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Two constant sign solutions for a nonhomogeneous Neumann boundary value problem

Abstract. We consider a nonlinear Neumann problem with a nonhomogeneous elliptic differential operator. With some natural conditions for its structure and some general assumptions on the growth of the reaction term we prove that the problem has two nontrivial solutions of constant sign. In the proof we use variational methods with truncation and minimization techniques.

1. Introduction

Let $\Omega \subseteq \mathbb{R}^N$ be a bounded domain with a $C^{1,\alpha}$ boundary $\partial\Omega$, where $\alpha \in (0,1]$ is a positive constant. In this paper we are looking for smooth solutions to the following Neumann problem

$$\begin{cases} -\operatorname{div} a(\nabla u(z)) = f(z, u(z)) & \text{a.e. in } \Omega, \\ \dfrac{\partial u}{\partial n_a} = 0 & \text{on } \partial\Omega, \end{cases} \tag{1.1}$$

where $\frac{\partial u}{\partial n_a} = (a(\nabla u(z)), n(z))_{\mathbb{R}^N}$ with $n(\cdot) = (n_1(\cdot), \dots, n_N(\cdot))$ the outward unit normal vector on $\partial\Omega$. On the continuous map $a = (a_i)_{i=1}^N \colon \mathbb{R}^N \to \mathbb{R}^N$ we impose certain conditions (see Section 3) to obtain a p-Laplacian type operator, which unifies several important differential operators. Similar conditions are studied widely in literature (see Damascelli [2], Montenegro [6], Motreanu-Papageorgiou [7]), as they allow us to apply the regularity results of Lieberman [5]. The reaction term $f \colon \Omega \times \mathbb{R} \to \mathbb{R}$ is a Carathéodory function. We assume that $f(z, \cdot)$ has a positive and negative z-dependant zero and we are interested in the existence of constant sign positive and negative solutions of Problem (1.1), imposing some growth conditions on $f(z, \cdot)$ only near zero, without any control in $\pm\infty$.

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The rest of this paper is organized as follows. In Section 2 we provide mathematical preliminaries and recall the main mathematical tools which will be employed in this paper. In Section 3 we formulate the assumptions on maps a and f and provide a basic example of the reaction term f. Next we prove the existence theorem using variational and truncation methods.

2. Mathematical background

In this paper we will denote by $(\cdot,\cdot)_{\mathbb{R}^N}$ the scalar product in $\mathbb{R}^N$. Also $\|\cdot\|$ denotes the norm in Sobolev space $W^{1,p}(\Omega)$. We will assume that $1 < p < \infty$.

In the analysis of Problem (1.1) we will use the positive cone

$$C_+ = \Big\{ u \in C^1(\bar{\Omega}) \mid \; u(z) \geq 0 \text{ for all } z \in \bar{\Omega} \text{ and } \frac{\partial u}{\partial n_a} = 0 \text{ on } \partial\Omega \Big\}$$

and its interior given by

$$\operatorname{int} C_+ = \{ u \in C_+ \mid \; u(x) > 0 \text{ for all } z \in \bar{\Omega} \}.$$

Below we present main mathematical tools which will be needed in the proofs of our results.

Definition 2.1
Let $\phi\colon X \supseteq M \to \mathbb{R}$ be a functional on a subset M of the Banach space X. We say that ϕ is

- weakly sequentially lower semicontinuous on M iff for each $u \in M$ and each sequence $\{u_n\}_n \subseteq M$ such that $u_n \to u$ weakly in X, we have

$$\phi(u) \leq \liminf_{n\to\infty} \phi(u_n),$$

- weakly coercive iff

$$\lim_{\|u\|\to\infty} \phi(u) = \infty \quad \text{on } M.$$

Theorem 2.2 (25.D in Zeidler [9])
Suppose that the functional $\phi\colon X \supseteq M \to \mathbb{R}$ has the following three properties:

i. M is nonempty closed convex set in reflexive Banach space X,

ii. ϕ is weakly sequentially lower semicontinuous,

iii. ϕ is weakly coercive.

Then ϕ has a minimum on M.

Theorem 2.3 (1.7 in Lieberman [5])
Let $h\colon \mathbb{R}_+ \to \mathbb{R}$ be a C^1-function satisfying

$$\delta < \frac{th'(t)}{h(t)} \leq c_0 \quad \textit{for all } t > 0$$

with some constants $\delta > 0$, $c_0 > 0$. We define $H(\xi) = \int_0^\xi h(t)\,\mathrm{d}t$. By $W^{1,H}(\Omega)$ we denote the class of functions which are weakly differentiable in the set Ω with

$$\int_\Omega H(|\nabla u|)\,\mathrm{d}z < \infty.$$

Let $\alpha \in (0,1]$, $\Lambda, \Lambda_1, M_0 > 0$ be positive constants and let $\Omega \subseteq \mathbb{R}^N$ be a bounded domain with $C^{1,\alpha}$ boundary. Suppose that $A = (A_1, \dots, A_N)\colon \Omega \times [-M_0, M_0] \times \mathbb{R}^N \to \mathbb{R}^N$ is differentiable, $B\colon \Omega \times [-M_0, M_0] \times \mathbb{R}^N \to \mathbb{R}$ is a Carathéodory function and functions A, B satisfy the following conditions

$$(\nabla_y A(z_1, \xi_1, y)x, x)_{\mathbb{R}^N} \geq \frac{h(|y|)}{|y|}|x|^2, \qquad y \neq 0_N, \tag{2.1a}$$

$$\Big|\frac{\partial}{\partial y_j} A_i(z, \xi, y)\Big| \leq \Lambda \frac{h(|y|)}{|y|}, \qquad y \neq 0_N, \tag{2.1b}$$

$$|A(z_1, \xi_1, y) - A(z_2, \xi_2, y)| \leq \Lambda_1(1 + h(|y|))(|z_1 - z_2|^\alpha + |\xi_1 - \xi_2|^\alpha), \tag{2.1c}$$

$$|B(z_1, \xi_1, y)| \leq \Lambda_1(1 + h(|y|)|y|) \tag{2.1d}$$

for all $z_1, z_2 \in \Omega$, $\xi_1, \xi_2 \in [-M_0, M_0]$ and $x, y \in \mathbb{R}^N$. Then any $W^{1,H}(\Omega)$ solution u of

$$\operatorname{div} A(z, u, \nabla u) + B(z, u, \nabla u) = 0$$

in Ω with $|u| \leq M_0$ in Ω is in $C^{1,\beta}(\Omega)$ for some positive β depending on α, Λ, δ, c_0, N.

