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On the superstability of generalized d'Alembert harmonic functions

Abstract. The aim of this paper is to study the superstability problem of the d'Alembert type functional equation

$$f(x+y+z)+f(x+y+\sigma(z))+f(x+\sigma(y)+z)+f(\sigma(x)+y+z) \\ =4f(x)f(y)f(z)$$

for all $x,y,z\in G$, where G is an abelian group and $\sigma\colon G\to G$ is an endomorphism such that $\sigma(\sigma(x))=x$ for an unknown function f from G into $\mathbb{C}$ or into a commutative semisimple Banach algebra.

1. Introduction

In 1940, Ulam [19] gave a wide ranging talk before the Mathematics Club of the University of Wisconsin in which he discussed a number of important unsolved problems and among those the following question concerning the stability of homomorphisms

Let G_1 be a group and let (G_2,d) be a metric group. Given $\delta>0$, does there exist $\epsilon>0$ such that if a mapping $h\colon G_1\to G_2$ satisfies the inequality

$$d(h(xy),h(x)h(y))\le\delta$$

for all $x,y\in G_1$, then there is a homomorphism $a\colon G_1\to G_2$ with

$$d(h(x),a(x))\le\epsilon$$

for all $x\in G_1$?

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In 1941, Hyers [10] considered the case of approximately additive mappings $f\colon E \to F$, where E and F are Banach spaces and f satisfies Hyers inequality

$$\|f(x+y) - f(x) - f(y)\| \le \epsilon$$

for all $x, y \in E$ and $\epsilon > 0$. He proved that then there exists a unique additive mapping $T\colon E \to F$ satisfying

$$\|f(x) - T(x)\| \le \epsilon$$

for all $x \in E$.

The above result was generalized by Bourgin [7] and Aoki [1] in 1949 and 1950. In 1978 and 1982, Hyers' result was improved by Th.M. Rassias [16] and J.M. Rassias [15]. Namely, the condition bounded by the constant was replaced by the condition bounded by two variables. Thereafter it was improved by Găvruta [9] to the condition bounded by the function.

In 1979, Baker et al. [4] and Bourgin [7] introduced that if f satisfies the inequality $|E_1(f) - E_2(f)| \le \epsilon$, then either f is bounded or $E_1(f) = E_2(f)$. This concept is now known as the superstability. In 1980, the superstability of the cosine functional equation (also called the d'Alembert functional equation)

$$f(x+y) + f(x-y) = 2f(x)f(y) \tag{A}$$

was investigated by Baker [5]; also by Badora [2] in 1998, and Badora and Ger [3] in 2002 under the condition $|f(x+y) + f(x-y) - 2f(x)f(y)| \le \epsilon$, $\varphi(x)$ or $\varphi(y)$, respectively. Also the stability of the d'Alembert functional equation is founded in papers [6, 11, 13, 18]. In [8] J. Brzdęk et al. gave the recent development of the conditional stability of the homomorphism equation. Recently, G.H. Kim [12] investigated the stability of the generalized d'Alembert type functional equation as follows

$$f(x+y) + f(x+\sigma(y)) = 2f(x)f(y), \tag{A_f}$$

where f is an unknown function. In [14] H.M. Kim, G.H. Kim and M.H. Han proved the superstability of approximate d'Alembert harmonic functions

$$f(x+y+z) + f(x+y-z) + f(x-y+z) + f(y+z-x) = 4f(x)f(y)f(z)$$

on an abelian group and on a commutative semisimple Banach algebra.

In this paper, let $(G, +)$ be an abelian group, $\mathbb{C}$ the field of complex numbers, $\mathbb{R}$ denote the set of real numbers, and let σ be an endomorphism of G with $\sigma(\sigma(x)) = x$ for all $x \in G$.

The aim of this paper is to investigate the superstability problem of the generalized d'Alembert type functional equation as follows

$$\begin{aligned} f(x+y+z) + f(x+y+\sigma(z)) + f(x+\sigma(y)+z) + f(\sigma(x)+y+z) \\ = 4f(x)f(y)f(z) \end{aligned} \tag{1}$$

for all $x, y, z \in G$, where G is an abelian group and $f\colon G \to \mathbb{C}$. Moreover, we extend all superstability results for equation (1) to the superstability on the commutative semisimple Banach algebra.

