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Affine analogues of the Sasaki-Shchepetilov connection

Abstract. For two-dimensional manifold M with locally symmetric connection ∇ and with ∇-parallel volume element vol one can construct a flat connection on the vector bundle $TM \oplus E$, where E is a trivial bundle. The metrizable case, when M is a Riemannian manifold of constant curvature, together with its higher dimension generalizations, was studied by A.V. Shchepetilov [J. Phys. A: **36** (2003), 3893-3898]. This paper deals with the case of non-metrizable locally symmetric connection. Two flat connections on $TM \oplus (\mathbb{R} \times M)$ and two on $TM \oplus (\mathbb{R}^2 \times M)$ are constructed. It is shown that two of those connections – one from each pair – may be identified with the standard flat connection in $\mathbb{R}^N$, after suitable local affine embedding of (M, ∇) into $\mathbb{R}^N$.

1. Introduction

In the article [9] R. Sasaki proposed to add the property of describing pseudospherical surfaces to other remarkable properties – such as applicability of the inverse scattering method, infinite number of conservation laws and Bäcklund transformations – which characterize soliton equations in $1+1$ dimensions. He expressed the $\mathbf{sl}(2, \mathbb{R})$-valued 1-form Ω, which arises in the corresponding linear scattering problem $dv = \Omega v$, $v = \binom{v_1}{v_2}$, by 1-forms ω^1, ω^2 and ${\omega^2}_1$

$$\Omega = \begin{pmatrix} -\frac{1}{2}\omega^2 & \frac{1}{2}({\omega^2}_1 + \omega^1) \\ \frac{1}{2}(-{\omega^2}_1 + \omega^1) & \frac{1}{2}\omega^2 \end{pmatrix}$$

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in such a way, that the integrability condition $d\Omega - \Omega \wedge \Omega = 0$ is equivalent to the structural equations $d\omega^1 = \omega^2{}_1 \wedge \omega^2$, $d\omega^2 = -\omega^2{}_1 \wedge \omega^1$ and $d\omega^2{}_1 = \omega^1 \wedge \omega^2$ of a pseudospherical surface ($K = -1$). This $\mathbf{sl}(2, \mathbb{R})$-valued 1-form Ω itself can be interpreted as the connection form of a connection on some principal $SL(2, \mathbb{R})$-bundle. The condition $d\Omega - \Omega \wedge \Omega = 0$ means that the curvature of this connection vanishes. In this respect the connection Ω differs from the Levi-Civita connection of the considered pseudospherical metric. On the other hand, Ω appeared to be somehow related to the Levi-Civita connection, because the Levi-Civita connection form $\begin{pmatrix} 0 & -\omega^2{}_1 \\ \omega^2{}_1 & 0 \end{pmatrix}$ "is contained" in Ω. As might be expected, the question of finding the geometric interpretation of Ω occurred.

In the paper [10] A.V. Shchepetilov explained the geometric meaning of the Sasaki connection. Using an equivalent representation of Ω, $\mathbf{so}(2, 1)$-valued, he constructed a flat connection $\widehat{\nabla}$ on the vector bundle $TM \oplus E$, where TM is the tangent bundle and $E = \mathbb{R} \times M$ is a trivial one-dimensional vector bundle (our notation is slightly different from that in [10])

$$\widehat{\nabla}_X(Y \oplus f) = \big(\nabla_X Y + fX\big) \oplus \big(X(f) + g(X, Y)\big). \tag{1}$$

Here g is a metric on M, ∇ is its Levi-Civita connection, $f \in \mathcal{C}^\infty(M)$ is a section of E and X, Y are vector fields on M.

Shchepetilov considered also manifolds with metric of constant positive curvature $K = +1$. The corresponding flat connection $\widehat{\nabla}$ on $TM \oplus E$ is

$$\widehat{\nabla}_X(Y \oplus f) = \big(\nabla_X Y + fX\big) \oplus \big(X(f) - g(X, Y)\big). \tag{2}$$

The aim of this paper is to construct a similar flat connection $\widehat{\nabla}$ for a two-dimensional manifold with non-metrizable locally symmetric connection ∇ and with ∇-parallel volume element. Our main motivation for research is as follows. Firstly, manifold with locally symmetric linear connection can be thought of as a generalization of a constant sectional curvature Riemannian manifold. Secondly, sometimes more important than (M, g) or (M, ∇) alone is an embedding of M into $\mathbb{R}^3$. For example, every isometric embedding of a pseudospherical surface (M, g) into $\mathbb{R}^3$ corresponds to some particular solution of the sine-Gordon equation. Therefore restriction to those non-flat locally symmetric connections which are induced on hypersurfaces in $\mathbb{R}^{n+1}$ is legitimated. If such hypersurface f is degenerate and its type number r is greater than 1, then around each generic point of M there exists a local cylinder decomposition which contains as a part a non-degenerate hypersurface in $\mathbb{R}^{r+1}$ with some locally symmetric connection (see [4]). On the other hand, if f is non-degenerate and $n > 2$, then ∇ is the Blaschke connection, $\nabla h = 0$, $S = \rho\,\mathrm{id}$, $\rho = \mathrm{const}$, $\rho \neq 0$ and $f(M)$ is an open part of a quadric with center [4]. Similarly as in the second proof of Berwald theorem in [3] one can then define a pseudo-scalar product G in $\mathbb{R}^{n+1}$ such that $G(f_*X, f_*Y) = h(X, Y)$, $G(f_*X, \xi) = 0$ and $G(\xi, \xi) = \rho$, where ξ is the affine normal. It is easy to check that relative to this pseudo-scalar product f is a hypersurface of constant sectional curvature ρ. If f is non-degenerate, $n = 2$ and the induced locally symmetric connection satisfies the condition $\dim \operatorname{im} R = 2$, then there also exists a pseudo-scalar product on $\mathbb{R}^{n+1} = \mathbb{R}^3$ relative to which f has constant Gaussian curvature and ξ is perpendicular to f [6].

