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Further investigations on a question of Zhang and Lü

Abstract. In the paper based on the question of Zhang and Lü [15], we present one theorem which will improve and extend results of Banerjee-Majumder [2] and a recent result of Li-Huang [9].

1. Introduction

Let f be a non-constant meromorphic function defined in the open complex plane $\mathbb{C}$. We adopt the standard notations of the Nevanlinna theory of meromorphic functions as explained in [6].

If for some $a \in \mathbb{C} \cup \{\infty\}$, f and g have the same set of a-points with the same multiplicities, we say that f and g share the value a CM (counting multiplicities) and if we do not consider the multiplicities, then f, g are said to share the value a IM (ignoring multiplicities). When $a = \infty$ the zeros of $f - a$ means the poles of f.

It will be convenient to denote by E any set of positive real numbers of finite linear measure, not necessarily the same at each occurrence. For any non-constant meromorphic function f, the symbol $S(r, f)$ stands for any quantity satisfying

$$S(r, f) = o(T(r, f)) \qquad (r \to \infty,\, r \notin E).$$

A meromorphic function $a = a(z)$ ($\not\equiv \infty$) is called a small function with respect to f provided that $T(r, a) = S(r, f)$ as $r \to \infty$, $r \notin E$. If $a = a(z)$ is a small function we say that f and g share a IM or a CM according if $f - a$ and $g - a$ share 0 IM or 0 CM, respectively. We use I to denote any set of infinite linear measure of $0 < r < \infty$.

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It is known that the hyper order of f, denoted by $\rho_2(f)$, is defined by

$$\rho_2(f) = \limsup_{r\to\infty} \frac{\log\log T(r,f)}{\log r}.$$

The subject on sharing values between entire functions and their derivatives was first studied by Rubel and Yang ([12]). In 1977, they proved that if a non-constant entire function f and f' share two distinct finite numbers a, b CM, then $f = f'$. In 1979, analogous result for IM sharing was obtained by Mues and Steinmetz in the following manner.

Theorem A ([11])
Let f be a non-constant entire function. If f and f' share two distinct values a, b IM, then $f' \equiv f$.

Subsequently, similar considerations have been made with respect to higher derivatives and more general differential expressions as well.

Above theorems motivate the researchers to study the relation between an entire function and its derivative counterpart for one CM shared value. In 1996, in this direction the following famous conjecture was proposed by Brück ([3]).

Conjecture
Let f be a non-constant entire function such that the hyper order $\rho_2(f)$ of f is not a positive integer or infinite. If f and f' share a finite value a CM, then $\frac{f'-a}{f-a} = c$, where c is a non-zero constant.

Brück himself proved the conjecture for $a = 0$. For $a \neq 0$, Brück ([3]) obtained the following result in which additional supposition was required.

Theorem B ([3])
Let f be a non-constant entire function. If f and f' share the value 1 CM and if $N(r,0;f') = S(r,f)$, then $\frac{f'-1}{f-1}$ is a non-zero constant.

Next we recall the following definitions.

Definition 1.1 ([8])
Let p be a positive integer and $a \in \mathbb{C} \cup \{\infty\}$.

(i) $N(r,a;f \mid \geq p)$ ($\overline{N}(r,a;f \mid \geq p)$) denotes the counting function (reduced counting function) of those a-points of f whose multiplicities are not less than p.

(ii) $N(r,a;f \mid \leq p)$ ($\overline{N}(r,a;f \mid \leq p)$) denotes the counting function (reduced counting function) of those a-points of f whose multiplicities are not greater than p.

Definition 1.2 ([14])
For $a \in \mathbb{C} \cup \{\infty\}$ and a positive integer p we denote by $N_p(r,a;f)$ the sum $\overline{N}(r,a;f) + \overline{N}(r,a;f \mid \geq 2) + \ldots + \overline{N}(r,a;f \mid \geq p)$. Clearly $N_1(r,a;f) = \overline{N}(r,a;f)$.

Definition 1.3 ([14])
For $a \in \mathbb{C} \cup \{\infty\}$ and a positive integer p we put

$$\delta_p(a, f) = 1 - \limsup_{r\to\infty} \frac{N_p(r, a; f)}{T(r, f)}.$$

Clearly, $0 \leq \delta(a, f) \leq \delta_p(a, f) \leq \delta_{p-1}(a, f) \leq \ldots \leq \delta_2(a, f) \leq \delta_1(a, f) = \Theta(a, f) \leq 1$.

Definition 1.4
For two positive integers n, p we define

$$\mu_p = \min\{n, p\} \quad \text{and} \quad \mu_p^* = p + 1 - \mu_p.$$

Then clearly

$$N_p(r, 0; f^n) \leq \mu_p N_{\mu_p^*}(r, 0; f).$$

Definition 1.5 ([2])
Let z_0 be a zero of $f - a$ of multiplicity p and a zero of $g - a$ of multiplicity q. We denote by $\overline{N}_L(r, a; f)$ the counting function of those a-points of f and g, where $p > q \geq 1$, by $N_E^{1)}(r, a; f)$ the counting function of those a-points of f and g, where $p = q = 1$ and by $\overline{N}_E^{(2}(r, a; f)$ the counting function of those a-points of f and g, where $p = q \geq 2$, each point in these counting functions is counted only once. In the same way we can define $\overline{N}_L(r, a; g)$, $N_E^{1)}(r, a; g)$, $\overline{N}_E^{(2}(r, a; g)$.

