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Comparative growth analysis of Wronskians in the light of their relative orders

Abstract. In this paper we study the comparative growth properties of a composition of entire and meromorphic functions on the basis of the relative order (relative lower order) of Wronskians generated by entire and meromorphic functions.

1. Introduction, definitions and notations

Let $\mathbb{C}$ be the set of all finite complex numbers. Also let f be a meromorphic function and g be an entire function defined in $\mathbb{C}$. The maximum modulus function relating to entire g is defined as $M_g(r) = \max\{|g(z)| : |z| = r\}$. For a meromorphic function f, $M_f(r)$ cannot be identified as f is not analytic. In this case one may characterize another function $T_f(r)$ known as Nevanlinna's characteristic function of f, playing the same role as the maximum modulus function in the following way

$$T_f(r) = N_f(r) + m_f(r),$$

where the function $N_f(r,a)$ resp. $\overline{N}_f(r,a)$, known as counting function of a-points (distinct a-points) of meromorphic f is defined as follows

$$N_f(r,a) = \int_0^r \frac{n_f(t,a) - n_f(0,a)}{t}\,dt + \overline{n}_f(0,a)\log r$$

resp.

$$\overline{N}_f(r,a) = \int_0^r \frac{\overline{n}_f(t,a) - \overline{n}_f(0,a)}{t}\,dt + \overline{n}_f(0,a)\log r.$$

In addition, we represent by $n_f(r,a)$ ($\overline{n}_f(r,a)$) the number of a-points (distinct a-points) of f in $|z| \le r$ and an ∞-point is a pole of f. In many occasions $N_f(r,\infty)$ and $\overline{N}_f(r,\infty)$ are symbolized by $N_f(r)$ and $\overline{N}_f(r)$, respectively.

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On the other hand, the function $m_f(r,\infty)$ alternatively indicated by $m_f(r)$, known as the proximity function of f, is defined as

$$m_f(r) = \frac{1}{2\pi}\int_0^{2\pi} \log^+ |f(re^{i\theta})|\, d\theta, \quad \text{where } \log^+ x = \max(\log x, 0) \text{ for all } x \geqslant 0.$$

Also we may imply $m(r, \frac{1}{f-a})$ by $m_f(r,a)$.

If f is entire, then the Nevanlinna's characteristic function $T_f(r)$ of f is defined as

$$T_f(r) = m_f(r).$$

Moreover, $M_f(r)$ and $T_f(r)$ are both strictly increasing and continuous functions of r when the entire function f is non-constant. Also their inverses $M_f^{-1}(r)\colon (|f(0)|,\infty) \to (0,\infty)$ and $T_f^{-1}\colon (T_f(0),\infty) \to (0,\infty)$ respectively exist, where $\lim_{s\to\infty} M_g^{-1}(s) = \infty$ and $\lim_{s\to\infty} T_f^{-1}(s) = \infty$.

In this connection we immediately remind the following definition which is relevant.

Definition 1.1 ([2])
A non-constant entire function f is said have the Property (A) if for any $\sigma > 1$ and for all sufficiently large r, $[M_f(r)]^2 \leq M_f(r^\sigma)$ holds. For the examples of functions with or without the Property (A), one may see [2].

However, in the case of any two entire functions f and g, the ratio $\frac{M_f(r)}{M_g(r)}$ as $r \to \infty$ is illustrated as the growth of f with respect to g in terms of their maximum moduli. Analogously, while f and g are both meromorphic functions, the ratio $\frac{T_f(r)}{T_g(r)}$ as $r \to \infty$ is illustrated as the growth of f with respect to g in terms of their Nevanlinna's characteristic functions. Also the concept of the growth measuring tools such as order and lower order which are conventional in complex analysis and the growth of entire or meromorphic functions can be studied in terms of their orders and lower orders – normally defined in terms of their growths with respect to the exp function which are shown in the following definition.