On the contrary, if $f\colon M \to \mathbb{R}^{n+1}$ is of type number 1 or if $f\colon M \to \mathbb{R}^3$ is nondegenerate and $\dim \operatorname{im} R = 1$, then the connection as a connection of 1-codimensional nullity ($\dim \ker R = n-1$) is not metrizable [7], therefore we have reason for generalizing Shchepetilov's construction. The present paper deals with the case $n = 2$.

2. Preliminaries

Let M be a connected two-dimensional real manifold and let ∇ be a locally symmetric connection on M, satisfying the condition $\dim \operatorname{im} R = 1$, where for $p \in M$

$$\operatorname{im} R|_p := \operatorname{span}\{R(X,Y)Z : \ X, Y, Z \in T_pM\}$$

and R is the curvature tensor of ∇. Such connections were studied by B. Opozda in [5]. Opozda proved that for every $p \in M$ there is a coordinate system (u, v) around p such that

$$\nabla_{\partial_u}\partial_u = \nabla_{\partial_u}\partial_v = 0 \qquad \text{and} \qquad \nabla_{\partial_v}\partial_v = \varepsilon u \partial_u, \tag{3}$$

where $\varepsilon \in \{1, -1\}$. A local coordinate system in which a locally symmetric connection ∇ is expressed by (3) will be called a canonical coordinate system for ∇ [5]. It is not unique. It is easy to check that if u, v and $\overline{u}, \overline{v}$ are canonical coordinate systems then on each connected component of the intersection of their domains we have $\overline{u} = Au + \chi(v)$, $\overline{v} = \delta v + B$, where A, B, δ are constants, $\delta^2 = 1$, and χ satisfies the differential equation $\chi'' + \varepsilon\chi = 0$.

The Ricci tensor $\operatorname{Ric}(X, Y) := \operatorname{trace}[V \mapsto R(V, X)Y]$ of such a connection is symmetric and for every $p \in M$ there exists a ∇-parallel volume element around p. Here we assume that a ∇-parallel volume element vol exists on the whole M.

It follows, that for every $p \in M$ we can find around p a local basis (X_1, X_2) of TM, satisfying the conditions:

$$X_1 \in \ker \operatorname{Ric}, \qquad \operatorname{Ric}(X_2, X_2) = \varepsilon \qquad \text{and} \qquad \operatorname{vol}(X_1, X_2) = 1. \tag{4}$$

For example, on the domain of canonical coordinates (u, v) as in (3) we may take $X_1 = \frac{1}{c}\partial_u$ and $X_2 = \partial_v$, where c is the non-zero constant such that $\operatorname{vol} = c\, du \wedge dv$. Let ω^1, ω^2 be the dual basis for (X_1, X_2). The local connection form is $(\omega^i{}_j) = \left(\begin{smallmatrix} 0 & \omega^1{}_2 \\ 0 & 0 \end{smallmatrix}\right)$ and the structural equations are $d\omega^1 = -\omega^1{}_2 \wedge \omega^2$, $d\omega^2 = 0$ and $d\omega^1{}_2 = \varepsilon\omega^1 \wedge \omega^2$.

The following proposition is easy to check.

Proposition 2.1
Let M be a two-dimensional manifold with locally symmetric connection ∇ satisfying condition $\dim \operatorname{im} R = 1$. Let ω^1, ω^2 and $\omega^i{}_j$ be the dual basis and the local connection forms for some local basis of TM satisfying the condition (4). Then each of the following four 1-forms Ω_i

$$\Omega_1 = \begin{pmatrix} 0 & -\omega^1{}_2 & \omega^1 \\ 0 & 0 & \omega^2 \\ 0 & -\varepsilon\omega^2 & 0 \end{pmatrix}, \qquad \Omega_2 = \begin{pmatrix} 0 & -\omega^1{}_2 & \varepsilon\omega^2 \\ 0 & 0 & 0 \\ -\omega^2 & \omega^1 & 0 \end{pmatrix},$$

$$\Omega_3 = \begin{pmatrix} 0 & -\omega^1{}_2 & \omega^1 & \omega^2 \\ 0 & 0 & \omega^2 & 0 \\ 0 & -\varepsilon\omega^2 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \qquad \Omega_4 = \begin{pmatrix} 0 & -\omega^1{}_2 & \varepsilon\omega^2 & 0 \\ 0 & 0 & 0 & 0 \\ -\omega^2 & \omega^1 & 0 & 0 \\ 0 & \omega^2 & 0 & 0 \end{pmatrix}$$

satisfies the condition $d\Omega_i - \Omega_i \wedge \Omega_i = 0$.

Those $\mathbf{gl}(N,\mathbb{R})$-valued ($N = 3$ or $N = 4$) 1-forms were obtained in [8] as the local connection forms of connections on some principal $GL(N,\mathbb{R})$-bundle P and seem to be analogous to the Sasaki connection form. The bundle $P(M,G)$, $G = GL(N,\mathbb{R})$, is an extension of the bundle $Q(M,H)$ consisting of all linear frames on M which satisfy (4). The structure group is $H := \{\left(\begin{smallmatrix} 1 & t \\ 0 & 1 \end{smallmatrix}\right) : t \in \mathbb{R}\} \cup \{\left(\begin{smallmatrix} -1 & t \\ 0 & -1 \end{smallmatrix}\right) : t \in \mathbb{R}\}$. Here we need not explain what the bundle $P(M,G)$ is. It suffices to know that there exists $f\colon Q \to P$ such that the triple $(f, \mathrm{id}_M, \iota)$ is a homomorphism of principal fibre bundles $Q(M,H)$ and $P(M,G)$. The homomorphism $\iota\colon H \to G$ of structure groups is given by $\iota(a) := \left(\begin{smallmatrix} a & 0 \\ 0 & I_{N-2} \end{smallmatrix}\right)$, where I_{N-2} is the identity $(N-2)\times(N-2)$ matrix. Each of the forms Ω_i is a local connection form associated with a local section $f \circ \sigma$ of P, where σ is some local section of Q.