Definition 1.6 ([7])
Let k be a nonnegative integer or infinity. For $a \in \mathbb{C} \cup \{\infty\}$ we denote by $E_k(a; f)$ the set of all a-points of f, where an a-point of multiplicity m is counted m times if $m \leq k$ and $k + 1$ times if $m > k$. If $E_k(a; f) = E_k(a; g)$, we say that f, g share the value a with weight k.

The definition implies that if f, g share a value a with weight k, then z_0 is an a-point of f with multiplicity $m\,(\leq k)$ if and only if it is an a-point of g with multiplicity $m\,(\leq k)$ and z_0 is an a-point of f with multiplicity $m\,(> k)$ if and only if it is an a-point of g with multiplicity $n\,(> k)$, where m is not necessarily equal to n.

We write f, g share (a, k) to mean that f, g share the value a with weight k. Clearly if f, g share (a, k), then f, g share (a, p) for any integer p, $0 \leq p < k$. Also we note that f, g share a value a IM or CM if and only if f, g share $(a, 0)$ or (a, ∞), respectively.

With the notion of weighted sharing of values Lahiri-Sarkar ([8]) improved the result of Zhang ([13]). In ([14]) Zhang extended the result of Lahiri-Sarkar ([8]) and replaced the concept of value sharing by small function sharing.

In 2008 Zhang and Lü ([15]) considered the uniqueness of the n-th power of a meromorphic function sharing a small function with its k-th derivative and proved the following theorem.

Theorem C ([15])
Let $k\,(\geq 1)$, $n\,(\geq 1)$ be integers and f be a non-constant meromorphic function. Also let $a(z)\,(\not\equiv 0,\infty)$ be a small function with respect to f. Suppose $f^n - a$ and $f^{(k)} - a$ share $(0,l)$. If $l=\infty$ and

$$(3+k)\Theta(\infty,f)+2\Theta(0,f)+\delta_{2+k}(0,f)>6+k-n$$

or $l=0$ and

$$(6+2k)\Theta(\infty,f)+4\Theta(0,f)+2\delta_{2+k}(0,f)>12+2k-n,$$

then $f^n \equiv f^{(k)}$.

In the same paper Zhang and Lü ([15]) raised the following question: What will happen if f^n and $[f^{(k)}]^s$ share a small function? In 2010, Chen and Zhang ([5]) gave a answer to the above question. Unfortunately there were some gaps in the proof of the theorems in ([5]) which was latter rectified by Banerjee and Majumder ([2]). In 2010 Banerjee and Majumder ([2]) proved two theorems one of which further improved Theorem C whereas the other answers the open question of Zhang and Lü ([15]) in the following manner.

Theorem D ([2])
Let $k\,(\geq 1)$, $n\,(\geq 1)$ be integers and f be a non-constant meromorphic function. Also let $a(z)\,(\not\equiv 0,\infty)$ be a small function with respect to f. Suppose $f^n - a$ and $f^{(k)} - a$ share $(0,l)$. If $l\geq 2$ and

$$(3+k)\Theta(\infty,f)+2\Theta(0,f)+\delta_{2+k}(0,f)>6+k-n$$

or $l=1$ and

$$\left(\frac{7}{2}+k\right)\Theta(\infty,f)+\frac{5}{2}\Theta(0,f)+\delta_{2+k}(0,f)>7+k-n$$

or $l=0$ and

$$(6+2k)\Theta(\infty,f)+4\Theta(0,f)+\delta_{2+k}(0,f)+\delta_{1+k}(0,f)>12+2k-n,$$

then $f^n = f^{(k)}$.

Theorem E ([2])
Let $k\,(\geq 1)$, $n\,(\geq 1)$, $m\,(\geq 2)$ be integers and f be a non-constant meromorphic function. Also let $a(z)\,(\not\equiv 0,\infty)$ be a small function with respect to f. Suppose $f^n - a$ and $[f^{(k)}]^m - a$ share $(0,l)$. If $l=2$ and

$$(3+2k)\Theta(\infty,f)+2\Theta(0,f)+2\delta_{1+k}(0,f)>7+2k-n \tag{1.1}$$

or $l=1$ and

$$\left(\frac{7}{2}+2k\right)\Theta(\infty,f)+\frac{5}{2}\Theta(0,f)+2\delta_{1+k}(0,f)>8+2k-n \tag{1.2}$$

or $l=0$ and

$$(6+3k)\Theta(\infty,f)+4\Theta(0,f)+3\delta_{1+k}(0,f)>13+3k-n, \tag{1.3}$$

then $f^n \equiv [f^{(k)}]^m$.

For $m = 1$ it can be easily proved that Theorem D is a better result than Theorem E. Also we observe that in the conditions (1.1)–(1.3) there was no influence of m.