Definition 1.2
The order ρ_f (resp. the lower order λ_f) of an entire function f is defined as

$$\rho_f = \limsup_{r\to\infty} \frac{\log\log M_f(r)}{\log\log M_{\exp z}(r)} = \limsup_{r\to\infty} \frac{\log\log M_f(r)}{\log(r)}$$

resp.

$$\lambda_f = \liminf_{r\to\infty} \frac{\log\log M_f(r)}{\log\log M_{\exp z}(r)} = \liminf_{r\to\infty} \frac{\log\log M_f(r)}{\log(r)}.$$

When f is meromorphic, one may easily prove that

$$\rho_f = \limsup_{r\to\infty} \frac{\log T_f(r)}{\log T_{\exp z}(r)} = \limsup_{r\to\infty} \frac{\log T_f(r)}{\log(\frac{r}{\pi})} = \limsup_{r\to\infty} \frac{\log T_f(r)}{\log(r) + O(1)}$$

resp.

$$\lambda_f = \liminf_{r\to\infty} \frac{\log T_f(r)}{\log T_{\exp z}(r)} = \liminf_{r\to\infty} \frac{\log T_f(r)}{\log(\frac{r}{\pi})} = \liminf_{r\to\infty} \frac{\log T_f(r)}{\log(r) + O(1)}.$$

Both entire and meromorphic functions have the regular growth if their order coincides with their lower orders.

Bernal [1, 2] initiated the idea of the relative order of an entire function f with respect to another entire function g, symbolized by $\rho_g(f)$ to keep away from comparing growth just with $\exp z$ which is as follows

$$\begin{aligned}\rho_g(f) &= \inf\{\mu > 0 : \ M_f(r) < M_g(r^\mu) \text{ for all } r > r_0(\mu) > 0\}\\ &= \limsup_{r\to\infty} \frac{\log M_g^{-1} M_f(r)}{\log r}.\end{aligned}$$

The definition agrees with the classical one [10] if $g(z) = \exp z$.

Likewise, one may define the relative lower order of an entire function f with respect to another entire function g symbolized by $\lambda_g(f)$ in the following way

$$\lambda_g(f) = \liminf_{r\to\infty} \frac{\log M_g^{-1} M_f(r)}{\log r}.$$

Widening this notion, Lahiri and Banerjee [9] established the definition of the relative order of a meromorphic function with respect to an entire function which is as follows.

Definition 1.3 ([9])
Let f be any meromorphic function and g be any entire function. The relative order of f with respect to g is defined as

$$\begin{aligned}\rho_g(f) &= \inf\{\mu > 0 : \ T_f(r) < T_g(r^\mu) \text{ for all large } r\}\\ &= \limsup_{r\to\infty} \frac{\log T_g^{-1} T_f(r)}{\log r}.\end{aligned}$$

Similarly, one may define the relative lower order of a meromorphic function f with respect to an entire function g in the following way

$$\lambda_g(f) = \liminf_{r\to\infty} \frac{\log T_g^{-1} T_f(r)}{\log r}.$$

It is known (cf. [9]) that if $g(z) = \exp z$, then Definition 1.3 coincides with the classical definition of the order of a meromorphic function f.

The following definitions are also well known.

Definition 1.4
A meromorphic function $a \equiv a(z)$ is called small with respect to f if $T(r, a) = S(r, f)$, where $S(r, f) = o\{T(r, f)\}$, i.e. $\frac{S(r,f)}{T(r,f)} \to 0$ as $r \to \infty$.

Definition 1.5
Let $a_1, a_2, \ldots, a_k$ be linearly independent meromorphic functions and small with respect to f. We denote by $L(f) = W(a_1, a_2, \ldots, a_k, f)$ the Wronskian determinant of $a_1, a_2, \ldots, a_k$, f, i.e.

$$L(f) = \begin{vmatrix} a_1 & a_2 & \ldots & a_k & f \\ a_1' & a_2' & \ldots & a_k' & f' \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ a_1^{(k)} & a_2^{(k)} & \ldots & a_k^{(k)} & f^{(k)} \end{vmatrix}.$$

Definition 1.6
If $a \in \mathbb{C} \cup \{\infty\}$, the quantity

$$\delta(a; f) = 1 - \limsup_{r\to\infty} \frac{N(r, a; f)}{T_f(r)} = \liminf_{r\to\infty} \frac{m(r, a; f)}{T_f(r)}$$

is called the Nevanlinna's deficiency of the value a.

From the second fundamental theorem, it follows that the set of values of $a \in \mathbb{C} \cup \{\infty\}$ for which $\delta(a; f) > 0$ is countable and $\sum_{a\neq\infty} \delta(a; f) + \delta(\infty; f) \leq 2$ (cf. [7, p.43]). If in particular, $\sum_{a\neq\infty} \delta(a; f) + \delta(\infty; f) = 2$, we say that f has the maximum deficiency sum.