In the construction of P and Ω in [8] and in the present paper we consider the left action of H on Q: $a * q := qa^{-1}$, where $(v_1, v_2)h := (h^1{}_1 v_1 + h^2{}_1 v_2, h^1{}_2 v_1 + h^2{}_2 v_2)$ for $h = \left(\begin{smallmatrix} h^1{}_1 & h^1{}_2 \\ h^2{}_1 & h^2{}_2 \end{smallmatrix}\right) \in H$, and some left action of G on P. Another possible way is to consider traditionally a right action, but we have then $-\Omega$ instead of Ω.

3. The connections on the vector bundle $TM \oplus E$

We will use the definition of the covariant derivative of a section of an associated bundle which comes from [1], and is described for example in [2]. Since we consider here the left action of G on P and the right action of G on $\mathbb{R}^N$, $z * c := c^{-1}z$, some details may be different from that of [1] and [2].

Let TM be the tangent bundle of M and let E be the trivial bundle, $E = \mathbb{R}^{N-2} \times M$.

Proposition 3.1
The bundle $TM \oplus E$ is a vector bundle associated to P with fibre $\mathbb{R}^N$

$$P \times_G \mathbb{R}^N = (P \times \mathbb{R}^N)/\sim,$$

with the equivalence relation $\sim$ given by $(cp, z * c^{-1}) \sim (p, z)$.

Proof. For $x \in M$ we take a basis $q = (v_1, v_2) \in Q$ of T_xM and identify $(z^1 v_1 + z^2 v_2) \oplus (z^3, \ldots, z^N)$ from $(TM \oplus E)|_x$ with $[(f(q), z)] \in (P \times \mathbb{R}^N)/\sim$. This identification is correct, because if we take another basis $q' = (v_1', v_2') \in Q_x$, then $q' = a * q = qa^{-1}$ for some $a \in H$ and $z^1 v_1 + z^2 v_2 = z'^1 v_1' + z'^2 v_2'$ with $z'^1 = a^1{}_1 z^1 + a^1{}_2 z^2$, $z'^2 = a^2{}_1 z^1 + a^2{}_2 z^2$. It follows that $(z'^1 v_1' + z'^2 v_2') \oplus (z'^3, \ldots, z'^N) = (z^1 v_1 + z^2 v_2) \oplus (z^3, \ldots, z^N)$ for $z' = \iota(a)z = z * (\iota(a))^{-1}$. We obtain $[(f(q'), z')] = [(f(a * q), z * (\iota(a))^{-1})] = [(\iota(a)f(q), z * (\iota(a))^{-1})] = [(f(q), z)]$.

Let $[(p, z)] \in P \times_G \mathbb{R}^N$ and let $\pi(p) = x$, where $\pi\colon P \to M$. Let $q = (v_1, v_2) \in Q_x$, then $f(q) \in P_x$. Since G acts transitively on fibres of P, there exists $b \in G$

such that $p = bf(q)$. It follows that $[(p,z)] = [(bf(q),z)] = [(bf(q),(z*b)*b^{-1})] = [(f(q),z*b)] = [(f(q),b^{-1}z)]$, therefore we have to identify $[(p,z)]$ with $(y^1v_1 + y^2v_2) \oplus (y^3,\dots,y^N)$, where $y = b^{-1}z$.

To each local section η of an associated vector bundle $P \times_G \mathbb{R}^N$ corresponds some mapping $\widetilde{\eta}\colon P|_U \to \mathbb{R}^N$ – called the Crittenden mapping – which satisfies the condition $\widetilde{\eta}(bp) = \widetilde{\eta}(p) * b^{-1}$. Since we have actually defined the right action of G on $\mathbb{R}^N$ using the left action, $x * c := c^{-1}x$, we can write this condition simply as $\widetilde{\eta}(bp) = b\widetilde{\eta}(p)$. By definition of the Crittenden mapping, $[(p,\widetilde{\eta}(p))] = \eta(\pi(p))$. Conversely, to each mapping $\widetilde{\eta}\colon P|_U \to \mathbb{R}^N$ satisfying the condition $\widetilde{\eta}(b*p) = \widetilde{\eta}(p) * b^{-1}$ corresponds a local section of the associated bundle.

Let X be a vector field on M. For every connection form Ω_i from Proposition 2.1 we will find the covariant derivative $\widehat{\nabla}_X\eta$ of a local section η of $TM \oplus E$.

Theorem 3.2

Let $\eta = Y \oplus \Psi$, with a vector field Y on $U \subset M$ and $\Psi\colon U \to \mathbb{R}^{N(i)}$, be a local section of $TM \oplus E$. Here $N(1) = N(2) = 1$ and $N(3) = N(4) = 2$. Let $\widehat{\nabla}^i_X\eta$ denote the covariant derivative of η with respect to the connection corresponding to local connection form Ω_i from Proposition 2.1. Then

$$\begin{aligned}
&\widehat{\nabla}^1_X(Y \oplus \Psi) = \big(\nabla_X Y - \Psi X\big) \oplus \big(X(\Psi) + \mathrm{Ric}(X,Y)\big),\\
&\widehat{\nabla}^2_X(Y \oplus \Psi) = \big(\nabla_X Y - \Psi LX\big) \oplus \big(X(\Psi) - \mathrm{vol}(X,Y)\big),\\
&\widehat{\nabla}^3_X(Y \oplus (\Psi^1,\Psi^2))\\
&\qquad\qquad = \big(\nabla_X Y - \Psi^1 X - \varepsilon\Psi^2 LX\big) \oplus \big(X(\Psi^1) + \mathrm{Ric}(X,Y), X(\Psi^2)\big)
\end{aligned}$$

and

$$\begin{aligned}
&\widehat{\nabla}^4_X(Y \oplus (\Psi^1,\Psi^2))\\
&\qquad = \big(\nabla_X Y - \Psi^1 LX\big) \oplus \big(X(\Psi^1) - \mathrm{vol}(X,Y), X(\Psi^2) - \varepsilon\,\mathrm{Ric}(X,Y)\big),
\end{aligned}$$

with the $(1,1)$ tensor field L such that $\mathrm{vol}(LX,Y) = \mathrm{Ric}(X,Y)$ for every X, Y.

