-3, 125–153. Cited on 150.

[18] D.H. Hyers, G.I. Isac, Th.M. Rassias, *Stability of functional equations in several variables*, (English summary) Progress in Nonlinear Differential Equations and their Applications 34, Birkhäuser Boston, Inc., Boston, MA, 1998. Cited on 150.

[19] D.H. Hyers, *Transformations with bounded m-th differences*, Pacific J. Math. **11** (1961), 591–602. Cited on 150, 153 and 155.

[20] S.-M. Jung, *Stability of the quadratic equation of Pexider type*, Abh. Math. Sem. Univ. Hamburg **70** (2000), 175–190. Cited on 150.

[21] S.-M. Jung, *On the Hyers-Ulam stability of the functional equations that have the quadratic property*, J. Math. Anal. Appl. **222** (1998), no. 1, 126–137. Cited on 150.

[22] S.-M. Jung, P.K. Sahoo, *Hyers-Ulam stability of the quadratic equation of Pexider type*, J. Korean Math. Soc. **38** (2001), no. 3, 645–656. Cited on 150.

[23] C.F.K. Jung, *On generalized complete metric spaces*, Bull. Amer. Math. Soc. **75** (1969), 113–116. Cited on 150.

[24] Yongjin Li, Liubin Hua, *Hyers-Ulam stability of a polynomial equation*, Banach J. Math. Anal. **3** (2009), no. 2, 86–90. Cited on 150.

[25] R. Łukasik, *Some generalization of Cauchy's and the quadratic functional equations*, Aequationes Math. **83** (2012), no. 1-2, 75–86. Cited on 150 and 157.

[26] R. Łukasik, *The stability of some generalization of the quadratic functional equation*, Georgian Math. J. **21** (2014), no. 4, 463–474. Cited on 151.

[27] S. Mazur, W. Orlicz, *Grundlegende Eigenschaften der polynomischen Operationen. Erst Mitteilung*, Studia Math. **5** (1934), 50–68. Cited on 151.

[28] J.C. Oxtoby, *Measure and category. A survey of the analogies between topological and measure spaces*, Second edition, Graduate Texts in Mathematics 2, Springer-Verlag, New York-Berlin, 1980. Cited on 157.

[29] J.M. Rassias, *On the Ulam stability of mixed type mappings on restricted domains*, J. Math. Anal. Appl. **276** (2002), 747–762. Cited on 150.

[30] Th.M. Rassias, *On the stability of the linear mapping in Banach spaces*, Proc. Amer. Math. Soc. **72** (1978), no. 2, 297–300. Cited on 149 and 150.

[31] Th.M. Rassias, *On the stability of functional equations and a problem of Ulam*, Acta Appl. Math. **62** (2000), no. 1, 23–130. Cited on 150.

[32] F. Skof, *Local properties and approximation of operators*, Rend. Sem. Mat. Fis. Milano **53** (1983), 113–129. Cited on 150.

[33] S.M. Ulam, *A collection of mathematical problems*, Interscience Tracts in Pure and Applied Mathematics 8, Interscience Publishers, New York-London 1960. Cited on 149.

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Received: June 14, 2015; final version: October 31, 2015; available online: December 14, 2015.