Theorem 2.4 (5.3.1 in Pucci-Serrin [8])
Let $\Omega \subseteq \mathbb{R}^N$ be a domain. Suppose that $A \in C^1(\mathbb{R}^+)$ is such that function $s \mapsto sA(s)$ is strictly increasing in $\mathbb{R}^+$ and $sA(s) \to 0$ as $s \to 0^+$. Let $B \in L^\infty_{loc}(\Omega \times \mathbb{R}^+ \times \mathbb{R}^N)$ satisfy the following condition

$$B(z, \xi, y) \geq -\kappa\Phi(|y|) - p(\xi)$$

for all $(z, \xi, y) \in \Omega \times [0, \infty) \times \mathbb{R}^N$ such that $|\xi| \leq 1$, where $\kappa > 0$ is a constant, $p\colon \mathbb{R}^+ \to \mathbb{R}$ is non-decreasing on some interval $(0, \delta)$, $\delta > 0$, $\Phi(s) := sA(s)$ when $s > 0$ and $\Phi(0) := 0$. For $s \geq 0$ we define

$$L(s) = s\Phi(s) - \int_0^s \Phi(t)\,\mathrm{d}t.$$

If either $p \equiv 0$ in $[0, d]$, $d > 0$, or the following condition is satisfied

$$\lim_{\epsilon \to 0^+} \int_0^\epsilon \frac{1}{L^{-1}(P(s))}\,\mathrm{d}s = \infty,$$

where $P(s) = \int_0^s p(t)\,\mathrm{d}t$, then the strong maximum principle for

$$\operatorname{div}(A(|\nabla u(z)|)\nabla u(z)) + B(z, u(z), \nabla u(z)) \leq 0 \tag{2.2}$$

holds, i.e. if u is a classical distribution solution of (2.2) with $u(z_0) = 0$ at some point $z_0 \in \Omega$, then $u \equiv 0$ in Ω. By classical distribution solution we mean a function $u \in C^1(\Omega)$, which satisfies (2.2) in the distribution sense.

To deal with the boundary condition in Problem (1.1), we introduce the following function space framework, due to Casas-Fernández [1]: for $p' \in (1,\infty)$ such that $\frac{1}{p}+\frac{1}{p'}=1$ we introduce a separable Banach space

$$W^{p'}(\operatorname{div},\Omega)=\{v\in L^{p'}(\Omega,\mathbb{R}^N)\ |\ \operatorname{div} v\in L^1(\Omega)\}$$

endowed with the norm

$$\|v\|_{W^{p'}(\operatorname{div},\Omega)}=\|v\|_{L^{p'}(\Omega,\mathbb{R}^N)}+\|\operatorname{div} v\|_{L^1(\Omega)}.$$

If Ω has a Lipschitz boundary $\partial\Omega$, we have that the space $C^\infty(\bar{\Omega},\mathbb{R}^N)$ is dense in $W^{p'}(\operatorname{div},\Omega)$ (see Lemma 1 in Casas-Fernández [1]). We denote the space of traces on $\partial\Omega$ by $W^{1/p',p}(\partial\Omega)$, endowed with the usual norm, and denote the trace of $u\in W^{1,p}(\Omega)$ on $\partial\Omega$ by $\gamma_0(u)$. Let us also consider the space

$$T^p(\partial\Omega)=W^{1/p',p}(\partial\Omega)\cap L^\infty(\Omega)$$

endowed with the norm $\|h\|_{T^p(\partial\Omega)}=\|h\|_{W^{1/p',p}(\partial\Omega)}+\|h\|_{L^\infty(\Omega)}$. We denote the dual space of $T^p(\partial\Omega)$ by $T^{-p'}(\partial\Omega)$ and the duality brackets by $\langle\cdot,\cdot\rangle_T$. We have

$$T^p(\partial\Omega)=\{\gamma_0(u)\ |\ \ u\in W^{1,p}(\Omega)\cap L^\infty(\Omega)\}.$$

Also there exists a unique linear continuous map

$$\gamma_n\colon W^{p'}(\operatorname{div},\Omega)\to T^{-p'}(\partial\Omega)$$

such that

$$\gamma_n(v)=(v,n)_{\mathbb{R}^N},\ \forall v\in C^\infty(\bar{\Omega},\mathbb{R}^N).$$

From this result one can obtain the following Green's formula.