Very recently, in order to improve the results of Zhang ([14]), Li-Huang ([9]) obtained the following theorem. In view of Lemma 2.1 proved latter on, we see that the following result obtained in ([9]) is better than that of Theorem D for $n = 1$.

Theorem F ([9])
Let f be a non-constant meromorphic function, $k\,(\geq 1)$, $l\,(\geq 0)$ be integers and also let $a(z)\,(\not\equiv 0, \infty)$ be a small function with respect to f. Suppose $f - a$ and $f^{(k)} - a$ share $(0, l)$. If $l \geq 2$ and

$$(3+k)\Theta(\infty, f) + \delta_2(0, f) + \delta_{2+k}(0, f) > k + 4$$

or $l = 1$ and

$$\Big(\frac{7}{2} + k\Big)\Theta(\infty, f) + \frac{1}{2}\Theta(0, f) + \delta_2(0, f) + \delta_{2+k}(0, f) > k + 5$$

or $l = 0$ and

$$(6+2k)\Theta(\infty, f) + 2\Theta(0, f) + \delta_2(0, f) + \delta_{1+k}(0, f) + \delta_{2+k}(0, f) > 2k + 10,$$

then $f \equiv f^{(k)}$.

Next we recall the following definition.

Definition 1.7 ([6])
Let $n_{0j}, n_{1j}, \ldots, n_{kj}$ be nonnegative integers. The expression

$$M_j[f] = (f)^{n_{0j}}(f^{(1)})^{n_{1j}} \ldots (f^{(k)})^{n_{kj}}$$

is called a differential monomial generated by f of degree $d_{M_j} = d(M_j) = \sum_{i=0}^{k} n_{ij}$ and weight $\Gamma_{M_j} = \sum_{i=0}^{k}(i+1)n_{ij}$.

The sum $P[f] = \sum_{j=1}^{t} b_j M_j[f]$ is called a differential polynomial generated by f of degree $\overline{d}(P) = \max\{d(M_j) :\ 1 \leq j \leq t\}$ and weight $\Gamma_P = \max\{\Gamma_{M_j} :\ 1 \leq j \leq t\}$, where $T(r, b_j) = S(r, f)$ for $j = 1, 2, \ldots, t$.

The numbers $\underline{d}(P) = \min\{d(M_j) :\ 1 \leq j \leq t\}$ and k (the highest order of the derivative of f in $P[f]$) are called respectively the lower degree and order of $P[f]$.

$P[f]$ is said to be homogeneous if $\overline{d}(P)$=$\underline{d}(P)$. Moreover, $P[f]$ is called a linear differential polynomial generated by f if $\overline{d}(P) = 1$. Otherwise, $P[f]$ is called a non-linear differential polynomial.

We denote by $Q = \max\{\Gamma_{M_j} - d(M_j) :\ 1 \leq j \leq t\} = \max\{n_{1j} + 2n_{2j} + \ldots + kn_{kj} :\ 1 \leq j \leq t\}$.

Also for the sake of convenience for a differential monomial $M[f]$ we denote by $\lambda = \Gamma_M - d_M$.

Recently Charak-Lal ([4]) considered the possible extension of Theorem D in the direction of the question of Zhang and Lü ([15]) up to differential polynomial. They proved the following result.

Theorem G ([4])
Let f be a non-constant meromorphic function and n be a positive integer and $a(z)\,(\not\equiv 0,\infty)$ be a meromorphic function satisfying $T(r,a)=o(T(r,f))$ as $r\to\infty$. Let $P[f]$ be a non-constant differential polynomial in f. Suppose f^n and $P[f]$ share (a,l). If $l\geq 2$ and

$$(3+Q)\Theta(\infty,f)+2\Theta(0,f)+\overline{d}(P)\delta(0,f)>Q+5+2\overline{d}(P)-\underline{d}(P)-n$$

or $l=1$ and

$$\Big(\frac{7}{2}+Q\Big)\Theta(\infty,f)+\frac{5}{2}\Theta(0,f)+\overline{d}(P)\delta(0,f)>Q+6+2\overline{d}(P)-\underline{d}(P)-n$$

or $l=0$ and

$$(6+2Q)\Theta(\infty,f)+4\Theta(0,f)+2\overline{d}(P)\delta(0,f)>2Q+4\overline{d}(P)-2\underline{d}(P)+10-n,$$

then $f^n\equiv P[f]$.

This is a supplementary result corresponding to Theorem D because putting $P[f]=f^{(k)}$ one cannot obtain Theorem D, rather in this case a set of stronger conditions are obtained as particular case of Theorem F. So it is natural to ask the next question.

Question 1.8
Is it possible to improve Theorem D in the direction of Theorem F up to differential monomial so that the result give a positive answer to the question of Zhang and Lü [15]?

To seek the possible answer of Question 1.1 is the motivation of the paper.

The following theorem is the main result of this paper which gives a positive answer of Zhang and Lü([15]).