In this paper we wish to prove some newly developed results based on the growth properties of the relative order and the relative lower order of Wronskians generated by entire and meromorphic functions. We do not explain the standard definitions and notations in the theory of entire and meromorphic functions as those are available in [7] and [11].

2. Lemmas

In this section we present some lemmas which will be needed in the sequel.

Lemma 2.1 ([3])
Let f be meromorphic and g be entire, then for all sufficiently large values of r

$$T_{f\circ g}(r) \leqslant \{1 + o(1)\} \frac{T_g(r)}{\log M_g(r)} T_f(M_g(r)).$$

Lemma 2.2 ([4])
Let f be meromorphic and g be entire and suppose that $0 < \mu < \rho_g \leq \infty$. Then for a sequence of values of r tending to infinity

$$T_{f\circ g}(r) \geq T_f(\exp(r^\mu)).$$

Lemma 2.3 ([8])
Let f be meromorphic and g be entire such that $0 < \rho_g < \infty$ and $0 < \lambda_f$. Then for a sequence of values of r tending to infinity

$$T_{f\circ g}(r) > T_g(\exp(r^\mu)),$$

where $0 < \mu < \rho_g$.

Lemma 2.4 ([6])
Let f be an entire function which satisfies the Property (A), $\beta > 0$, $\delta > 1$ and $\alpha > 2$. Then

$$\beta T_f(r) < T_f(\alpha r^{\delta}).$$

Lemma 2.5 ([5])
If f be a transcendental meromorphic function with the maximum deficiency sum and g be a transcendental entire function of regular growth having non zero finite order and $\sum_{a\neq\infty}\delta(a;g)+\delta(\infty;g)=2$, then the relative order and relative lower order of $L(f)$ with respect to $L(g)$ are same as those of f with respect to g, i.e.

$$\rho_{L[g]}(L[f]) = \rho_g(f) \quad \text{and} \quad \lambda_{L[g]}(L[f]) = \lambda_g(f).$$

3. Theorems

In this section we present the main results of the paper.

Theorem 3.1
Let f be a transcendental meromorphic function with the maximum deficiency sum and h be a transcendental entire function of the regular growth having non zero finite order with $\sum_{a\neq\infty}\delta(a;h)+\delta(\infty;h)=2$ and $0<\lambda_h(f)\leq\rho_h(f)<\infty$. Also let g be an entire function with finite order. If h satisfies the Property (A), then for every positive constant μ and each $\alpha\in(-\infty,\infty)$

$$\lim_{r\to\infty}\frac{\{\log T_h^{-1}T_{f\circ g}(r)\}^{1+\alpha}}{\log T_{L[h]}^{-1}T_{L[f]}(\exp r^{\mu})} = 0, \quad \text{where } \mu > (1+\alpha)\rho_g.$$

Proof. Let us suppose that $\beta > 1+o(1)$ and $\delta > 1$. If $1+\alpha\leq 0$, then the theorem is obvious. We consider $1+\alpha>0$.

Since $T_h^{-1}(r)$ is an increasing function of r, it follows from Lemma 2.1, Lemma 2.4 and the inequality $T_g(r)\leq\log M_g(r)$ (cf. [7]) for all sufficiently large values of r that

$$T_h^{-1}T_{f\circ g}(r) \leqslant T_h^{-1}[\{1+o(1)\}T_f(M_g(r))],$$

i.e.

$$T_h^{-1}T_{f\circ g}(r) \leqslant \beta[T_h^{-1}T_f(M_g(r))]^{\delta},$$

i.e.

$$\log T_h^{-1}T_{f\circ g}(r) \leqslant \delta\log T_h^{-1}T_f(M_g(r)) + O(1), \tag{1}$$

i.e.

$$\log T_h^{-1}T_{f\circ g}(r) \leqslant \delta(\rho_h(f)+\varepsilon)r^{\rho_g+\varepsilon} + O(1). \tag{2}$$

Again, for all sufficiently large values of r, we get in view of Lemma 2.5 that

$$\log T_{L[h]}^{-1}T_{L[f]}(\exp r^{\mu}) \geq (\lambda_{L[h]}(L[f])-\varepsilon)r^{\mu},$$

i.e.

$$\log T_{L[h]}^{-1}T_{L[f]}(\exp r^{\mu}) \geq (\lambda_h(f)-\varepsilon)r^{\mu}. \tag{3}$$