Proof. By definition of the covariant derivative, the Crittenden mapping corresponding to $\widehat{\nabla}_X\eta$ is equal to $X^H(\widetilde{\eta})$, where X^H is the horizontal lift of X to $P|_U$.

We use a local section $\tau = f \circ \sigma$ of P, where $\sigma = (V_1,V_2)$ is a local section of Q. Let $Y = Y^1V_1 + Y^2V_2$, then $\widetilde{\eta} \circ \tau = (Y^1,Y^2,\Psi)$.

Let $\widetilde{\Omega}$ be the connection form on P. The local connection form is $\tau^*\widetilde{\Omega} = \Omega_\sigma$. We have

$$\widetilde{\widehat{\nabla}_X\eta}(\tau(x)) = X^H_{\tau(x)}(\widetilde{\eta}), \qquad X^H_{\tau(x)} = d_x\tau(X_x) + B^*_{\tau(x)},$$

where the right-invariant vector field $B = -\Omega_\sigma(X_x)$, which we easily obtain from the condition $\widetilde{\Omega}(X^H_{\tau(x)}) = 0$:

$$0 = \widetilde{\Omega}_{\tau(x)}(d_x\tau(X_x)) + \widetilde{\Omega}_{\tau(x)}(B^*_{\tau(x)}) = (\tau^*\widetilde{\Omega})_x(X_x) + B = \Omega_\sigma(X_x) + B.$$

The first part of $X^H_{\tau(x)}(\widetilde{\eta})$ is equal to

$$(d_x\tau(X_x))(\widetilde{\eta}) = X_x(\widetilde{\eta} \circ \tau) = (X_x(Y^1), X_x(Y^2), X_x(\Psi)).$$

The second part is

$$B^*_{\tau(x)}(\widetilde{\eta}) = \frac{d}{dt}\widetilde{\eta}(b_t\tau(x))\Big|_{t=0} = \frac{d}{dt}b_t\widetilde{\eta}(\tau(x))\Big|_{t=0} = \frac{d}{dt}b_t\Big|_{t=0}\widetilde{\eta}(\tau(x)) = B\widetilde{\eta}(\tau(x)).$$

Here (b_t) is 1-parameter subgroup of G generated by B. It follows that

$$\widetilde{\widehat{\nabla}_X\eta}(\tau(x)) = \begin{pmatrix} X_x(Y^1) \\ X_x(Y^2) \\ X_x(\Psi) \end{pmatrix} - \Omega_\sigma(X_x)\begin{pmatrix} Y^1(x) \\ Y^2(x) \\ \Psi(x) \end{pmatrix}. \tag{5}$$

For $\Omega_\sigma = \Omega_1$ we obtain

$$\widetilde{\widehat{\nabla}_X\eta}\circ\tau = \begin{pmatrix} X(Y^1) \\ X(Y^2) \\ X(\Psi) \end{pmatrix} - \begin{pmatrix} 0 & -\omega^1{}_2(X) & \omega^1(X) \\ 0 & 0 & \omega^2(X) \\ 0 & -\varepsilon\omega^2(X) & 0 \end{pmatrix}\begin{pmatrix} Y^1 \\ Y^2 \\ \Psi \end{pmatrix}$$

and

$$\begin{aligned} \widehat{\nabla}_X\eta = &\big((X(Y^1) + \omega^1{}_2(X)Y^2 - \omega^1(X)\Psi)V_1 + (X(Y^2) - \omega^2(X)\Psi)V_2\big) \\ &\oplus \big(X(\Psi) + \varepsilon\omega^2(X)Y^2\big). \end{aligned}$$

Since $\nabla_X V_1 = 0$, we have

$$\begin{aligned} \nabla_X Y &= \nabla_X(Y^1V_1 + Y^2V_2) \\ &= X(Y^1)V_1 + Y^1\nabla_X V_1 + X(Y^2)V_2 + Y^2\nabla_X V_2 \\ &= X(Y^1)V_1 + X(Y^2)V_2 + Y^2\omega^1{}_2(X)V_1. \end{aligned}$$

We have also

$$\begin{aligned} \mathrm{Ric}(X,Y) &= \mathrm{Ric}(\omega^1(X)V_1 + \omega^2(X)V_2, Y^1V_1 + Y^2V_2) \\ &= \omega^2(X)Y^2\,\mathrm{Ric}(V_2,V_2) \\ &= \omega^2(X)Y^2\varepsilon, \end{aligned}$$

because V_1 is a local section of $\ker\mathrm{Ric}$.

We obtain finally

$$\widehat{\nabla}_X(Y\oplus\Psi) = \big(\nabla_X Y - \Psi X\big)\oplus\big(X(\Psi) + \mathrm{Ric}(X,Y)\big). \tag{6}$$

If we take $\Omega_\sigma = \Omega_2$, then we obtain from (5)

$$\widetilde{\widehat{\nabla}_X\eta}\circ\tau = \begin{pmatrix} X(Y^1) \\ X(Y^2) \\ X(\Psi) \end{pmatrix} - \begin{pmatrix} 0 & -\omega^1{}_2(X) & \varepsilon\omega^2(X) \\ 0 & 0 & 0 \\ -\omega^2(X) & \omega^1(X) & 0 \end{pmatrix}\begin{pmatrix} Y^1 \\ Y^2 \\ \Psi \end{pmatrix},$$

which gives

$$\begin{aligned} \widehat{\nabla}_X(Y\oplus\Psi) = &\big((X(Y^1) + \omega^1{}_2(X)Y^2 - \varepsilon\omega^2(X)\Psi)V_1 + X(Y^2)V_2\big) \\ &\oplus\big(X(\Psi) + \omega^2(X)Y^1 - \omega^1(X)Y^2\big) \end{aligned}$$

$$= \big(\nabla_X Y - \Psi\varepsilon\omega^2(X)V_1\big) \oplus \big(X(\Psi) - \mathrm{vol}(X,Y)\big),$$

because $\mathrm{vol}(V_1, V_2) = 1$.