Theorem 2.5 (1 in Casas-Fernández [1])
Let $a=(a_i)_{i=1}^N\colon\Omega\times(\mathbb{R}\times\mathbb{R}^N)\to\mathbb{R}^N$ be a Carathéodory map, which satisfies

$$|a_i(z,s,\xi)|\le k_1(|s|^{p-1}+|\xi|^{p-1})+k_2(z),\qquad i=1,\dots,N$$

with some constant $k_1>0$ and a function $k_2\in L^{p'}(\Omega)$. Then if $u\in W^{1,p}(\Omega)$ and $-\operatorname{div} a(\cdot,u,\nabla u)\in L^1(\Omega)$, then there exists a unique element of $T^{-p'}(\partial\Omega)$, which by extension we denote $\frac{\partial u}{\partial n_a}$, satisfying the Green's formula

$$\sum_{i=1}^N\int_\Omega a_i(z,u(z),\nabla u(z))\frac{\partial v}{\partial z_i}\,\mathrm{d}z=\int_\Omega -\operatorname{div} a(z,u(z),\nabla u(z))v(z)\,\mathrm{d}z+\Big\langle\frac{\partial u}{\partial n_a},\gamma_0(v)\Big\rangle_T$$

for all $v\in W^{1,p}(\Omega)\cap L^\infty(\Omega)$.

3. Problem setting

In this section we formulate our assumptions on the continuous map a and the Carathéodory reaction term f in Problem (1.1).

Assumption $H(a)_1$
The function $a\colon \mathbb{R}^N \to \mathbb{R}^N$ *is such that* $a(y) = a_0(|y|)y$ *for all* $y \in \mathbb{R}^N$*, where* $a_0\colon [0,\infty) \to [0,\infty)$*,* $a_0 \in C^1(0,\infty)$*,* $a_0 \in C([0,\infty))$ *is such that* $a_0(t) > 0$ *for* $t > 0$*.*

Assumption $H(a)_2$
There exist some constants $\delta, c_0, c_1, c_2, c_3 > 0$*,* $q \in (1,p)$ *and there exists a function* $h \in C^1(0,\infty) p = \frac{p'}{p'-1}$ *satisfying*

$$\delta < \frac{th'(t)}{h(t)} \le c_0 \quad \text{for all } t > 0, \tag{3.1}$$

$$c_1 t^{p-1} \le h(t) \le c_2(t^{q-1} + t^{p-1}) \quad \text{for all } t > 0 \tag{3.2}$$

such that

$$|\nabla a(y)| \le c_3 \frac{h(|y|)}{|y|} \quad \text{for all } y \in \mathbb{R}^N \backslash \{0\}. \tag{3.3}$$

Assumption $H(a)_3$
For all $y, \xi \in \mathbb{R}^N$ *such that* $y \neq 0$ *we have*

$$(\nabla a(y)\xi, \xi)_{\mathbb{R}^N} \ge \frac{h(|y|)}{|y|}|\xi|^2. \tag{3.4}$$

Assumption $H(a)_4$
There exists some constants $\mu \in (1,q]$ *and* $\tau \in (1,p]$ *such that the map* $t \mapsto G_0(t^{\frac{1}{\tau}})$ *is convex on* $(0,\infty)$ *and* $\lim_{t\to 0^+} \frac{G_0(t)}{t^\mu} = 0$*, where* $G_0(t) = \int_0^t a_0(s)s \, \mathrm{d}s$*.*

Proposition 3.1
If assumptions $H(a)_1$*–*$H(a)_4$ *hold, then* G_0 *is strictly convex and strictly increasing. Let*

$$G(y) := G_0(|y|), \qquad y \in \mathbb{R}^N.$$

Then G *is strictly convex,* $G(0) = 0$ *and* $\nabla G(y) = a(y)$ *for* $y \in \mathbb{R}^N \backslash \{0\}$*. Moreover, for all* $y \in \mathbb{R}^N$

$$G(y) \le (a(y), y)_{\mathbb{R}^N}. \tag{3.5}$$

Proof. As $a_0 > 0$, G_0 is strictly increasing. Thus the function $t \mapsto G_0(t^{\frac{1}{\tau}})$ is strictly inceasing on $(0,\infty)$. So G_0 is also strictly convex, because from $H(a)_4$ we have for any $\lambda \in (0,1)$, $s, t > 0$, $s \neq t$

$$\begin{aligned} G_0((1-\lambda)t + \lambda s) &= G_0\big(((1-\lambda)t + \lambda s)^{\tau \cdot \frac{1}{\tau}}\big) < G_0\big(((1-\lambda)t^\tau + \lambda s^\tau)^{\frac{1}{\tau}}\big) \\ &\le (1-\lambda)G_0\big((t^\tau)^{\frac{1}{\tau}}\big) + \lambda G_0\big((s^\tau)^{\frac{1}{\tau}}\big). \end{aligned}$$

Here we have used the fact, that for $\tau > 1$ the function $t \mapsto t^\tau$ is strictly convex on $(0,\infty)$. To show that G is also strictly convex, let $\lambda \in (0,1)$. Then by the definition of the norm and the properties of G_0 (strict monotonicity and strict convexity) we get

$$\begin{aligned} G((1-\lambda)u + \lambda v) &= G_0(|(1-\lambda)u + \lambda v|) \le G_0((1-\lambda)|u| + \lambda|v|) \\ &< (1-\lambda)G_0(|u|) + \lambda G_0(|v|) \\ &= (1-\lambda)G(u) + \lambda G(v). \end{aligned}$$

To obtain the gradient of G, for $i = 1, \dots, N$ and $y \in \mathbb{R}^N \setminus \{0\}$ we compute

$$\frac{\partial G}{\partial y_i}(y) = \frac{\partial G_0}{\partial y_i}(|y|) = G_0'(|y|)\frac{y_i}{|y|}.$$