Hence, for all sufficiently large values of r, we obtain from (2) and (3) that

$$\frac{\{\log T_h^{-1} T_{f\circ g}(r)\}^{1+\alpha}}{\log T_{L[h]}^{-1} T_{L[f]}(\exp r^{\mu})} \leq \frac{[\delta(\rho_h(f)+\varepsilon) r^{\rho_g+\varepsilon} + O(1)]^{1+\alpha}}{(\lambda_h(f)-\varepsilon) r^{\mu}}, \tag{4}$$

where we choose $0 < \varepsilon < \min\{\lambda_h(f), \frac{\mu}{1+\alpha} - \rho_g\}$. So from (4) we obtain that

$$\lim_{r\to\infty} \frac{\{\log T_h^{-1} T_{f\circ g}(r)\}^{1+\alpha}}{\log T_{L[h]}^{-1} T_{L[f]}(\exp r^{\mu})} = 0.$$

This proves the theorem.

Remark 3.2
In Theorem 3.1 if we take the condition $0 < \rho_h(f) < \infty$ instead of $0 < \lambda_h(f) \leq \rho_h(f) < \infty$, the theorem remains true with "limit inferior" in place of "limit".

In view of Theorem 3.1, the following theorem can be carried out.

Theorem 3.3
Let f be a meromorphic function and g, h be any two transcendental entire functions with the maximum deficiency sum where g is with finite order, h is of the regular growth having non zero finite order, $\lambda_h(g) > 0$ and $\rho_h(f) < \infty$. If h satisfy the Property (A), then for every positive constant μ and each $\alpha \in (-\infty, \infty)$

$$\lim_{r\to\infty} \frac{\{\log T_h^{-1} T_{f\circ g}(r)\}^{1+\alpha}}{\log T_{L[h]}^{-1} T_{L[g]}(\exp r^{\mu})} = 0, \quad \textit{where } \mu > (1+\alpha)\rho_g.$$

The proof is omitted.

Remark 3.4
If we take in Theorem 3.3 the condition $\rho_h(g) > 0$ instead of $\lambda_h(g) > 0$, the theorem remains true with "limit" replaced by "limit inferior".

Theorem 3.5
Let f be a transcendental meromorphic function with $\sum_{a\neq\infty} \delta(a; f) + \delta(\infty; f) = 2$ and g be an entire function with $\lambda_g < \mu < \infty$. Also let h be any transcendental entire function of the regular growth having non zero finite order with the maximum deficiency and satisfies Property (A) and $0 < \lambda_h(f) \leq \rho_h(f) < \infty$. Then for a sequence of values of r tending to infinity

$$T_h^{-1} T_{f\circ g}(r) < T_{L[h]}^{-1} T_{L[f]}(\exp r^{\mu}).$$

Proof. Let us consider $\delta > 1$. Since $T_h^{-1}(r)$ is an increasing function of r, it follows from (1) that for a sequence of values of r tending to infinity

$$\log T_h^{-1} T_{f\circ g}(r) \leqslant \delta(\rho_h(f)+\varepsilon) r^{\lambda_g+\varepsilon} + O(1). \tag{5}$$

Now, (3) and (5), for a sequence of values of r tending to infinity, yield

$$\frac{\log T_{L[h]}^{-1} T_{L[f]}(\exp r^{\mu})}{\log T_h^{-1} T_{f\circ g}(r)} \geq \frac{(\lambda_h(f)-\varepsilon) r^{\mu}}{\delta(\rho_h(f)+\varepsilon) r^{\lambda_g+\varepsilon}+O(1)}. \tag{6}$$

As $\lambda_g < \mu$, we can choose $\varepsilon\,(>0)$ in such a way that

$$\lambda_g+\varepsilon<\mu<\rho_g. \tag{7}$$

Thus, from (6) and (7), we obtain that

$$\limsup_{r\to\infty} \frac{\log T_{L[h]}^{-1} T_{L[f]}(\exp r^{\mu})}{\log T_h^{-1} T_{f\circ g}(r)} = \infty. \tag{8}$$

From (8), we obtain for a sequence of values of r tending to infinity and also for $K>1$

$$T_{L[h]}^{-1} T_{L[f]}(\exp r^{\mu}) > T_h^{-1} T_{f\circ g}(r).$$

Thus the theorem follows.

In the line of Theorem 3.5, we may state the following result without its proof.