Theorem 1.9
Let $k\,(\geq 1)$, $n\,(\geq 1)$ be integers and f be a non-constant meromorphic function and $M[f]$ be a differential monomial of degree d_M and weight Γ_M and k is the highest derivative in $M[f]$. Also let $a(z)\,(\not\equiv 0,\infty)$ be a small function with respect to f. Suppose f^n-a and $M[f]-a$ share $(0,l)$. If $l\geq 2$ and

$$(3+\lambda)\Theta(\infty,f)+\mu_2\delta_{\mu_2^*}(0,f)+d_M\delta_{2+k}(0,f)>3+\Gamma_M+\mu_2-n \tag{1.4}$$

or $l=1$ and

$$\Big(\frac{7}{2}+\lambda\Big)\Theta(\infty,f)+\frac{1}{2}\Theta(0,f)+\mu_2\delta_{\mu_2^*}(0,f)+d_M\delta_{2+k}(0,f)>4+\Gamma_M+\mu_2-n$$

or $l=0$ and

$$\begin{aligned}(6+2\lambda)\Theta(\infty,f)+2\Theta(0,f)+\mu_2\delta_{\mu_2^*}(0,f)+d_M\delta_{2+k}(0,f)+d_M\delta_{1+k}(0,f)\\ >8+2\Gamma_M+\mu_2-n,\end{aligned} \tag{1.5}$$

then $f^n\equiv M[f]$.

However the following question is still open.

Question 1.10
Is it possible to extend Theorem 1.9 up to differential polynomial instead of differential monomial?

Following example shows that in Theorem 1.9 $a(z) \not\equiv 0, \infty$ is necessary.

Example 1.11
Let us take $f(z) = e^{e^z}$ and $M = f'$, then M and f share 0 (or, ∞) and the deficiency conditions stated in Theorem 1.9 is satisfied as 0, ∞ both are exceptional values of f but $f \not\equiv M$.

The next example shows that the deficiency conditions stated in Theorem 1.9 are not necessary.

Example 1.12
Let $f(z) = Ae^z + Be^{-z}$, $AB \neq 0$. Then $\overline{N}(r, f) = S(r, f)$ and $\overline{N}(r, 0; f) = \overline{N}(r, -\frac{B}{A}; e^{2z}) \sim T(r, f)$. Thus $\Theta(\infty, f) = 1$ and $\Theta(0, f) = \delta_p(0, f) = 0$.

It is clear that $M[f] = f''$ and f share $a(z) = \frac{1}{z}$ and the deficiency conditions in Theorem 1.9 is not satisfied, but $M \equiv f$.

In the next example we see that f^n cannot be replaced by arbitrary polynomial $P[f] = a_0 f^n + a_1 f^{n-1} + \ldots + a_n$ in Theorem 1.9 for IM sharing ($l = 0$) case.

Example 1.13
If we take $f(z) = e^z$, $P[f] = f^2 + 2f$ and $M[f] = f^{(3)}$, then $P + 1 = (M + 1)^2$. Thus P and M share $(-1, 0)$. Also $\Theta(0, f) = \Theta(\infty, f) = \delta_p(0, f) = \delta(0, f) = 1$ as 0 and ∞ are exceptional values of f. Thus (1.5) of Theorem 1.9 is satisfied but $P \not\equiv M$.

In view of Example 1.13 the following question is inevitable.

Question 1.14
Is it possible to replace f^n by arbitrary polynomial $P[f] = a_0 f^n + a_1 f^{n-1} + \ldots + a_n$ in Theorem 1.9 for $l \geq 1$?

2. Lemmas

In this section we present some lemmas needed in the sequel. Let F, G be two non-constant meromorphic functions. Henceforth we shall denote by H the following function.

$$H = \Big(\frac{F''}{F'} - \frac{2F'}{F-1}\Big) - \Big(\frac{G''}{G'} - \frac{2G'}{G-1}\Big).$$

Lemma 2.1
$1 + \delta_2(0, f) \geq 2\Theta(0, f)$.

Proof.

$$\begin{aligned}2\Theta(0,f)-\delta_2(0,f)-1 &= \limsup_{r\to\infty}\frac{N_2(r,0;f)}{T(r,f)}-\limsup_{r\to\infty}\frac{2\overline{N}(r,0;f)}{T(r,f)}\\ &\leq \limsup_{r\to\infty}\frac{N_2(r,0;f)-2\overline{N}(r,0;f)}{T(r,f)}\\ &\leq 0.\end{aligned}$$

The following three lemmas can be proved using Milloux Theorem ([6]). So we omit the details.

Lemma 2.2
Let f be a non-constant meromorphic function and $M[f]$ be a differential monomial of degree d_M and weight Γ_M. Then $T(r,M)\leq d_MT(r,f)+\lambda\overline{N}(r,\infty;f)+S(r,f)$.

Lemma 2.3
$N(r,0;M)\leq T(r,M)-d_MT(r,f)+d_MN(r,0;f)+S(r,f)$.

Lemma 2.4
$N(r,0;M)\leq d_MN(r,0;f)+\lambda\overline{N}(r,\infty;f)+S(r,f)$.

Lemma 2.5 ([10])
Let f be a non-constant meromorphic function and let

$$R(f)=\frac{\sum_{i=0}^{n}a_if^i}{\sum_{j=0}^{m}b_jf^j}$$

be an irreducible rational function in f with constant coefficients $\{a_i\}$ and $\{b_j\}$, where $a_n\neq 0$ and $b_m\neq 0$. Then

$$T(r,R(f))=pT(r,f)+S(r,f),$$

where $p=\max\{n,m\}$.