So

$$\nabla G(u) = \frac{G_0'(|y|)}{|y|} y = a_0(|y|)y = a(y)$$

(see the definition of G_0 in $H(a)_4$). Next we show that inequality (3.5) holds. For $y = 0$ the inequality is true. Let $y \in \mathbb{R}^N \setminus \{0\}$, $b \in \mathbb{R}^N$. It follows from convexity of G that

$$G(b) \geq G(y) + (\nabla G(y), b - y)_{\mathbb{R}^N}.$$

As $\nabla G(y) = a(y)$ and $G(0) = 0$, for $b = 0$ we get

$$0 \geq G(y) - (a(y), y)_{\mathbb{R}^N}.$$

Lemma 3.2 (Properties of a)
If assumptions $H(a)_1$–$H(a)_4$ hold, then

(a) *the map $a \colon \mathbb{R}^N \to \mathbb{R}^N$ is maximal monotone and strictly monotone, i.e.*

$$\big[(b - a(y), x - y)_{\mathbb{R}^N} > 0 \quad \forall y \in \mathbb{R}^N\big] \implies b = a(x),$$
$$(a(x) - a(y), x - y)_{\mathbb{R}^N} > 0 \quad \text{for all } x, y \in \mathbb{R}^N,\ x \neq y,$$

respectively,

(b) *there exists $c_4 > 0$ such that for all $y \in \mathbb{R}^N$*

$$|a(y)| \leq c_4(|y|^{q-1} + |y|^{p-1}), \tag{3.6}$$

(c) *for all $y \in \mathbb{R}^N$ we have*

$$(a(y), y)_{\mathbb{R}^N} \geq \frac{c_1}{p-1}|y|^p. \tag{3.7}$$

Proof. (a) Strict monotonicity of a is equivalent to the strict convexity of G (see Zeidler [9], Proposition 25.10). As a is monotone and continuous, it is also maximal monotone (Zeidler [9]).

(b) and (c) The proof is similar to the proof of Lemma 2.1 in Damascelli [2]. We use the fact that

$$a_j(y_1) - a_j(y_2) = \int_0^1 \sum_{i=1}^N \frac{\partial a_j}{\partial x_i}(y_1 + t(y_1 - y_2))((y_1)_i - (y_2)_i)\,\mathrm{d}t$$

and, as $a((0, \dots, 0)) = 0$ by $H(a)_1$, from (3.3) and (3.2) (see $H(a)_2$) we get (3.6). Similarly, from $H(a)_3$ and (3.2) in $H(a)_2$ we get (3.7).

Corollary 3.3
If hypotheses $H(a)_1$–$H(a)_4$ *hold, then for all* $y \in \mathbb{R}^N$

$$\frac{c_1}{p(p-1)}|y|^p \le G(y) \le c_4(|y|^q + |y|^p). \tag{3.8}$$

Proof. To prove the first inequality we observe, that from (3.7) we have

$$a_0(|y|)|y|^2 \ge \frac{c_1}{p-1}|y|^p \quad \text{for all } y \in \mathbb{R}^N.$$

So in particular

$$a_0(s)s \ge \frac{c_1}{p-1}s^{p-1} \quad \text{for all } s \ge 0.$$

Thus

$$G(y) = G_0(|y|) = \int_0^{|y|} a_0(s)s\,\mathrm{d}s \ge \int_0^{|y|} \frac{c_1}{p-1}s^{p-1}\,\mathrm{d}s = \frac{c_1}{p(p-1)}|y|^p.$$

We prove the second inequality using the Cauchy-Schwarz inequality, Lemma 3.2 and Proposition 3.1:

$$G(y) \le (a(y), y)_{\mathbb{R}^N} \le |a(y)||y| \le c_4(|y|^q + |y|^p)$$

(see (3.5), (3.6)).

Example 3.4
Here we present some examples of maps satisfying hypotheses $H(a)$:

(a) $a(y) = |y|^{p-2}y$ with $1 < p < \infty$. This map corresponds to the p-Laplacian operator defined by

$$\Delta_p u = \operatorname{div}(|\nabla u|^{p-2}\nabla u), \qquad u \in W^{1,p}(\Omega).$$

(b) $a(y) = |y|^{p-2}y + |y|^{q-2}y$ with $1 < q < p < \infty$. This map corresponds to the (p,q)-differential operator defined by

$$\Delta_p u + \Delta_q u, \qquad u \in W^{1,p}(\Omega).$$

(c) $a(y) = (1 + |y|^2)^{(p-2)/2}y$ with $1 < p < \infty$. This map corresponds to the generalized p-mean curvature differential operator defined by

$$\operatorname{div}((1 + |\nabla u|^2)^{(p-2)/2}\nabla u), \qquad u \in W^{1,p}(\Omega).$$

Notation 3.5
Let $A\colon W^{1,p}(\Omega) \to W^{1,p}(\Omega)^*$ be defined by

$$\langle A(u), v\rangle = \int_\Omega (a(\nabla u(x)), \nabla v(x))_{\mathbb{R}^N}\,\mathrm{d}x, \qquad u, v \in W^{1,p}(\Omega), \tag{3.9}$$

where $\langle\cdot,\cdot\rangle$ denotes duality brackets for $(W^{1,p}(\Omega)^*, W^{1,p}(\Omega))$. We will also denote the duality brackets for $(W_0^{1,p}(\Omega)^*, W_0^{1,p}(\Omega))$ by $\langle\cdot,\cdot\rangle_0$. For a Carathéodory function $f\colon \Omega \times \mathbb{R} \to \mathbb{R}$ let $N_f\colon W^{1,p}(\Omega) \to \mathcal{M}(\Omega, \mathbb{R})$, where $\mathcal{M}(\Omega, \mathbb{R})$ is the set of all measurable functions on Ω, be defined by

$$N_f(u)\colon \Omega \ni x \mapsto N_f(u)(x) = f(x, u(x)) \in \mathbb{R} \quad \text{for } u \in W^{1,p}(\Omega).$$

Our assumptions on the Carathéodory map $f\colon \Omega \times \mathbb{R} \to \mathbb{R}$ are the following.