In the special case, if $\sigma(x) = -x$ we obtain the result that is in [14].

2. Superstability of equation (1)

In this section, we will investigate the superstability of (1). The functional equation (1) is connected with the d'Alembert functional equation (A_f) as follows [12].

Lemma 2.1
Let f be a complex-valued function on an abelian group G such that $f(0) > 0$. Then, f satisfies (1) *on G if and only if f satisfies* (A_f) *on G.*

Proof. Assume that f satisfies (A_f) on G. Then, we have

$$\begin{aligned} f(x+y+z) &+ f(x+y+\sigma(z)) + f(x+\sigma(y)+z) + f(\sigma(x)+y+z) \\ &= 2f(x+y)f(z) + f(x+\sigma(y)+z) + f(\sigma(x+\sigma(y))+z) \\ &= 2f(x+y)f(z) + 2f(x+\sigma(y))f(z) \\ &= 2f(z)(f(x+y)+f(x+\sigma(y)) \\ &= 4f(x)f(y)f(z) \end{aligned}$$

for all $x, y, z \in G$.

For the converse, we consider f satisfying (1) on G. Putting $x = y = z = 0$ in (1) and as $f(0) > 0$, we get $f(0) = 1$. Setting $y = z = 0$ in (1), we obtain

$$f(\sigma(x)) = f(x)$$

for all $x \in G$. Next, taking $z := 0$ in (1), we get

$$\begin{aligned} 2f(x+y)+2f(x+\sigma(y)) &= f(x+y)+f(x+y)+f(x+\sigma(y))+f(\sigma(x+\sigma(y)) \\ &= f(x+y)+f(x+y)+f(x+\sigma(y))+f(\sigma(x)+y) \\ &= 4f(x)f(y)f(0) \end{aligned}$$

for all $x, y \in G$. Then, f satisfies the d'Alembert functional equation (A_f) on G. This completes the proof.

Theorem 2.2
Let $f\colon G \to \mathbb{C}$ be a function and let $\varphi\colon G \to [0, +\infty[$ satisfy the inequality

$$\begin{aligned} |f(x+y+z) &+ f(x+y+\sigma(z)) + f(x+\sigma(y)+z) \\ &\quad + f(\sigma(x)+y+z) - 4f(x)f(y)f(z)| \\ &\le \varphi(x) \end{aligned} \tag{2}$$

for all $x, y, z \in G$. Then, either f is bounded or f satisfies the functional equation (1).

Proof. If f is unbounded, then we can choose a sequence $\{y_n\}_{n\in\mathbb{N}}$ in G such that

$$\lim_{n\to\infty} |f(y_n)| = \lim_{n\to\infty} |f(\sigma(y_n))| = \lim_{n\to\infty} |f(\sigma(-y_n))| = \infty$$

and

$$|f(y_n)f(\sigma(-y_n))| > 1.$$

Taking $y = y_n$ and $z = \sigma(-y_n)$ in (2), we get

$$|f(x + y_n - \sigma(y_n)) + f(\sigma(x) + y_n - \sigma(y_n)) - f(x)(4f(y_n)f(\sigma(-y_n)) - 2)| \leq \varphi(x)$$

for all $x \in G$. Then

$$\begin{aligned}\Big|\frac{f(x + y_n - \sigma(y_n)) + f(\sigma(x) + y_n - \sigma(y_n))}{4f(y_n)f(\sigma(-y_n)) - 2} - f(x)\Big| \\ \leq \frac{\varphi(x)}{|4f(y_n)f(\sigma(-y_n)) - 2|}\end{aligned} \tag{3}$$

for all $x \in G$. Passing to the limit as $n \to \infty$ in (3), we obtain the following

$$f(x) = \lim_{n\to\infty} \frac{f(x + y_n - \sigma(y_n)) + f(\sigma(x) + y_n - \sigma(y_n))}{4f(y_n)f(\sigma(-y_n)) - 2} \tag{4}$$

for all $x \in G$. From (4), we will see that

$$f(\sigma(x)) = f(x) \tag{5}$$

for all $x \in G$.