Lemma 2.6
$N(r,\infty;\frac{M}{f^{d_M}})\leq d_MN(r,0;f)+\lambda\overline{N}(r,\infty;f)+S(r,f)$.

Proof. Let z_0 be a pole of f of order t. Then it is a pole of $\frac{M}{f^{d_M}}$ of order $n_1+2n_2+\ldots+kn_k=\lambda$.

Let z_0 be a zero of f of order s. Then it is a pole of $\frac{M}{f^{d_M}}$ of order at most sd_M. So, $N(r,\infty;\frac{M}{f^{d_M}})\leq d_MN(r,0;f)+\lambda\overline{N}(r,\infty;f)+S(r,f)$.

Lemma 2.7
For any two non-constant meromorphic functions f_1 and f_2,

$$N_p(r,\infty;f_1f_2)\leq N_p(r,\infty;f_1)+N_p(r,\infty;f_2).$$

Proof. Let z_0 be a pole of f_i of order t_i for $i = 1, 2$. Then z_0 be a pole of $f_1 f_2$ of order at most $t_1 + t_2$.

Case 1. Let $t_1 \geq p$ and $t_2 \geq p$. Then $t_1 + t_2 \geq p$. So z_0 is counted at most p times in the left hand side of the above counting function, whereas the same is counted $p + p$ times in the right hand side of the above counting function.

Case 2. Let $t_1 \geq p$ and $t_2 < p$.

Subcase 2.1. Let $t_1 + t_2 \geq p$. So z_0 is counted at most p times in the left hand side of the above counting function, whereas the same is counted as $p + \max\{0, t_2\}$ times in the right hand side of the above counting function.

Subcase 2.2. Let $t_1 + t_2 < p$. This case is occurred if t_2 is negative, i.e. if z_0 is a zero of f_2. Then z_0 is counted at most $\max\{0, t_1 + t_2\}$ times whereas the same is counted p times in the right hand side of the above expression.

Case 3. Let $t_1 < p$ and $t_2 \geq p$. Then $t_1 + t_2 \geq p$. This case can be disposed off as done in Case 2.

Case 4. Let $t_1 < p$ and $t_2 < p$.

Subcase 4.1. Let $t_1 + t_2 \geq p$. Then z_0 is counted at most p times whereas the same is counted $\max\{0, t_1\} + \max\{0, t_2\}$ times in the right hand side of the above expression.

Subcase 4.2. Let $t_1 + t_2 < p$. Then z_0 is counted at most $\max\{0, t_1 + t_2\}$ times whereas z_0 is counted $\max\{0, t_1\} + \max\{0, t_2\}$ times in the right hand side of the above counting functions. Combining all the cases, Lemma 2.7 follows.

Lemma 2.8 ([8])
$N_p(r, 0; f^{(k)}) \leq N_{p+k}(r, 0; f) + k\overline{N}(r, \infty; f) + S(r, f)$.

Lemma 2.9
For the differential monomial $M[f]$,

$$N_p(r, 0; M[f]) \leq d_M N_{p+k}(r, 0; f) + \lambda\overline{N}(r, \infty; f) + S(r, f).$$

Proof. Clearly for any non-constant meromorphic function f, $N_p(r, f) \leq N_q(r, f)$ if $p \leq q$.

Now by using the above fact and Lemma 2.7, Lemma 2.8, we get

$$\begin{aligned}
N_p(r, 0; M[f]) &\leq \sum_{i=0}^{k} n_i N_p(r, 0; f^{(i)}) + S(r, f) \\
&\leq \sum_{i=0}^{k} n_i \{N_{p+i}(r, 0; f) + i\overline{N}(r, \infty; f)\} + S(r, f) \\
&\leq \sum_{i=0}^{k} n_i N_{p+i}(r, 0; f) + \lambda\overline{N}(r, \infty; f) + S(r, f) \\
&\leq \sum_{i=0}^{k} n_i N_{p+k}(r, 0; f) + \lambda\overline{N}(r, \infty; f) + S(r, f) \\
&\leq d_M N_{p+k}(r, 0; f) + \lambda\overline{N}(r, \infty; f) + S(r, f).
\end{aligned}$$

Lemma 2.10
Let f be a non-constant meromorphic function and $a(z)$ be a small function in f. Let us define $F = \frac{f^n}{a}$, $G = \frac{M}{a}$. Then $FG \not\equiv 1$.

Proof. On contrary assume $FG \equiv 1$. Then in view of Lemma 2.6 and the First Fundamental Theorem, we get

$$\begin{aligned}(n + d_M)T(r, f) &= T\Big(r, \frac{M}{f^{d_M}}\Big) + S(r, f)\\ &\le d_M N(r, 0; f) + \lambda \overline{N}(r, \infty; f) + S(r, f)\\ &= S(r, f),\end{aligned}$$

which is a contradiction.