Assumption $H(f)_1$
For a.e. $z \in \Omega$, $f(z,0) = 0$ and for every $\rho > 0$ there exists $a_\rho \in L^\infty(\Omega)_+$ such that for a.e. $z \in \Omega$ and every $\xi \in \mathbb{R}$ we have

$$|\xi| \leq \rho \implies |f(z,\xi)| \leq a_\rho(z).$$

Assumption $H(f)_2$
There exist functions $w_\pm \in W^{1,p}(\Omega) \cap C(\bar{\Omega})$ and constants c_-, c_+ such that

$$\begin{aligned} w_-(z) \leq c_- < 0 < c_+ \leq w_+(z) &\quad \text{for all } z \in \bar{\Omega}, \\ f(z, w_+(z)) \leq 0 \leq f(z, w_-(z)) &\quad \text{for a.e. } z \in \Omega, \\ A(w_-) \leq 0 \leq A(w_+) &\quad \text{in } (W^{1,p}(\Omega))^*, \end{aligned} \tag{3.10}$$

where A is defined by (3.9) and by (3.10) we mean that for any $u \in W^{1,p}(\Omega)$ with $u \geq$, the following inequalities hold

$$\langle A(w_-), u\rangle \leq 0 \quad \text{and} \quad \langle A(w_+), u\rangle \geq 0.$$

Assumption $H(f)_3$
There exists $\delta_0 > 0$ such that for a.e. $z \in \Omega$ and for all $\xi \in \mathbb{R}$ such that $0 < |\xi| \leq \delta_0$, we have

$$0 < f(z,\xi)\xi \leq \mu F(z,\xi) \quad \text{and} \quad \operatorname*{ess\,inf}_{\Omega} F(\cdot, \delta_0) > 0$$

with $F(z,\xi) = \int_0^\xi f(z,t)\,\mathrm{d}t$ and μ is as in $H(a)_4$.

Assumption $H(f)_4$
There exist $\widehat{c}_0, \widehat{c}_1 > 0$ and $s, r \in \mathbb{R}$ with $s \neq r$ and $s < \mu$, $s \leq \tau \leq p \leq r \leq p^*$ (where τ and μ are the same as in $H(a)_4$) such that for a.e. $z \in \Omega$ and for all $\xi \in \mathbb{R}$

$$f(z,\xi)\xi \geq \widehat{c}_0|\xi|^s - \widehat{c}_1|\xi|^r. \tag{3.11}$$

Remark 3.6
Hypothesis $H(f)_3$ implies that

$$\widehat{c}_2|\xi|^\mu \leq F(z,\xi) \quad \text{for a.e. } z \in \Omega \text{ and for } |\xi| \leq \delta_0,$$

with some $\widehat{c}_2 > 0$.

Remark 3.7
Let us consider

$$\psi(\xi) = \widehat{c}_0|\xi|^{s-2}\xi - \widehat{c}_1|\xi|^{r-2}\xi, \qquad \xi \in \mathbb{R}. \tag{3.12}$$

Then inequality (3.11) becomes $f(z,\xi)\xi \geq \psi(\xi)\xi$ for $\xi \in \mathbb{R}$ and a.e. $z \in \Omega$, so for a.e. $z \in \Omega$ we have

$$\begin{aligned} f(z,\xi) \geq \psi(\xi), &\quad \text{if } \xi \geq 0, \\ f(z,\xi) \leq \psi(\xi), &\quad \text{if } \xi < 0. \end{aligned} \tag{3.13}$$

Let $\xi_0 = (\frac{\hat{c}_0}{\hat{c}_1})^{\frac{1}{r-s}}$. We can observe that $\psi > 0$ on $(-\infty, -\xi_0) \cup (0, \xi_0)$ and $\psi < 0$ on $(-\xi_0, 0) \cup (\xi_0, \infty)$. Also ψ is strictly increasing on $(\infty, -\xi_0)$ and strictly decreasing on (ξ_0, ∞). Thus, from $H(f)_2$ and (3.13) we infer that for a.e. $z \in \Omega$ we have $\psi(w_+(z)) < 0$ and $\psi(w_-(z)) > 0$.

Lemma 3.8
Let $1 < s \leq p \leq r$ with $s < r$. For any constants α, β, γ, given, if $\alpha, \beta > 0$, then we can find $M_1, M_2 > 0$ such that for any $\xi > 0$ we have

$$\alpha\xi^s - \beta\xi^r + \gamma\xi \leq M_1 - M_2\xi^p.$$

Example 3.9
$f(z,\xi) = \psi(\xi) = \widehat{c}_0|\xi|^{s-2}\xi - \widehat{c}_1|\xi|^{r-2}\xi$ satisfies conditions $H(f)_1$–$H(f)_4$. Indeed, we can easily observe that $H(f)_1$ and $H(f)_4$ are satisfied. We have that $\frac{\mu}{s} > 1$ and $\frac{\mu}{r} < 1$ (recall that $1 < s < \mu \leq q < p \leq r$), so

$$f(z,\xi)\xi = \widehat{c}_0|\xi|^s - \widehat{c}_1|\xi|^r < \frac{\mu}{s} \cdot \widehat{c}_0|\xi|^s - \frac{\mu}{r} \cdot \widehat{c}_1|\xi|^r = \mu F(z,\xi),$$

thus $H(f)_3$ is fulfilled. We set $w_\pm(z) = \pm\xi_0 = \pm(\frac{\hat{c}_0}{\hat{c}_1})^{\frac{1}{r-s}}$ for $z \in \Omega$ and $c_\pm = \pm\frac{\xi_0}{2}$. We have $w_\pm \in W^{1,p}(\Omega)$ and $\nabla w_\pm = 0$, so

$$\langle A(w_\pm), v\rangle = \int_\Omega (a(\nabla w_\pm(x)), \nabla v(x))_{\mathbb{R}^N}\, \mathrm{d}x = 0$$

for any $v \in W^{1,p}(\Omega)$. Also

$$f(z, w_\pm(z)) = \psi(w_\pm(z)) = \psi(\pm\xi_0) = 0$$

(see Remark 3.7). Thus $H(f)_2$ is also fulfilled.