Theorem 3.6

Let g be any transcendental entire function with $\sum_{a\neq\infty}\delta(a;g)+\delta(\infty;g)=2$, $\lambda_g<\mu<\infty$ and h be any transcendental entire function of the regular growth having non zero finite order with the maximum deficiency sum and satisfy the Property (A). Let moreover, f be a meromorphic function with finite relative order with respect to h. Then for a sequence of values of r tending to infinity

$$T_h^{-1} T_{f\circ g}(r) < T_{L[h]}^{-1} T_{L[g]}(\exp r^{\mu})$$

when $\lambda_h(g)>0$.

Theorem 3.7

Let f be a meromorphic function and h, g be any two transcendental entire functions with $\sum_{a\neq\infty}\delta(a;h)+\delta(\infty;h)=2$, $\sum_{a\neq\infty}\delta(a;g)+\delta(\infty;g)=2$, $\lambda_h(f)>0$ and $0<\rho_h(g)<\infty$. If h is of the regular growth having non zero finite order, then

$$\limsup_{r\to\infty} \frac{\log T_h^{-1} T_{f\circ g}(r)}{\log T_{L[h]}^{-1} T_{L[g]}(\exp r^{\mu})} = \infty,$$

where $0<\mu<\rho_g$.

Proof. Let $0<\mu<\mu'<\rho_g$. As $T_h^{-1}(r)$ is an increasing function of r, it follows from Lemma 2.2 for a sequence of values of r tending to infinity that

$$\log T_h^{-1} T_{f\circ g}(r) \geq \log T_h^{-1} T_f(\exp(r^{\mu'})),$$

i.e.

$$\log T_h^{-1} T_{f\circ g}(r) \geq (\lambda_h(f)-\varepsilon) r^{\mu'}. \tag{9}$$

Again for all sufficiently large values of r we get in view of Lemma 2.5 that

$$\log T^{-1}_{L[h]} T_{L[g]}(\exp r^{\mu}) \le (\rho_{L[h]}(L[g]) + \varepsilon) r^{\mu},$$

i.e.

$$\log T^{-1}_{L[h]} T_{L[g]}(\exp r^{\mu}) \le (\rho_h(g) + \varepsilon) r^{\mu}. \tag{10}$$

Therefore combining (9) and (10), we obtain for a sequence of values of r tending to infinity that

$$\frac{\log T_h^{-1} T_{f\circ g}(r)}{\log T^{-1}_{L[h]} T_{L[g]}(\exp r^{\mu})} \ge \frac{(\lambda_h(f) - \varepsilon) r^{\mu'}}{(\rho_h(g) + \varepsilon) r^{\mu}}. \tag{11}$$

Since $\mu < \mu'$, the theorem follows from (11).

Corollary 3.8
Under the assumptions of Theorem 3.7,

$$T_h^{-1} T_{f\circ g}(r) \ge T^{-1}_{L[h]} T_{L[g]}(\exp r^{\mu}), \qquad 0 < \mu < \rho_g.$$

Proof. In view of Theorem 3.7, we get for a sequence of values of r tending to infinity that

$$\log T_h^{-1} T_{f\circ g}(r) \ge K \log T^{-1}_{L[h]} T_{L[g]}(\exp r^{\mu}) \quad \text{for } K > 1,$$

i.e.

$$\log T_h^{-1} T_{f\circ g}(r) \ge \log\{T^{-1}_{L[h]} T_{L[g]}(\exp r^{\mu})\}^K,$$

i.e.

$$\log T_h^{-1} T_{f\circ g}(r) \ge \log T^{-1}_{L[h]} T_{L[g]}(\exp r^{\mu}),$$

i.e.

$$T_h^{-1} T_{f\circ g}(r) \ge T^{-1}_{L[h]} T_{L[g]}(\exp r^{\mu})$$

from which the corollary follows.

Similarly one may state the following theorem and corollary without their proofs as those can be carried out in the line of Theorem 3.7 and Corollary 3.8, respectively.

Theorem 3.9
Let f be a transcendental meromorphic function with the maximum deficiency sum and h be an transcendental entire function of regular growth having non zero finite order with $\sum_{a\neq\infty} \delta(a;)h + \delta(\infty; h) = 2$. If h satisfies $0 < \lambda_h(f) \le \rho_h(f) < \infty$, then for any entire function g

$$\limsup_{r\to\infty} \frac{\log T_h^{-1} T_{f\circ g}(r)}{\log T^{-1}_{L[h]} T_{L[f]}(\exp r^{\mu})} = \infty,$$

where $0 < \mu < \rho_g$.