Lemma 2.11 ([2])
Let F and G share $(1, l)$ and $\overline{N}(r, F) = \overline{N}(r, G)$ and $H \not\equiv 0$, where F, G and H are defined as earlier. Then

$$\begin{aligned}N(r, \infty; H) \le \overline{N}(r, \infty; F) &+ \overline{N}(r, 0; F| \ge 2) + \overline{N}(r, 0; G| \ge 2) + \overline{N}_0(r, 0; F')\\ &+ \overline{N}_0(r, 0; G') + \overline{N}_L(r, 1; F) + \overline{N}_L(r, 1; G) + S(r, f).\end{aligned}$$

Lemma 2.12
Let F and G share $(1, l)$. Then $\overline{N}_L(r, 1; F) \le \frac{1}{2}\overline{N}(r, \infty; F) + \frac{1}{2}\overline{N}(r, 0; F) + S(r, F)$ if $l \ge 1$ and $\overline{N}_L(r, 1; F) \le \overline{N}(r, \infty; F) + \overline{N}(r, 0; F) + S(r, F)$ if $l = 0$.

Proof. Let $l \ge 1$. Then multiplicity of any 1-point of F counted in $\overline{N}_L(r, 1; F)$ is at least 3 as $l \ge 1$. Therefore, $\overline{N}_L(r, 1; F) \le \frac{1}{2}\overline{N}(r, 0; F'|F \ne 0) \le \frac{1}{2}\overline{N}(r, \infty; F) + \frac{1}{2}\overline{N}(r, 0; F) + S(r, F)$.

Let $l = 0$. Then multiplicity of any 1-point of F counted in $\overline{N}_L(r, 1; F)$ is at least 2 as $l = 0$. So, $\overline{N}_L(r, 1; F) \le \overline{N}(r, 0; F'|F \ne 0) \le \overline{N}(r, \infty; F) + \overline{N}(r, 0; F) + S(r, F)$.

Lemma 2.13
Let F and G share $(1, l)$ and $H \not\equiv 0$. Then

$$\begin{aligned}\overline{N}(r, 1; F) + \overline{N}(r, 1; G) \le N(r, \infty; H) &+ \overline{N}_E^{(2}(r, 1; F) + \overline{N}_L(r, 1; F)\\ &+ \overline{N}_L(r, 1; G) + \overline{N}(r, 1; G) + S(r, f).\end{aligned}$$

Proof. Clearly,

$$\overline{N}(r, 1; F) = N(r, 1; F| = 1) + \overline{N}_E^{(2}(r, 1; F) + \overline{N}_L(r, 1; F) + \overline{N}_L(r, 1; G)$$

and by simple calculation,

$$N(r, 1; F| = 1) \le N(r, 0; H) + S(r, f) \le N(r, \infty; H) + S(r, f).$$

Lemma 2.14
Let f be a non-constant meromorphic function and $a(z)$ be a small function of f. Let $F = \frac{f^n}{a}$ and $G = \frac{M}{a}$ such that F and G shares $(1, \infty)$. Then one of the following cases holds:

(1) $T(r) \leq N_2(r, 0; F) + N_2(r, 0; G) + \overline{N}(r, \infty; F) + \overline{N}(r, \infty; G) + \overline{N}_L(r, \infty; F) + \overline{N}_L(r, \infty; G) + S(r)$,

(2) $F \equiv G$,

(2) $FG \equiv 1$,

where $T(r) = \max\{T(r, F), T(r, G)\}$ and $S(r) = o(T(r))$, $r \in I$, I is a set of infinite linear measure of $r \in (0, \infty)$.

Proof. Let z_0 be a pole of f which is not a pole or zero of $a(z)$. Then z_0 is a pole of F and G simultaneously. Thus F and G share those pole of f which is not zero or pole of $a(z)$. Clearly,

$$\begin{aligned} N(r, H) \leq\ & \overline{N}(r, 0; F \geq 2) + \overline{N}(r, 0; G \geq 2) + \overline{N}_L(r, \infty; F) + \overline{N}_L(r, \infty; G) \\ & + \overline{N}_0(r, 0; F') + \overline{N}_0(r, 0; G') + S(r, f). \end{aligned}$$

The rest of proof can be carried out in the line of proof of Lemma 2.13 of [1]. So we omit the details.

3. Proof of the theorem

Proof. Let $F = \frac{f^n}{a}$ and $G = \frac{M[f]}{a}$. Then $F - 1 = \frac{f^n - a}{a}$, $G - 1 = \frac{M[f] - a}{a}$. Since f^n and $M[f]$ share (a, l), it follows that F and G share $(1, l)$ except the zeros and poles of $a(z)$. Now we consider the following cases.