Let $(\widetilde{V}_1, \widetilde{V}_2)$ be another local basis of TM satisfying (4). Then in the intersection of the corresponding domains we have $\widetilde{V}_1 = \delta V_1$, $\widetilde{V}_2 = tV_1 + \delta V_2$ with $\delta \in \{1, -1\}$. For the new dual basis we obtain $\widetilde{\omega}^1 = \delta\omega^1 - t\omega^2$, $\widetilde{\omega}^2 = \delta\omega^2$. It follows that $\widetilde{\omega}^2\widetilde{V}_1 = \omega^2 V_1$, therefore the vector field $LX := \varepsilon\omega^2(X)V_1$ is defined on the whole M and L is a $(1,1)$ tensor field.

Note that for every Z we have

$$\begin{aligned} \mathrm{vol}(LX, Z) &= \mathrm{vol}(\varepsilon\omega^2(X)V_1, Z) = \varepsilon\omega^2(X)\omega^2(Z)\,\mathrm{vol}(V_1, V_2) = \varepsilon\omega^2(X)\omega^2(Z) \\ &= \mathrm{Ric}(X, Z). \end{aligned} \tag{7}$$

For the second connection we finally obtain the global formula

$$\widehat{\nabla}_X\big(Y \oplus \Psi\big) = \big(\nabla_X Y - \Psi LX\big) \oplus \big(X(\Psi) - \mathrm{vol}(X,Y)\big). \tag{8}$$

For $\Omega_\sigma = \Omega_3$ we have

$$\widetilde{\widehat{\nabla}_X \eta} \circ \tau = \begin{pmatrix} X(Y^1) \\ X(Y^2) \\ X(\Psi^1) \\ X(\Psi^2) \end{pmatrix} - \begin{pmatrix} 0 & -\omega^1{}_2(X) & \omega^1(X) & \omega^2(X) \\ 0 & 0 & \omega^2(X) & 0 \\ 0 & -\varepsilon\omega^2(X) & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} Y^1 \\ Y^2 \\ \Psi^1 \\ \Psi^2 \end{pmatrix},$$

hence

$$\begin{aligned} &\widehat{\nabla}_X(Y \oplus (\Psi^1, \Psi^2)) \\ &\quad = \big((X(Y^1) + \omega^1{}_2(X)Y^2 - \omega^1(X)\Psi^1 - \omega^2(X)\Psi^2)V_1 + (X(Y^2) - \omega^2(X)\Psi^1)V_2\big) \\ &\qquad \oplus \big(X(\Psi^1) + \varepsilon\omega^2(X)Y^2, X(\Psi^2)\big), \end{aligned}$$

which gives

$$\begin{aligned} &\widehat{\nabla}_X(Y \oplus (\Psi^1, \Psi^2)) \\ &\qquad = \big(\nabla_X Y - \Psi^1 X - \varepsilon\Psi^2 LX\big) \oplus \big(X(\Psi^1) + \mathrm{Ric}(X,Y), X(\Psi^2)\big). \end{aligned} \tag{9}$$

For $\Omega_\sigma = \Omega_4$ we obtain

$$\widetilde{\widehat{\nabla}_X \eta} \circ \tau = \begin{pmatrix} X(Y^1) \\ X(Y^2) \\ X(\Psi^1) \\ X(\Psi^2) \end{pmatrix} - \begin{pmatrix} 0 & -\omega^1{}_2(X) & \varepsilon\omega^2(X) & 0 \\ 0 & 0 & 0 & 0 \\ -\omega^2(X) & \omega^1(X) & 0 & 0 \\ 0 & \omega^2(X) & 0 & 0 \end{pmatrix} \begin{pmatrix} Y^1 \\ Y^2 \\ \Psi^1 \\ \Psi^2 \end{pmatrix}$$

and

$$\begin{aligned} &\widehat{\nabla}_X(Y \oplus (\Psi^1, \Psi^2)) \\ &\qquad = \big(\nabla_X Y - \Psi^1 LX\big) \oplus \big(X(\Psi^1) - \mathrm{vol}(X,Y), X(\Psi^2) - \varepsilon\,\mathrm{Ric}(X,Y)\big). \end{aligned} \tag{10}$$

4. Flatness of $\widehat{\nabla}$

Theorem 4.1
Each of four connections $\widehat{\nabla}^i$ in Theorem 3.2 is flat.

Proof. We will compute

$$\widehat{R}(X,Y)(Z\oplus\Psi)=(\widehat{\nabla}_X\widehat{\nabla}_Y-\widehat{\nabla}_Y\widehat{\nabla}_X-\widehat{\nabla}_{[X,Y]})(Z\oplus\Psi)$$

for each of four connections (6), (8), (9) and (10).

If we use $\nabla_X Y-\nabla_Y X-[X,Y]=T(X,Y)=0$, then for the connection (6) we obtain

$$\begin{aligned}&\widehat{R}(X,Y)(Z\oplus\Psi)\\&\quad=\big(R(X,Y)Z-(\operatorname{Ric}(Y,Z)X-\operatorname{Ric}(X,Z)Y)\big)\\&\qquad\oplus\big((\nabla_X\operatorname{Ric})(Y,Z)-(\nabla_Y\operatorname{Ric})(X,Z)-\Psi(\operatorname{Ric}(X,Y)-\operatorname{Ric}(Y,X))\big)\end{aligned}$$

But Ric is symmetric, $\nabla R=0$ implies $\nabla\operatorname{Ric}=0$, and for each two-dimensional manifold

$$R(X,Y)Z=\operatorname{Ric}(Y,Z)X-\operatorname{Ric}(X,Z)Y. \tag{11}$$

Therefore $\widehat{R}(X,Y)(Z\oplus\Psi)=0\oplus 0$.