Now, we will apply (3) to derive functional equation (1). Putting $y_n - \sigma(y_n) + y$ in the place of y in (2), we get

$$\begin{aligned}\big|&f(x + y_n - \sigma(y_n) + y + z) + f(x + y_n - \sigma(y_n) + y + \sigma(z)) \\ &+ f(x + \sigma(y_n - \sigma(y_n) + y) + z) + f(\sigma(x) + y_n - \sigma(y_n) + y + z) \\ &- 4f(x)f(y_n - \sigma(y_n) + y)f(z)\big| \\ &\leq \varphi(x)\end{aligned} \tag{6}$$

for all $x, y, z \in G$. Putting $y_n - \sigma(y_n) + \sigma(y)$ in the place of y in (2), we obtain

$$\begin{aligned}\big|&f(x + y_n - \sigma(y_n) + \sigma(y) + z) + f(x + y_n - \sigma(y_n) + \sigma(y) + \sigma(z)) \\ &+ f(x + \sigma(y_n - \sigma(y_n) + \sigma(y)) + z) + f(\sigma(x) + y_n - \sigma(y_n) + \sigma(y) + z) \\ &- 4f(x)f(y_n - \sigma(y_n) + \sigma(y))f(z)\big| \\ &\leq \varphi(x)\end{aligned} \tag{7}$$

for all $x, y, z \in G$. Combining (6) and (7) gives

$$\begin{aligned}\big|&f(x + y_n - \sigma(y_n) + y + z) + f(x + \sigma(y_n - \sigma(y_n) + \sigma(y)) + z) \\ &+ f(x + y_n - \sigma(y_n) + y + \sigma(z)) + f(\sigma(x) + y_n - \sigma(y_n) + \sigma(y) + z) \\ &+ f(x + \sigma(y_n - \sigma(y_n) + y) + z) + f(x + y_n - \sigma(y_n) + \sigma(y) + z) \\ &+ f(\sigma(x) + y_n - \sigma(y_n) + y + z) + f(x + y_n - \sigma(y_n) + \sigma(y) + \sigma(z)) \\ &- 4f(x)f(z)(f(y_n - \sigma(y_n) + y) + f(y_n - \sigma(y_n) + \sigma(y)))\big| \\ &\leq 2\varphi(x)\end{aligned} \tag{8}$$

for all $x, y, z \in G$. Using the fact (4) and (5), we see that

$$\begin{aligned}\lim_{n\to\infty} &\frac{f(x + y + z + y_n - \sigma(y_n)) + f(\sigma(x) + y_n - \sigma(y_n) + \sigma(y) + \sigma(z))}{4f(y_n)f(\sigma(-y_n)) - 2} \\ &= f(x + y + z)\end{aligned}$$

for all $x, y, z \in G$. Similarly,

$$\lim_{n\to\infty}\frac{f(x+y_n-\sigma(y_n)+y+\sigma(z))+f(\sigma(x)+y_n-\sigma(y_n)+\sigma(y)+z)}{4f(y_n)f(\sigma(-y_n))-2} = f(x+y+\sigma(z)),$$

$$\lim_{n\to\infty}\frac{f(x+\sigma(y_n-\sigma(y_n)+y)+z)+f(x+y_n-\sigma(y_n)+\sigma(y)+z)}{4f(y_n)f(\sigma(-y_n))-2} = f(x+\sigma(y)+z)$$

and

$$\lim_{n\to\infty}\frac{f(\sigma(x)+y_n-\sigma(y_n)+y+z)+f(x+y_n-\sigma(y_n)+\sigma(y)+\sigma(z))}{4f(y_n)f(\sigma(-y_n))-2} = f(\sigma(x)+y+z)$$

for all $x,y,z\in G$. Therefore, dividing inequality (8) by $|4f(y_n)f(\sigma(-y_n))-2|$ and taking the limit as $n\to\infty$, we get

$$f(x+y+z)+f(x+y+\sigma(z))+f(x+\sigma(y)+z)+f(\sigma(x)+y+z) = 4f(x)f(y)f(z)$$

for all $x,y,z\in G$. This completes the proof.

Corollary 2.3
Let $f\colon G\to\mathbb{C}$ be a function and $\varphi\colon G\to[0,+\infty[$ satisfy the inequality

$$\begin{aligned}|f(x+y+z)+f(x+y+\sigma(z))+f(x+\sigma(y)+z)\\ +f(\sigma(x)+y+z)-4f(x)f(y)f(z)|\\ \le\varphi(y)\ \textit{or}\ \varphi(z)\end{aligned}$$

for all $x,y,z\in G$. Then, either f is bounded or f satisfies the functional equation (1).