4. Existence of two constant sign solutions

In this section we state the main result of this paper.

Theorem 4.1
If hypotheses $H(a)_1$–$H(a)_4$ and $H(f)_1$–$H(f)_4$ hold, then Problem (1.1) *has at least two nontrivial constant sign smooth solutions*

$$u_0 \in \operatorname{int} C_+ \quad \textit{and} \quad v_0 \in -\operatorname{int} C_+.$$

Proof. First we prove the existence of nontrivial positive smooth solution u_0. We introduce the following truncation of the reaction term

$$\widehat{f}_+(z,\xi) = \begin{cases} 0, & \text{if } \xi < 0,\\ f(z,\xi) + \xi^{p-1}, & \text{if } 0 \leq \xi \leq w_+(z),\\ f(z,w_+(z)) + \xi^{p-1} + \psi(\xi) - \psi(w_+(z)), & \text{if } \xi > w_+(z), \end{cases} \tag{4.1}$$

where ψ is given by (3.12). This is a Carathéodory function. Let

$$\widehat{F}_+(z,\xi) = \int_0^\xi \widehat{f}_+(z,t)\,\mathrm{d}t$$

and consider the C^1-functional $\widehat{\varphi}_+\colon W^{1,p}(\Omega) \to \mathbb{R}$, defined by

$$\widehat{\varphi}_+(u) = \int_\Omega G(\nabla u(z))\,\mathrm{d}z + \frac{1}{p}\|u\|_p^p - \int_\Omega \widehat{F}_+(z,u(z))\,\mathrm{d}z, \qquad u \in W^{1,p}(\Omega).$$

Claim

There exists $u_0 \in W^{1,p}(\Omega)$ such that

$$\widehat{\varphi}_+(u_0) = \min_{u \in W^{1,p}(\Omega)} \widehat{\varphi}_+(u) =: \widehat{m}_+.$$

It is clear that $\widehat{\varphi}_+$ is weakly sequentially lower semicontinuous. We have

$$\widehat{F}_+(z,\xi) = \begin{cases} 0, & \text{if } \xi < 0,\\ \int_0^\xi f(z,t)\,\mathrm{d}t + \frac{1}{p}\xi^p, & \text{if } 0 \le \xi \le w_+(z),\\ \int_0^{w_+(z)} f(z,t)\,\mathrm{d}t + (\xi - w_+(z))f(z,w_+(z)) & \\ \quad + \int_{w_+(z)}^{\xi} \psi(t)\,\mathrm{d}t - (\xi - w_+(z))\psi(w_+(z)) + \frac{1}{p}\xi^p, & \text{if } \xi > w_+(z). \end{cases} \tag{4.2}$$

As $w_+ \in C(\bar{\Omega})$, there exists $z_0, z_1 \in \bar{\Omega}$ such that

$$w_+(z_0) = \max_{z \in \bar{\Omega}} w_+(z)$$

and

$$\psi(w_+(z_1)) = \min_{z \in \bar{\Omega}} \left(\widehat{c}_0 w_+(z)^{s-1} - \widehat{c}_1 w_+(z)^{r-1}\right) = \min_{z \in \bar{\Omega}} \psi(w_+(z)).$$

Also let us denote $\Omega_< := \{z \in \Omega \mid \ 0 \le u(z) \le w_+(z)\}$ and $\Omega_> := \{z \in \Omega \mid \ u(z) > w_+(z)\}$. Then (see (4.2))

$$\int_\Omega \widehat{F}_+(z,u(z))\,\mathrm{d}z = \int_\Omega \widehat{F}_+(z,u^+(z))\,\mathrm{d}z = \sum_{i=1}^{5} \mathrm{I}_i + \frac{1}{p}\|u^+\|_p^p,$$

where

$$\begin{aligned} I_1 &= \int_{\Omega_<} \int_0^{u^+(z)} f(z,t)\,\mathrm{d}t\,\mathrm{d}z,\\ I_2 &= \int_{\Omega_>} \int_0^{w_+(z)} f(z,t)\,\mathrm{d}t\,\mathrm{d}z,\\ I_3 &= \int_{\Omega_>} u^+(z) f(z,w_+(z))\,\mathrm{d}z + \int_{\Omega_>} w_+(z)(\psi(w_+(z)) - f(z,w_+(z)))\,\mathrm{d}z,\\ I_4 &= -\int_{\Omega_>} \left(\frac{\widehat{c}_0}{s} w_+(z)^s - \frac{\widehat{c}_1}{r} w_+(z)^r\right)\mathrm{d}z,\\ I_5 &= \int_{\Omega_>} \left(\frac{\widehat{c}_0}{s} u^+(z)^s - \frac{\widehat{c}_1}{r} u^+(z)^r - u^+(z)\psi(w_+(z))\right)\mathrm{d}z. \end{aligned}$$

From $H(f)_2$, $H(f)_1$ we have

$$\begin{aligned} I_1 &\le \int_{\Omega_<} \int_0^{u_+(z)} f^+(z,t)\,\mathrm{d}t\,\mathrm{d}z \le \int_{\Omega_<} \int_0^{w_+(z_0)} f^+(z,t)\,\mathrm{d}t\,\mathrm{d}z\\ &\le \int_{\Omega_<} \int_0^{w_+(z_0)} a_{w_+(z_0)}(z)\,\mathrm{d}t\,\mathrm{d}z\\ &\le w_+(z_0) \cdot |\Omega_<|_N \cdot \|a_{w_+(z_0)}\|_{L^\infty} \end{aligned}$$

and similarly,

$$I_2 \leq w_+(z_0) \cdot |\Omega_>|_N \cdot \|a_{w_+(z_0)}\|_{L^\infty}.$$

From $H(f)_2$ and Remark 3.7, we have

$$I_3 \leq \int_{\Omega_>} w_+(z)(\psi(w_+(z)) - f(z, w_+(z)))\,\mathrm{d}z \leq 0.$$