For the connection (8) we obtain

$$\begin{aligned}&\widehat{R}(X,Y)(Z\oplus\Psi)\\&\quad=\big(R(X,Y)Z+\operatorname{vol}(Y,Z)LX-\operatorname{vol}(X,Z)LY-\Psi((\nabla_X L)Y-(\nabla_Y L)X)\big)\\&\qquad\oplus\big((\nabla_Y\operatorname{vol})(X,Z)-(\nabla_X\operatorname{vol})(Y,Z)+\Psi(\operatorname{vol}(X,LY)-\operatorname{vol}(Y,LX))\big).\end{aligned}$$

From $\nabla\operatorname{vol}=0$ it follows that $R\cdot\operatorname{vol}=0$, therefore

$$\begin{aligned}0&=(R(X,Y)\cdot\operatorname{vol})(Z,W)=-\operatorname{vol}(R(X,Y)Z,W)-\operatorname{vol}(Z,R(X,Y)W)\\&=-\operatorname{vol}(R(X,Y)Z,W)+\operatorname{vol}(R(X,Y)W,Z),\end{aligned}$$

hence

$$\operatorname{vol}(R(X,Y)Z,W)=\operatorname{vol}(R(X,Y)W,Z). \tag{12}$$

For an arbitrary vector field W using (12), (7) and (11) we obtain

$$\begin{aligned}&\operatorname{vol}(R(X,Y)Z+\operatorname{vol}(Y,Z)LX-\operatorname{vol}(X,Z)LY,W)\\&\quad=\operatorname{vol}(R(X,Y)W,Z)+\operatorname{vol}(Y,Z)\operatorname{Ric}(X,W)-\operatorname{vol}(X,Z)\operatorname{Ric}(Y,W)\\&\quad=\operatorname{vol}(R(X,Y)W+\operatorname{Ric}(X,W)Y-\operatorname{Ric}(Y,W)X,Z)\\&\quad=0.\end{aligned}$$

From the non-degeneracy of vol it follows that

$$R(X,Y)Z+\operatorname{vol}(Y,Z)LX-\operatorname{vol}(X,Z)LY=0. \tag{13}$$

Moreover, $\nabla\operatorname{Ric}=0$, $\nabla\operatorname{vol}=0$ and (7) imply $\nabla L=0$. We have also $\operatorname{vol}(X,LY)-\operatorname{vol}(Y,LX)=-\operatorname{vol}(LY,X)+\operatorname{vol}(LX,Y)=-\operatorname{Ric}(Y,X)+\operatorname{Ric}(X,Y)=0$. Hence $\widehat{R}(X,Y)(Z\oplus\Psi)=0\oplus 0$ for $\widehat{\nabla}$ given by (8).

For the connection (9) we obtain

$$
\begin{aligned}
&\widehat{R}(X,Y)(Z\oplus(\Psi^1,\Psi^2))\\
&\quad=\big(R(X,Y)Z-\mathrm{Ric}(Y,Z)X+\mathrm{Ric}(X,Z)Y-\varepsilon\Psi^2((\nabla_X L)(Y)-(\nabla_Y L)(X))\big)\\
&\qquad\oplus\big((\nabla_X \mathrm{Ric})(Y,Z)-(\nabla_Y \mathrm{Ric})(X,Z)-\Psi^1(\mathrm{Ric}(X,Y)-\mathrm{Ric}(Y,X))\\
&\qquad\quad+\varepsilon\Psi^2(\mathrm{Ric}(Y,LX)-\mathrm{Ric}(X,LY)),0\big)\\
&\quad=0\oplus(0,0).
\end{aligned}
$$

Note that $\operatorname{im} L\subset\ker\mathrm{Ric}$.

For (10) we have

$$
\begin{aligned}
&\widehat{R}(X,Y)(Z\oplus(\Psi^1,\Psi^2))\\
&\quad=\big(R(X,Y)Z+\mathrm{vol}(Y,Z)LX-\mathrm{vol}(X,Z)LY-\Psi^1((\nabla_X L)(Y)-(\nabla_Y L)(X))\big)\\
&\qquad\oplus\big((\nabla_Y \mathrm{vol})(X,Z)-(\nabla_X \mathrm{vol})(Y,Z)+\Psi^1(\mathrm{vol}(X,LY)-\mathrm{vol}(Y,LX)),\\
&\qquad\quad\varepsilon(\nabla_Y \mathrm{Ric})(X,Z)-\varepsilon(\nabla_X \mathrm{Ric})(Y,Z)+\varepsilon\Psi^1(\mathrm{Ric}(X,LY)-\mathrm{Ric}(Y,LX))\big)\\
&\quad=0\oplus(0,0).
\end{aligned}
$$

5. Some remarks about interpretation of $\widehat{\nabla}$

As is shown in [10], in the metric case using (at least local) embedding of (M,g) with $K=\pm1$ into Euclidean or pseudoeuclidean space $\mathbf{E}$ we may identify $\widehat{\nabla}$ with the restriction of the flat connection on $T\mathbf{E}=\mathbf{E}\times\mathbf{E}$ to $\mathbf{E}\times M$ and identify the trivial one-dimensional summand E with the normal bundle of the surface.

We consider now the case of non-metrizable locally symmetric connection on M, $\dim M=2$. Let $f\colon M\to\mathbb{R}^3$ be an immersion and let ∇ be the connection induced on M by f and the transversal vector field ξ. If we identify the bundle $f_*(TM)\oplus\mathbb{R}\,\xi$ with $TM\oplus E$, then to the vector field $f_*(Y)+\Psi\xi$ corresponds the section $Y\oplus\Psi$ of $TM\oplus E$. The Gauss and Weingarten formulae yield that to $D_X(f_*Y+\Psi\xi)$ corresponds

$$
\widehat{D}_X(Y\oplus\Psi)=\big(\nabla_X Y-\Psi SX\big)\oplus\big(X(\Psi)+h(X,Y)+\Psi\tau(X)\big), \tag{14}
$$

where h is the affine fundamental form, S is the shape operator and τ is the transversal connection form (see [3] for the definitions). We look for f and ξ such that $\widehat{D}=\widehat{\nabla}$. Comparing the right-hand side of (14) with that of (6) and (8) for the section $0\oplus1$ gives $\tau=0$, which means that we may restrict ourselves to equiaffine transversal vector fields.