Proof. Similarly as in the proof of Theorem 2.2, we conclude the desired result.

Corollary 2.4 ([14, Theorem 2.2])
Let $f\colon G\to\mathbb{C}$ be a function and $\varphi\colon G\to[0,+\infty[$ satisfy the inequality

$$\begin{aligned}|f(x+y+z)+f(x+y-z)+f(x-y+z)\\ +f(y+z-x)-4f(x)f(y)f(z)|\\ \le\varphi(x)\ \textit{or}\ \varphi(y)\ \textit{or}\ \varphi(z)\end{aligned}$$

for all $x,y,z\in G$. Then, either f is bounded or f satisfies the functional equation $f(x+y+z)+f(x+y-z)+f(x-y+z)+f(y+z-x)=4f(x)f(y)f(z)$ on G.

Proof. It suffices to take $\sigma(x)=-x$ in Theorem 2.2.

Corollary 2.5
Let $f\colon G\to\mathbb{C}$ be a function and $\varphi\colon G\to[0,+\infty[$ satisfy the inequality

$$|f(x+y+z)-f(x)f(y)f(z)|\le\varphi(x)\ \textit{or}\ \varphi(y)\ \textit{or}\ \varphi(z)$$

for all $x,y,z\in G$. Then, either f is bounded or f satisfies the functional equation $f(x+y+z)=f(x)f(y)f(z)$ on G.

Proof. It suffices to take $\sigma(x)=x$ in Theorem 2.2.

3. Extension to Banach algebra

In this section, let $(G, +)$ be an abelian group and $(E, \|\cdot\|)$ be a commutative semisimple Banach algebra. All the results in Section 2 can be extended to the superstability of (1) on the commutative semisimple Banach algebra.

Theorem 3.1
Let $f\colon G \to E$ be a function and $\varphi\colon G \to [0, +\infty[$ satisfy the inequality

$$\begin{aligned} \|f(x+y+z) + f(x+y+\sigma(z)) + f(x+\sigma(y)+z) \\ + f(\sigma(x)+y+z) - 4f(x)f(y)f(z)\| \\ \leq \varphi(x) \end{aligned} \tag{9}$$

for all $x, y, z \in G$. For an arbitrary linear multiplicative functional $x^* \in E^*$, if the superposition $x^* \circ f$ is unbounded, then f satisfies (1).

Proof. Suppose that (9) holds, and fix an arbitrary linear multiplicative functional $x^* \in E^*$. Let $\|x^*\| = 1$ without loss of generality. Then, for every $x, y, z \in G$, we get

$$\begin{aligned} \varphi(x) &\geq \|f(x+y+z) + f(x+y+\sigma(z)) + f(x+\sigma(y)+z) \\ &\quad + f(\sigma(x)+y+z) - 4f(x)f(y)f(z)\| \\ &= \sup_{\|z^*\|=1} |z^*(f(x+y+z) + f(x+y+\sigma(z)) + f(x+\sigma(y)+z) \\ &\quad + f(\sigma(x)+y+z) - 4f(x)f(y)f(z))| \\ &\geq |(x^*\circ f)(x+y+z) + (x^*\circ f)(x+y+\sigma(z)) + (x^*\circ f)(x+\sigma(y)+z) \\ &\quad + (x^*\circ f)(\sigma(x)+y+z) - 4(x^*\circ f)(x)(x^*\circ f)(y)(x^*\circ f)(z))| \end{aligned}$$

which states that the superposition $x^* \circ f\colon G \to \mathbb{C}$ yields a solution of the inequality (2), since the superposition $x^* \circ f$ is unbounded, Theorem 2.2 shows that the superposition $x^* \circ f$ is a solution of equation (1). In other words, bearing the linear multiplicativity of x^* in mind, for all $x, y, z \in G$, the difference $D_\sigma f(x, y, z)\colon G \times G \times G \to \mathbb{C}$ falls into the kernel of x^*, where $D_\sigma f(x, y, z) := f(x+y+z) + f(x+y+\sigma(z)) + f(x+\sigma(y)+z) + f(\sigma(x)+y+z) - 4f(x)f(y)f(z)$. Therefore, in view of the unrestricted choice of x^*, we infer that

$$D_\sigma f(x, y, z) \in \bigcap \{\mathrm{Ker}\, x^* : \ x^* \text{ is a linear multiplicative member of } E^*\}$$

for all $x, y, z \in G$. Since the algebra E has been assumed to be semisimple, the last term of the above formula coincides with the singleton $\{0\}$, i.e

$$\forall x, y, z \in G \quad D_\sigma f(x, y, z) = 0$$

as claimed. This completes the proof.