Corollary 3.10
Under the assumptions of Theorem 3.9,

$$T_h^{-1}T_{f\circ g}(r) \geq T_{L[h]}^{-1}T_{L[f]}(\exp r^{\mu}), \qquad 0 < \mu < \rho_g.$$

As an application of Theorem 3.5 and Corollary 3.10, we may state the following result.

Theorem 3.11
Let f be a transcendental meromorphic function with $\sum_{a\neq\infty}\delta(a;f)+\delta(\infty;f)=2$ and g be an entire function with $\lambda_g < \mu < \rho_g$. Let moreover, h be any transcendental entire function of the regular growth having non zero finite order with the maximum deficiency and satisfying the Property (A) and let $0 < \lambda_h(f) \leq \rho_h(f) < \infty$. Then

$$\liminf_{r\to\infty}\frac{T_h^{-1}T_{f\circ g}(r)}{T_{L[h]}^{-1}T_{L[f]}(\exp r^{\mu})} \leq 1 \leq \limsup_{r\to\infty}\frac{T_h^{-1}T_{f\circ g}(r)}{T_{L[h]}^{-1}T_{L[f]}(\exp r^{\mu})}.$$

The proof is omitted.

Similarly, in view of Theorem 3.6 and Corollary 3.8, the following theorem can be carried out.

Theorem 3.12
Let h be any transcendental entire function of the regular growth having non zero finite order with the maximum deficiency sum and satisfying the Property (A) and let g be any transcendental entire function with $\sum_{a\neq\infty}\delta(a;g)+\delta(\infty;g)=2$, $0 < \lambda_g < \mu < \rho_g < \infty$ and $0 < \lambda_h(g) \leq \rho_h(g) < \infty$. Moreover, let f be a meromorphic function with $0 < \lambda_h(f) \leq \rho_h(f) < \infty$. Then

$$\liminf_{r\to\infty}\frac{T_h^{-1}T_{f\circ g}(r)}{T_{L[h]}^{-1}T_{L[g]}(\exp r^{\mu})} \leq 1 \leq \limsup_{r\to\infty}\frac{T_h^{-1}T_{f\circ g}(r)}{T_{L[h]}^{-1}T_{L[g]}(\exp r^{\mu})}.$$

The proof is omitted.

Theorem 3.13
Let f be a transcendental meromorphic function with $\sum_{a\neq\infty}\delta(a;f)+\delta(\infty;f)=2$ and let h be any transcendental entire function of the regular growth having non zero finite order with the maximum deficiency sum and $0 < \lambda_h(f) \leq \rho_h(f) < \infty$. Then for any entire function g

$$\limsup_{r\to\infty}\frac{\log^{[2]} T_h^{-1}T_{f\circ g}(\exp r^{B})}{\log^{[2]} T_{L[h]}^{-1}T_{L[f]}(\exp r^{\mu})} = \infty,$$

where $0 < \mu < \rho_g$ and $B > 0$.

Proof. Let $0 < \mu' < \rho_g$. As $T_h^{-1}(r)$ is an increasing function of r, it follows from (9), for a sequence of values of r tending to infinity that

$$\log^{[2]} T_h^{-1}T_{f\circ g}(r) \geq O(1) + \mu'\log r.$$

Hence, for a sequence of values of r tending to infinity, we get

$$\log^{[2]} T_h^{-1} T_{f\circ g}(\exp r^B) \geq O(1) + \mu' r^B. \tag{12}$$

Again in view of Lemma 2.5, we have for all sufficiently large values of r that

$$\log T_{L[h]}^{-1} T_{L[f]}(\exp r^{\mu}) \leq (\rho_{L[h]}(L[f]) + \varepsilon) r^{\mu},$$

i.e.

$$\log T_{L[h]}^{-1} T_{L[f]}(\exp r^{\mu}) \leq (\rho_h(f) + \varepsilon) r^{\mu},$$

i.e.

$$\log^{[2]} T_{L[h]}^{-1} T_{L[f]}(\exp r^{\mu}) \leq O(1) + \mu \log r. \tag{13}$$

Now combining (12) with (13) we obtain for a sequence of values of r tending to infinity that

$$\frac{\log^{[2]} T_h^{-1} T_{f\circ g}(\exp r^B)}{\log^{[2]} T_{L[h]}^{-1} T_{L[f]}(\exp r^{\mu})} \geq \frac{O(1) + \mu' r^B}{O(1) + \mu \log r},$$

which completes the proof.