Case 1. Let $H \not\equiv 0$.
Subcase 1.1. Suppose $l \geq 1$, then using the Second Fundamental Theorem and Lemmas 2.13, 2.11 we have

$$\begin{aligned} T(r, F) + T(r, G) \leq\ & \overline{N}(r, \infty; F) + \overline{N}(r, \infty; G) + \overline{N}(r, 0; F) + \overline{N}(r, 0; G) \\ & + N(r, H) + \overline{N}_E^{(2}(r, 1; F) + \overline{N}_L(r, 1; F) + \overline{N}_L(r, 1; G) \\ & + \overline{N}(r, 1; G) - \overline{N}_0(r, 0; F') - \overline{N}_0(r, 0; G') + S(r, f) \\ \leq\ & 2\overline{N}(r, \infty; F) + \overline{N}(r, \infty; G) + N_2(r, 0; F) + N_2(r, 0; G) \\ & + \overline{N}_E^{(2}(r, 1; F) + 2\overline{N}_L(r, 1; F) + 2\overline{N}_L(r, 1; G) \\ & + \overline{N}(r, 1; G) + S(r, f). \end{aligned} \tag{3.1}$$

Subsubcase 1.1.1. For $l = 1$ from inequality (3.1) and in view of Lemmas 2.12, 2.9

we obtain

$$\begin{aligned}
T(r,F)+T(r,G) &\le 2\overline{N}(r,\infty;F)+\overline{N}(r,\infty;G)+N_2(r,0;F)+N_2(r,0;G)\\
&\quad+\overline{N}_E^{(2}(r,1;F)+2\overline{N}_L(r,1;F)+2\overline{N}_L(r,1;G)\\
&\quad+\overline{N}(r,1;G)+S(r,f)\\
&\le \frac{5}{2}\overline{N}(r,\infty;F)+\overline{N}(r,\infty;G)+\frac{1}{2}\overline{N}(r,0;F)+\mu_2N_{\mu_2^*}(r,0;f)\\
&\quad+N_2(r,0;G)+\overline{N}_E^{(2}(r,1;F)+\overline{N}_L(r,1;F)+2\overline{N}_L(r,1;G)\\
&\quad+\overline{N}(r,1;G)+S(r,f)\\
&\le \frac{5}{2}\overline{N}(r,\infty;F)+\overline{N}(r,\infty;G)+\frac{1}{2}\overline{N}(r,0;F))+\mu_2N_{\mu_2^*}(r,0;f)\\
&\quad+N_2(r,0;G)+N(r,1;G)+S(r,f),
\end{aligned}$$

i.e. for any $\varepsilon>0$

$$\begin{aligned}
nT(r,f) &\le \Big(\lambda+\frac{7}{2}\Big)\overline{N}(r,\infty;f)+\frac{1}{2}\overline{N}(r,0;f)+\mu_2N_{\mu_2^*}(r,0;f)\\
&\quad+d_MN_{2+k}(r,0;f)+S(r,f)\\
&\le \Big\{\Big(\lambda+\frac{7}{2}\Big)-\Big(\lambda+\frac{7}{2}\Big)\Theta(\infty,f)+\frac{1}{2}-\frac{1}{2}\Theta(0,f)+\mu_2-\mu_2\delta_{\mu_2^*}(0,f)\\
&\quad+d_M-d_M\delta_{2+k}(0,f)+\varepsilon\Big\}T(r,f)+S(r,f).
\end{aligned}$$

i.e.

$$\begin{aligned}
&\Big\{\Big(\lambda+\frac{7}{2}\Big)\Theta(\infty,f)+\frac{1}{2}\Theta(0,f)+\mu_2\delta_{\mu_2^*}(0,f)+d_M\delta_{2+k}(0,f)-\varepsilon\Big\}T(r,f)\\
&\qquad\le(\Gamma_M+\mu_2+4-n)T(r,f)+S(r,f),
\end{aligned}$$

which is a contradiction.
Subsubcase 1.1.2. Let $l\ge 2$. Using the inequality (3.1) and Lemma 2.9, we get

$$\begin{aligned}
T(r,F)+T(r,G) &\le 2\overline{N}(r,\infty;F)+\overline{N}(r,\infty;G)+N_2(r,0;F)+N_2(r,0;G)\\
&\quad+\overline{N}_E^{(2}(r,1;F)+2\overline{N}_L(r,1;F)+2\overline{N}_L(r,1;G)\\
&\quad+\overline{N}(r,1;G)+S(r,f)\\
&\le 2\overline{N}(r,\infty;F)+\overline{N}(r,\infty;G)+\mu_2N_{\mu_2^*}(r,0;f)\\
&\quad+N_2(r,0;G)+N(r,1;G)+S(r,f),
\end{aligned}$$

i.e. for any $\varepsilon>0$

$$\begin{aligned}
nT(r,f) &\le (\lambda+3)\overline{N}(r,\infty;f)+\mu_2N_{\mu_2^*}(r,0;f)+d_MN_{2+k}(r,0;f)+S(r,f)\\
&\le \{(\lambda+3)-(\lambda+3)\Theta(\infty,f)+\mu_2-\mu_2\delta_{\mu_2^*}(0,f)\\
&\quad+d_M-d_M\delta_{2+k}(0,f)+\varepsilon\}T(r,f)+S(r,f),
\end{aligned}$$

i.e.

$$\begin{aligned}\{(\lambda+3)\Theta(\infty,f)+\mu_2\delta_{\mu_2^*}(0,f)+d_M\delta_{2+k}(0,f)-\varepsilon\}T(r,f)\\ \le(\Gamma_M+3+\mu_2-n)T(r,f)+S(r,f),\end{aligned}$$

which is a contradiction.