Remark 3.2
By the similar manner, we can prove that if the difference $D_\sigma f(x, y, z)$ is bounded by $\varphi(y)$ or $\varphi(z)$, we obtain the same result as in Theorem 3.1.

Corollary 3.3
Let $\varphi\colon G \to [0, +\infty[$ be a function and $f\colon G \to E$ satisfy the inequality

$$\|f(x+y+z) + f(x+y-z) + f(x-y+z) + f(y+z-x) - 4f(x)f(y)f(z)\| \\ \le \varphi(x) \text{ or } \varphi(y) \text{ or } \varphi(z)$$

for all $x, y, z \in G$. For an arbitrary linear multiplicative functional $x^* \in E^*$, if the superposition $x^* \circ f$ is unbounded, then f satisfies

$$f(x+y+z) + f(x+y-z) + f(x-y+z) + f(y+z-x) = 4f(x)f(y)f(z)$$

for all $x, y, z \in G$.

Theorem 3.4
Let $\varphi\colon \mathbb{R} \to [0, +\infty[$ and $f\colon \mathbb{R} \to \mathbb{C}$ satisfy the inequality

$$\begin{aligned} |f(x+y+z) &+ f(x+y+\sigma(z)) + f(x+\sigma(y)+z) \\ &+ f(\sigma(x)+y+z) - 4f(x)f(y)f(z)| \\ \le \varphi(x) \end{aligned}$$

for all $x, y, z \in \mathbb{R}$. If f is an unbounded harmonic function, then there is a constant $\alpha \in \mathbb{C}^* \setminus i\mathbb{R}$ such that $f(x) = \frac{e^{\alpha x} + e^{\alpha\sigma(x)}}{2}$ and f is a solution of the equation (A_f).

Proof. By Theorem 2.2, f satisfies the functional equation (1). Suppose that f is unbounded and $f(0) = 0$. Putting $y = z := 0$ in (1), we get

$$3f(x) + f(\sigma(x)) = 4f(x)f(0)^2 = 0 \tag{10}$$

for all $x \in \mathbb{R}$. Replacing x by $\sigma(x)$ in (10) and then combining the equalities, we see that $f(\sigma(x)) = -f(x)$ for all $x \in \mathbb{R}$, so $f(x) = 0$ for all $x \in \mathbb{R}$. This is a contradiction. Therefore, $|f(0)| > 0$. Hence, f satisfies also the equation (A_f) by Lemma 2.1. It is well known that a harmonic solution $f\colon \mathbb{R} \to \mathbb{C}$ of the d'Alembert functional equation (A_f) has to have the form $f(x) = \frac{e^{\alpha x} + e^{\alpha\sigma(x)}}{2}$ for all $x \in \mathbb{R}$, where α is a complex number (see [17, Theorem 1]). Since f is unbounded, the constant α of that form falls into the set $\alpha \in \mathbb{C}^* \setminus i\mathbb{R}$. This completes the proof.

Remark 3.5
Similarly, one can prove that if the difference $D_\sigma f(x, y, z)$ is bounded by $\varphi(y)$ or $\varphi(z)$, we obtain the same result as in Theorem 3.4.

Corollary 3.6
Assume that $\varphi\colon \mathbb{R} \to [0, +\infty[$ and $f\colon \mathbb{R} \to \mathbb{C}$ satisfy the inequality

$$|f(x+y+z) + f(x+y-z) + f(x-y+z) + f(y+z-x) - 4f(x)f(y)f(z)| \\ \le \varphi(x) \text{ or } \varphi(y) \text{ or } \varphi(z)$$

for all $x, y, z \in \mathbb{R}$. If f is an unbounded harmonic function, then there is a constant $\beta \in \mathbb{C} \setminus \mathbb{R}$ such that $f(x) = \cos(\beta x)$ and f is a solution of the equation (A).

Proof. It suffices to take $\sigma(x) = -x$ and $\beta = i\alpha$ in Theorem 3.4.

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