In view of Theorem 3.13, we can state the following result.

Theorem 3.14

Let h be any transcendental entire function of the regular growth having non zero finite order with the maximum deficiency sum and let g be any transcendental entire function with $\sum_{a\neq\infty} \delta(a; g) + \delta(\infty; g) = 2$, $\lambda_h(f) > 0$ and $0 < \rho_h(g) < \infty$. Then for any meromorphic function f

$$\limsup_{r\to\infty} \frac{\log^{[2]} T_h^{-1} T_{f\circ g}(\exp r^B)}{\log^{[2]} T_{L[h]}^{-1} T_{L[g]}(\exp r^{\mu})} = \infty,$$

where $0 < \mu < \rho_g$ and $B > 0$.

The proof is omitted.

Theorem 3.15

Let f be a transcendental meromorphic function with $\sum_{a\neq\infty} \delta(a; f) + \delta(\infty; f) = 2$ and let h be any transcendental entire function of the regular growth having non zero finite order with the maximum deficiency sum and $\lambda_h(f) > 0$. Then for any entire function g with $0 < \rho_g \leq \infty$

$$\limsup_{r\to\infty} \frac{\log T_h^{-1} T_{f\circ g}(r)}{\log T_{L[h]}^{-1} T_{L[f]}(r)} = \infty.$$

Proof. Since $T_h^{-1}(r)$ is an increasing function of r, we get from Lemma 2.2 for a sequence of values of r tending to infinity that

$$\log T_h^{-1} T_{f\circ g}(r) \geq \log T_h^{-1} T_f(\exp(r^{\mu})),$$

i.e.

$$\log T_h^{-1} T_{f\circ g}(r) \geq (\lambda_h(f) - \varepsilon) r^{\mu}, \tag{14}$$

where $0 < \mu < \rho_g \leq \infty$.

Also for all sufficiently large values of r, in view of Lemma 2.5, we obtain that

$$\log T_{L[h]}^{-1} T_{L[f]}(r) \leq (\rho_{L[h]}(L[f]) + \varepsilon) \log r,$$

i.e.

$$\log T_{L[h]}^{-1} T_{L[f]}(r) \leq (\rho_h(f) + \varepsilon) \log r. \tag{15}$$

Therefore from (14) and (15), for a sequence of values of r tending to infinity, we obtain that

$$\frac{\log T_h^{-1} T_{f\circ g}(r)}{\log T_{L[h]}^{-1} T_{L[f]}(r)} \geq \frac{(\lambda_h(f) - \varepsilon) r^{\mu'}}{(\rho_h(f) + \varepsilon) \log r},$$

i.e.

$$\limsup_{r\to\infty} \frac{\log T_h^{-1} T_{f\circ g}(r)}{\log T_{L[h]}^{-1} T_{L[f]}(r)} = \infty.$$

Thus the theorem follows.

Theorem 3.16

Let h be any transcendental entire function of the regular growth having non zero finite order with the maximum deficiency sum and let g be any transcendental entire function with $\sum_{a\neq\infty} \delta(a;g) + \delta(\infty;g) = 2$ and $\lambda_h(g) > 0$. Then for any meromorphic function f

$$\limsup_{r\to\infty} \frac{\log T_h^{-1} T_{f\circ g}(r)}{\log T_{L[h]}^{-1} T_{L[g]}(r)} = \infty,$$

where $0 < \rho_g < \infty$ and $0 < \lambda_f$.

The proof of Theorem 3.16 is omitted as it can be carried out in the line of Theorem 3.15 and with the help of Lemma 2.3.

Theorem 3.17

Let f be a transcendental meromorphic function with $\sum_{a\neq\infty} \delta(a;f) + \delta(\infty;f) = 2$ and let h be any transcendental entire function of the regular growth having non zero finite order with the maximum deficiency sum and $0 < \lambda_h(f) \leq \rho_h(f) < \infty$. Moreover, let g be an entire function with non zero order. Then for every positive constant A and every real number α

$$\limsup_{r\to\infty} \frac{\log T_h^{-1} T_{f\circ g}(r)}{\{\log T_{L[h]}^{-1} T_{L[f]}(r^A)\}^{1+\alpha}} = \infty.$$

Proof. If α be such that $1+\alpha \leq 0$, then the theorem is trivial. So we suppose that $1+\alpha > 0$.