Subcase 1.2. Let $l=0$. Then by using the Second Fundamental Theorem and Lemma 2.13, 2.11, 2.12, 2.9 we get

$$\begin{aligned}
&T(r,F)+T(r,G)\\
&\quad\le \overline{N}(r,\infty;F)+\overline{N}(r,0;F)+\overline{N}(r,1;F)+\overline{N}(r,\infty;G)+\overline{N}(r,0;G)\\
&\qquad+\overline{N}(r,1;G)-\overline{N}_0(r,0;F')-\overline{N}_0(r,0;G')+S(r,F)+S(r,G)\\
&\quad\le \overline{N}(r,\infty;F)+\overline{N}(r,0;F)+\overline{N}(r,\infty;G)+\overline{N}(r,0;G)+N(r,\infty;H)\\
&\qquad+\overline{N}_E^{(2}(r,1;F)+\overline{N}_L(r,1;F)+\overline{N}_L(r,1;G)+\overline{N}(r,1;G)\\
&\qquad-\overline{N}_0(r,0;F')-\overline{N}_0(r,0;G')+S(r,F)+S(r,G)\\
&\quad\le 2\overline{N}(r,\infty;F)+\overline{N}(r,\infty;G)+N_2(r,0;F)+N_2(r,0;G) \qquad (3.2)\\
&\qquad+\overline{N}_E^{(2}(r,1;F)+2\overline{N}_L(r,1;F)+2\overline{N}_L(r,1;G)+\overline{N}(r,1;G)+S(r,f)\\
&\quad\le 2\overline{N}(r,\infty;F)+\overline{N}(r,\infty;G)+\mu_2N_{\mu_2^*}(r,0,f)+N_2(r,0;G)\\
&\qquad+2(\overline{N}(r,\infty;F)+\overline{N}(r,0;F))+\overline{N}(r,\infty;G)+\overline{N}(r,0;G)\\
&\qquad+\overline{N}_E^{(2}(r,1;F)+\overline{N}_L(r,1;G)+\overline{N}(r,1;G)+S(r,f)\\
&\quad\le 4\overline{N}(r,\infty;F)+\mu_2N_{\mu_2^*}(r,0,f)+N_2(r,0;G)+2\overline{N}(r,\infty;G)\\
&\qquad+\overline{N}(r,0;G)+2\overline{N}(r,0;F)+T(r,G)+S(r,f),
\end{aligned}$$

i.e. for any $\varepsilon>0$

$$\begin{aligned}
nT(r,f)&\le(2\lambda+6)\overline{N}(r,\infty;f)+2\overline{N}(r,0;f)+\mu_2N_{\mu_2^*}(r,0,f)\\
&\quad+d_MN_{1+k}(r,0;f)+d_MN_{2+k}(r,0;f)+S(r,f)\\
&\le\big\{(2\lambda+6)-(2\lambda+6)\Theta(\infty,f)+2-2\Theta(0,f)+\mu_2-\mu_2\delta_{\mu_2^*}(0,f)\\
&\quad+2d_M-d_M\delta_{1+k}(0,f)-d_M\delta_{2+k}(0,f)+\varepsilon\big\}T(r,f)+S(r,f),
\end{aligned}$$

i.e.

$$\begin{aligned}\big\{(2\lambda+6)\Theta(\infty,f)+2\Theta(0,f)+\mu_2\delta_{\mu_2^*}(0,f)+d_M\delta_{1+k}(0,f)\\ +d_M\delta_{2+k}(0,f)-\varepsilon\big\}T(r,f)\\ \le(2\Gamma_M+8+\mu_2-n)T(r,f)+S(r,f),\end{aligned}$$

which is a contradiction.

Case 2. Let $H\equiv0$. On integration we get

$$\frac{1}{G-1}\equiv\frac{A}{F-1}+B,$$

where $A(\ne0)$, B are complex constants. Then F and G share $(1,\infty)$. Moreover, by the construction of F and G we see that F and G share $(\infty,0)$ also.

So using Lemma 2.9 and condition (1.4), we obtain

$$\begin{aligned}
&N_2(r,0;F)+N_2(r,0;G)+\overline{N}(r,\infty;F)+\overline{N}(r,\infty;G)\\
&\qquad+\overline{N}_L(r,\infty;F)+\overline{N}_L(r,\infty;G)+S(r)\\
&\quad\le \mu_2 N_{\mu_2^*}(r,0;f)+d_M N_{2+k}(r,0;f)+(\lambda+3)\overline{N}(r,\infty;f)+S(r)\\
&\quad\le \big\{(3+\lambda+d_M+\mu_2)-((\lambda+3)\Theta(\infty,f)+\delta_{\mu_2^*}(0,f)\\
&\qquad+d_M\delta_{2+k}(0,f))\big\}T(r,f)+S(r)\\
&\quad< T(r,F)+S(r).
\end{aligned}$$

Hence inequality (1) of Lemma 2.14 does not hold. Again in view of Lemma 2.10, we get $F\equiv G$, i.e. $f^n\equiv M[f]$.

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