Furthermore, since h is always symmetric and vol is anti-symmetric, we see that there are no f and ξ which allow to identify in the above described way the connection (8) with the standard connection D on the bundle $\mathbb{R}^3\times M$.

As concerns (6), it should be $h=\mathrm{Ric}$, which implies that we should consider some realization of ∇ on a degenerate surface f with the type number tf equal to 1. Such realizations were described by B. Opozda in [7]. Using a general description given in Proposition 6.2 of [7] and claiming that $\xi = -f$, we easily obtain the following particular local realizations of ∇

$$f(u,v) = (u, \cos v, \sin v) \in \mathbb{R}^3 \qquad \text{for } \varepsilon = 1 \tag{15}$$

and

$$f(u,v) = \Big(u, \frac{\sqrt{2}}{2}e^{-v}, \frac{\sqrt{2}}{2}e^{v}\Big) \in \mathbb{R}^3 \qquad \text{for } \varepsilon = -1. \tag{16}$$

Here u, v is some fixed local canonical coordinate system for ∇. The volume element $\mathrm{vol} = du \wedge dv$ is the element induced by (f, ξ) from $\mathbb{R}^3$.

For a centro-affine immersion $(f, \xi = -f)$ and $n = 2$ we have $SX = X$ and $\mathrm{Ric}(X,Y) = h(X,Y)\mathrm{tr}\, S - h(SX,Y) = (n-1)h(X,Y) = h(X,Y)$. It follows that using the immersion (15) or (16) we may identify (6) with the standard connection D.

To obtain $\widehat{\nabla} = \widehat{D}$ for $\widehat{\nabla}$ given by (9) we also choose and fix some local canonical coordinate system u, v for ∇ and use for example the immersion $f\colon M \to \mathbb{R}^4$, $f(u,v) = (u, \cos v, \sin v, 0)$ if $\varepsilon = 1$ and $f(u,v) = (u, \frac{\sqrt{2}}{2}e^{-v}, \frac{\sqrt{2}}{2}e^{v}, 0)$ if $\varepsilon = -1$, and the two-dimensional transversal bundle spanned by $\xi_1(u,v) = -f(u,v)$ and $\xi_2(u,v) = (-v, 0, 0, 1)$. The induced connection (which is equal to ∇), the affine fundamental forms h^1, h^2, the shape operators S_1, S_2, and the normal connection forms ${\tau^i}_j$ are defined by the following decompositions (cf [3])

$$\begin{aligned} D_X f_* Y &= f_* \nabla_X Y + h^1(X,Y)\xi_1 + h^2(X,Y)\xi_2, \\ D_X \xi_1 &= -f_* S_1 X + {\tau^1}_1(X)\xi_1 + {\tau^2}_1(X)\xi_2, \\ D_X \xi_2 &= -f_* S_2 X + {\tau^1}_2(X)\xi_1 + {\tau^2}_2(X)\xi_2. \end{aligned}$$

We obtain ${\tau^i}_j = 0$, $S_1 X = X$, $S_2 = dv(\cdot)\partial_u = \varepsilon L$, $h^2 = 0$ and $h^1(\partial_u, \partial_u) = h^1(\partial_u, \partial_v) = 0$, $h^1(\partial_v, \partial_v) = \varepsilon$. The volume element $\mathrm{vol} = du \wedge dv$ is induced from $\mathbb{R}^4$, $\mathrm{vol}(X,Y) = \det(f_*X, f_*Y, \xi_1, \xi_2)$. Identifying the vector field $f_*(Y) + \Psi^1\xi_1 + \Psi^2\xi_2$ with the section $Y \oplus (\Psi^1, \Psi^2)$ of $TM \oplus E$ we obtain $\widehat{\nabla}_X(Y \oplus (\Psi^1, \Psi^2))$ as in (9) from $D_X(f_*Y + \Psi^1\xi_1 + \Psi^2\xi_2)$.

Similarly as it was for (6), the above immersion f is degenerate. By definition (see [3]), an immersion $f\colon M^2 \to \mathbb{R}^4$ is non-degenerate if the symmetric bilinear function G_σ is non-degenerate. For a local frame field $\sigma = (X_1, X_2)$ the function G_σ is defined by the formula (cf [3])

$$\begin{aligned} G_\sigma(Y,Z) = \frac{1}{2}\big(&\det(f_*(X_1), f_*(X_2), D_Y f_*(X_1), D_Z f_*(X_2)) \\ &+ \det(f_*(X_1), f_*(X_2), D_Z f_*(X_1), D_Y f_*(X_2))\big). \end{aligned}$$

For $\sigma = (\partial_u, \partial_v)$ we obtain $G_\sigma = 0$.

It is impossible to obtain in a similar way the connection (10), because vol is anti-symmetric.

6. Some further remarks

In general, to each immersion (f, ξ) and to each local basis $\sigma = (X_1, X_2)$ of TM corresponds some $GL(3,\mathbb{R})$-valued 1-form Ω_σ

$$\Omega_\sigma = \begin{pmatrix} -\omega^1{}_1 & -\omega^1{}_2 & S^1(\cdot) \\ -\omega^2{}_1 & -\omega^2{}_2 & S^2(\cdot) \\ -h(\cdot, X_1) & -h(\cdot, X_2) & -\tau \end{pmatrix}.$$

Here $\omega^i{}_j$ are local connection forms of the induced connection and $S = S^1(\cdot)X_1 + S^2(\cdot)X_2$ is the shape operator. The condition $d\Omega_\sigma - \Omega_\sigma \wedge \Omega_\sigma = 0$ is equivalent to the fundamental Gauss, Codazzi and Ricci equations. The formula (5) gives on $TM \oplus E$ a flat connection $\widehat{D}$ described by formula (14).