As ψ and w_+ are continuous functions (see $H(f)_2$ and Remark 3.7) and Ω is bounded, there exists constant $\widehat{c}_3 \in \mathbb{R}$, such that

$$\mathrm{I}_4 \leq \widehat{c}_3.$$

Using Lemma 3.8 we can find $\widehat{c}_4, \widehat{c}_5, \widehat{c}_6 > 0$, such that

$$\begin{aligned}
\mathrm{I}_5 &\leq \int_{\Omega_>} \Big(\frac{\widehat{c}_0}{s} u^+(z)^s - \frac{\widehat{c}_1}{r} u^+(z)^r + u^+(z)|\psi(w_+(z_1))|\Big)\,\mathrm{d}z \\
&\leq \int_{\Omega_>} \big(\widehat{c}_4 - \widehat{c}_5 u^+(z)^p\big)\,\mathrm{d}z = \widehat{c}_4|\Omega_>|_N - \widehat{c}_5 \int_{\Omega_>} u_+(z)^p\,\mathrm{d}z \\
&\leq \widehat{c}_6 - \widehat{c}_5\|u^+\|_p^p.
\end{aligned}$$

Then

$$\begin{aligned}
\widehat{\varphi}_+(u) &= \int_\Omega G(\nabla u(z))\,\mathrm{d}z + \frac{1}{p}\|u\|_p^p - \int_\Omega \widehat{F}_+(z, u(z))\,\mathrm{d}z \\
&\geq \frac{c_1}{p(p-1)} \int_\Omega \|\nabla u^+(z)\|^p\,\mathrm{d}z + \frac{c_1}{p(p-1)} \int_\Omega \|\nabla u^-(z)\|^p\,\mathrm{d}z \\
&\quad + \frac{1}{p}\|u^+\|_p^p + \frac{1}{p}\|u^-\|_p^p + \widehat{c}_5\|u^+\|_p^p - \frac{1}{p}\|u^+\|_p^p + \widehat{c}_7 \\
&\geq \widehat{c}_8 + \widehat{c}_9\|u\|^p,
\end{aligned}$$

with some constants $\widehat{c}_7, \widehat{c}_8 \in \mathbb{R}$, $\widehat{c}_9 > 0$ (see (3.8)). Hence $\widehat{\varphi}_+$ is also weakly coercive, so we can apply Theorem 2.2 and obtain that there exists $u_0 \in W^{1,p}(\Omega)$ such that

$$\widehat{\varphi}_+(u_0) = \min_{u \in W^{1,p}(\Omega)} \widehat{\varphi}_+(u).$$

This proves the Claim.

By the Claim we get that $(\widehat{\varphi}_+)'(u_0) = 0$ (see Zeidler [9], Proposition 25.11, p.510). This implies

$$A(u_0) + |u_0|^{p-2}u_0 = N_{\widehat{f}_+}(u_0). \tag{4.3}$$

We will show that $\widehat{\varphi}_+(u_0) < 0$, so $u_0 \neq 0$. By virtue of hypothesis $H(a)_4$ for a given $\varepsilon > 0$ we can find $\delta_{1,\varepsilon} \in (0, \delta_0]$ such that

$$G_0(t) \leq \varepsilon t^\mu \quad \forall t \in (0, \delta_{1,\varepsilon}],$$

so by the definition of G for $y \in \mathbb{R}^N$ such that $|y| \leq \delta_{1,\varepsilon}$ we have

$$G(y) \leq \varepsilon |y|^\mu.$$

For $\tilde{u} \in \operatorname{int} C_+$ and $t \in (0,1)$ such that $|\nabla(t\tilde{u})(z)| \leq \delta_{1,\varepsilon}$ we get

$$\int_\Omega G(|\nabla(t\tilde{u})(z)|)\,\mathrm{d}z \leq t^\mu \varepsilon \|\nabla(\tilde{u})\|_\mu^\mu,$$

so by Remark 3.6 for $t \in (0,1)$ such that $t\tilde{u}(z) \leq \min\{\delta_{1,\varepsilon}, c_+\}$ (see $H(f)_2$) and $|\nabla(t\tilde{u})(z)| \leq \min\{\delta_{1,\varepsilon}, c_+\}$ for a.e. $z \in \Omega$, we have

$$\widehat{\varphi}_+(t\tilde{u}) \leq t^\mu \varepsilon \|\nabla(\tilde{u})\|_\mu^\mu - \widehat{c}_2 t^\mu \|\tilde{u}\|_\mu^\mu$$

(see (4.2)). Choosing $\varepsilon < \frac{\hat{c}_2 \|\tilde{u}\|_\mu^\mu}{\|\nabla(\tilde{u})\|_\mu^\mu}$, we see that

$$\widehat{\varphi}_+(u_0) \leq \widehat{\varphi}_+(t\tilde{u}) < 0 = \widehat{\varphi}_+(0),$$

hence $u_0 \neq 0$.

In the next step we will show that

$$A(u_0) = N_f(u_0). \tag{4.4}$$

First we act on (4.3) with $-u_0^- \in W^{1,p}(\Omega)$. Then

$$\begin{aligned}\int_\Omega \big(a(\nabla u_0(z)), \nabla(-u_0^-(z))\big)_{\mathbb{R}^N}\,\mathrm{d}z + \int_\Omega |u_0(z)|^{p-2} u_0(z)(-u_0^-)(z)\,\mathrm{d}z \\ = \int_\Omega \widehat{f}_+(z, u_0(z))(-u_0)^-(z)\,\mathrm{d}z = 0\end{aligned}$$

(see (4.1)). By (3.7) we get

$$\frac{c_1}{p-1}\|\nabla(-u_0^-))\|_p^p + \|(-u_0^-)\|_p^p \leq 0,$$

so $u_0 \geq 0$ (because $u_0 \neq 0$).