From the definition of $\rho_{L[h]}(L[f])$ and from from Lemma 2.5 it follows that for all sufficiently large values of r

$$\log T_{L[h]}^{-1} T_{L[f]}(r^A) \le (\rho_{L[h]}(L[f]) + \varepsilon) A \log r,$$

i.e.

$$\log T_{L[h]}^{-1} T_{L[f]}(r^A) \le (\rho_h(f) + \varepsilon) A \log r,$$

i.e.

$$\{\log T_{L[h]}^{-1} T_{L[f]}(r^A)\}^{1+\alpha} \le (\rho_h(f) + \varepsilon)^{1+\alpha} A^{1+\alpha} (\log r)^{1+\alpha}. \tag{16}$$

Now from (14) and (16), it follows that for a sequence of values of r tending to infinity

$$\frac{\log T_h^{-1} T_{f\circ g}(r)}{\{\log T_{L[h]}^{-1} T_{L[f]}(r^A)\}^{1+\alpha}} \geqslant \frac{(\lambda_h(f) - \varepsilon) r^\mu}{(\rho_h(f) + \varepsilon)^{1+\alpha} A^{1+\alpha} (\log r)^{1+\alpha}}.$$

Since $\frac{r^\mu}{(\log r)^{1+\alpha}} \to \infty$ as $r \to \infty$, the proof is completed.

In the line of Theorem 3.17 and with the help of Lemma 2.2, one may state the following result (the proof will be omitted).

Theorem 3.18
Let f be a transcendental meromorphic function with $\sum_{a\neq\infty} \delta(a; f) + \delta(\infty; f) = 2$ and non zero finite lower order and let g be an entire function with non zero finite order. Moreover, let h be any transcendental entire function of the regular growth having non zero finite order with the maximum deficiency sum, $\rho_h(f) < \infty$ and $\lambda_h(g) > 0$. Then for every positive constant A and every real number α

$$\limsup_{r\to\infty} \frac{\log T_h^{-1} T_{f\circ g}(r)}{\{\log T_{L[h]}^{-1} T_{L[f]}(r^A)\}^{1+\alpha}} = \infty.$$

Theorem 3.19
Let h be any transcendental entire function of the regular growth having non zero finite order with the maximum deficiency sum and let g be any transcendental entire function with $\sum_{a\neq\infty} \delta(a; g) + \delta(\infty; g) = 2$ and $\rho_h(g) < \infty$. Also let f be a meromorphic function with $0 < \lambda_h(f)$. Then for every positive constant A and every real number α

$$\limsup_{r\to\infty} \frac{\log T_h^{-1} T_{f\circ g}(r)}{\{\log T_{L[h]}^{-1} T_{L[g]}(r^A)\}^{1+\alpha}} = \infty.$$

Theorem 3.20
Let h be any transcendental entire function of the regular growth having non zero finite order with the maximum deficiency sum and let g be any transcendental entire function with non zero finite order, $\sum_{a\neq\infty} \delta(a; g) + \delta(\infty; g) = 2$ and $0 < \lambda_h(g) \le \rho_h(g) < \infty$. Let moreover, f be a meromorphic function with non zero finite lower order. Then for every positive constant A and every real number α

$$\limsup_{r\to\infty} \frac{\log T_h^{-1} T_{f\circ g}(r)}{\{\log T_{L[h]}^{-1} T_{L[g]}(r^A)\}^{1+\alpha}} = \infty.$$

We omit the proofs of Theorem 3.19 and Theorem 3.20 as those can be carried out in the line of Theorem 3.17 and Theorem 3.18, respectively.

4. Conclusion

The main aim of this paper is to extend the notion of order to the relative order in case of the growth properties of Wronskians generated by transcendental meromorphic functions. In fact, the relative order of growth gives a quantitative assessment of how different functions scale each other and until what extent they are self-similar in growth. Actually, in this paper we have established some theorems in this connection. Here, we are trying to extend the notion of growth properties of Wronskians on the basis of the relative order and the relative lower order. But still there are some problems to be investigated further. One of such problems is the study of the growth properties of any type of differential polynomial of higher dimension on the basis of the generalized relative order and the generalized relative lower order characterized by slowly changing functions. These type of studies can be regarded as open problems for the future workers in this branch.

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