The considered in the present paper 1-forms Ω_i were constructed as satisfying additional condition $\Omega_i = A\omega^1 + B\omega^2 + C\omega^i{}_j$ with constant A, B and C. For given Ω_σ such constant A, B and C may not exist, in such a case the connection $\widehat{D}$ is always different from $\widehat{\nabla}$. For example, (M, ∇) can be affinely immersed also as a non-degenerate surface in $\mathbb{R}^3$. Such immersions and transversal fields are described in [5]. If we use one of them, then we obtain $\widehat{D}$ different from (6) and (8).

For each given connection ∇ on M, for each $(1,1)$ tensor field A and $(0,2)$ tensor field α we can define some connection $\widehat{\nabla}^{A,\alpha}$ on $TM \oplus E$ by the formula

$$\widehat{\nabla}^{A,\alpha}(Y \oplus \Psi) = (\nabla_X Y + \Psi AX) \oplus (X(\Psi) + \alpha(X,Y)).$$

We may look for such connections ∇ for which there exist A and α such that $\widehat{\nabla}^{A,\alpha}$ is flat.

It is easy to compute

$$\begin{aligned} &\widehat{R}^{A,\alpha}(X,Y)(Y \oplus \Psi) \\ &\quad = \big(R(X,Y)Z + \alpha(Y,Z)AX - \alpha(X,Z)AY + \Psi((\nabla_X A)(Y) - (\nabla_Y A)(X))\big) \\ &\qquad \oplus \big((\nabla_X \alpha)(Y,Z) - (\nabla_Y \alpha)(X,Z) + \Psi(\alpha(X,AY) - \alpha(Y,AX))\big) \end{aligned}$$

7. The case of indefinite metric

To complete the description we consider now a two-dimensional manifold with indefinite metric g of constant curvature κ. We can assume, by replacing g by $-g$ if necessary, that $\kappa > 0$. Let $\kappa = \frac{1}{\rho^2}$. We take a local basis X_1, X_2 such that $g(X_1,X_1) = 1 = -g(X_2,X_2)$, $g(X_1,X_2) = 0$. The local connection forms are $\omega^1{}_1 = \omega^2{}_2 = 0$, $\omega^1{}_2 = \omega^2{}_1 =: \omega$. The structural equations are $d\omega^1 = -\omega \wedge \omega^2$, $d\omega^2 = -\omega \wedge \omega^1$, $d\omega = -\kappa\omega^1 \wedge \omega^2$ and the 1-form

$$\Omega_\sigma = \begin{pmatrix} 0 & -\omega & -\frac{1}{\rho}\omega^1 \\ -\omega & 0 & -\frac{1}{\rho}\omega^2 \\ \frac{1}{\rho}\omega^1 & -\frac{1}{\rho}\omega^2 & 0 \end{pmatrix}$$

satisfies the condition $d\Omega_\sigma - \Omega_\sigma \wedge \Omega_\sigma = 0$ [8]. Using (5) we obtain

$$\begin{aligned}\widehat{\nabla}_X(Y \oplus \Psi) &= \Big(\Big(X(Y^1) + \omega(X)Y^2 + \frac{1}{\rho}\omega^1(X)\Psi\Big)X_1 + \Big(X(Y^2) + \omega(X)Y^1 \\ &\quad + \frac{1}{\rho}\omega^2(X)\Psi\Big)X_2\Big) \oplus \Big(X(\Psi) - \frac{1}{\rho}(\omega^1(X)Y^1 - \omega^2(X)Y^2)\Big) \\ &= \Big(\nabla_X Y + \frac{1}{\rho}\Psi X\Big) \oplus \Big(X(\Psi) - \frac{1}{\rho}g(X,Y)\Big).\end{aligned} \tag{17}$$

Let $\mathbb{R}^{2,1} = \mathbb{R}^3$ with the scalar product $\langle (v^1,v^2,v^3),(w^1,w^2,w^3)\rangle = v^1w^1 + v^2w^2 - v^3w^3$. Let $Q = \{x \in \mathbb{R}^3 : \langle x,x\rangle = \rho^2\}$. Let $f\colon M \to Q \subset \mathbb{R}^{2,1}$ be a local isometric immersion. Then $g(X,Y) = \langle f_*(X), f_*(Y)\rangle$ and the connection induced by f and the normal vector field $\xi = \frac{1}{\rho}f$ is the Levi-Civita connection of g. We have $h(X,Y) = g(SX,Y)$ and $SX = -\frac{1}{\rho}X$. From (14) we obtain

$$\widehat{D}_X(Y \oplus \Psi) = \Big(\nabla_X Y + \frac{1}{\rho}\Psi(X)\Big) \oplus \Big(X(\Psi) - \frac{1}{\rho}g(X,Y)\Big)$$

and we see that $\widehat{D} = \widehat{\nabla}$.

If $\kappa = -\frac{1}{\rho^2}$, then to $-g$ corresponds the positive curvature $-\kappa = \frac{1}{\rho^2}$ and the formula (17) gives the flat connection

$$\begin{aligned}\widehat{\nabla}_X(Y \oplus \Psi) &= \Big(\nabla_X Y + \frac{1}{\rho}\Psi X\Big) \oplus \Big(X(\Psi) - \frac{1}{\rho}(-g)(X,Y)\Big) \\ &= \Big(\nabla_X Y + \frac{1}{\rho}\Psi X\Big) \oplus \Big(X(\Psi) + \frac{1}{\rho}g(X,Y)\Big).\end{aligned} \tag{18}$$

If $\rho = 1$, then from (18) we obtain (1) and from (17) we obtain (2). It follows that Shchepetilov's formulae hold also for indefinite metric g.

8. Summary

For a locally symmetric connection ∇ with one-dimensional $\operatorname{im} R$ we have constructed two flat connections on the vector bundle $TM \oplus (\mathbb{R} \times M)$ and two flat connections on $TM \oplus (\mathbb{R}^2 \times M)$. From each pair only one connection may be identified with the standard connection in $\mathbb{R}^N$, $N = 3$ or $N = 4$, after suitable local embedding of (M, ∇) into $\mathbb{R}^N$. Those embeddings are degenerate.

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