Next on (4.3) we act with $(u_0 - w_+)^+ \in W^{1,p}(\Omega)$. Suppose that $|\{u_0 > w_+\}|_N > 0$. Then

$$\begin{aligned}
&\big\langle A(u_0), (u_0 - w_+)^+\big\rangle + \int_\Omega |u_0(z)|^{p-1}(u_0 - w_+)^+(z)\,\mathrm{d}z \\
&\quad = \int_\Omega \widehat{f}_+(z, u_0(z))(u_0 - w_+)^+(z)\,\mathrm{d}z \\
&\quad = \int_\Omega (f(z, w_+(z)) + u_0(z)^{p-1} + \psi(u_0(z)) - \psi(w_+(z)))(u_0 - w_+)^+(z)\,\mathrm{d}z \\
&\quad = \int_\Omega |u_0(z)|^{p-1}(u_0 - w_+)^+(z)\,\mathrm{d}z + \int_{\{u_0 > w_+\}} f(z, w_+(z))(u_0 - w_+)(z)\,\mathrm{d}z \\
&\qquad + \int_{\{u_0 > w_+\}} (\psi(u_0(z)) - \psi(w_+(z)))(u_0 - w_+)(z)\,\mathrm{d}z \\
&\quad < \int_\Omega |u_0(z)|^{p-1}(u_0 - w_+)^+(z)\,\mathrm{d}z + \int_{\{u_0 > w_+\}} f(z, w_+(z))(u_0 - w_+)(z)\,\mathrm{d}z \\
&\quad \leq \int_\Omega |u_0(z)|^{p-1}(u_0 - w_+)^+(z)\,\mathrm{d}z \\
&\quad \leq \big\langle A(w_+), (u_0 - w_+)^+\big\rangle + \int_\Omega |u_0(z)|^{p-1}(u_0 - w_+)^+(z)\,\mathrm{d}z,
\end{aligned}$$

(see (4.1), $H(f)_2$ and Remark 3.7), so

$$\int_{\{u_0>w_+\}} (a(\nabla u_0(z)) - a(\nabla w_+(z)), \nabla((u_0 - w_+)(z)))_{\mathbb{R}^N} < 0,$$

a contradiction with the strict monotonicity of a (see Lemma 3.2). Thus $|\{u_0 > w_+\}|_N = 0$, hence $u_0 \le w_+$ a.e. in Ω. We have proved that

$$u_0 \in [0, w_+] \quad \text{and} \quad u_0 \ne 0,$$

where $[0, w_+] = \{u \in W^{1,p}(\Omega) \mid \ 0 \le u(z) \le w_+(z) \text{ for a.e. } z \in \Omega\}$. Then, by virtue of (4.1), we have that

$$N_{\widehat{f}_+}(u_0) = N_f(u_0) + |u_0|^{p-2}u_0,$$

so by (4.3) we obtain (4.4). Also $u_0 \in L^\infty(\Omega)$, as for a.e. $z \in \Omega$ we have $|u(z)| \le w_+(z) \le \|w_+\|_{C(\bar{\Omega})}$. By virtue of (3.6) and the representation theorem for the elements of $W^{-1,p'}(\Omega) = (W_0^{1,p}(\Omega))^*$ (see Gasiński-Papageorgiou [3], p.212), we can obtain

$$\operatorname{div} a(\cdot, \nabla u_0(\cdot)) \in W^{-1,p'}(\Omega).$$

Next we act on (4.4) with $v \in C_c^1(\Omega)$ and obtain

$$\langle -\operatorname{div} a(\cdot, \nabla u_0(\cdot)), v\rangle_0 = \langle N_f(u_0), v\rangle_0 .$$

Thus

$$-\operatorname{div} a(\nabla u_0(z)) = f(z, u_0(z)) \quad \text{a.e. in } \Omega \tag{4.5}$$

(recall that $C_c^1(\Omega)$ is dense in $W_0^{1,p}(\Omega)$). By Theorem 2.5 we have $\frac{\partial u}{\partial n_a} \in T^{-p'}(\partial\Omega)$ satisfying the Green's formula for operator a

$$\sum_{i=1}^N \int_\Omega a_i(\nabla u(z)) \frac{\partial v}{\partial z_i} \, \mathrm{d}z = \int_\Omega -\operatorname{div} a(\nabla u(z)) v(z) \, \mathrm{d}z + \left\langle \frac{\partial u}{\partial n_a}, \gamma_0(v) \right\rangle_T$$

for all $v \in W^{1,p}(\Omega) \cap L^\infty(\Omega)$. Combining this with (4.5) and (4.4) we obtain

$$\left\langle \frac{\partial u}{\partial n_a}, \gamma_0(v) \right\rangle_T = 0, \quad \forall v \in W^{1,p}(\Omega) \cap L^\infty(\Omega),$$

hence

$$\frac{\partial u}{\partial n_a} = 0 \quad \text{in } T^{-p'}(\partial\Omega)$$

(recall that $\gamma_0(W^{1,p}(\Omega) \cap L^\infty(\Omega)) = T^p(\partial\Omega)$).

As will be proven below, by the regularity result of Lieberman (Theorem 2.3), we have $u_0 \in C_+ \backslash \{0\}$. Indeed, let $h \in C^1(0, \infty)$ satisfy (3.1) and (3.2). We take $A(z, \xi, y) = a(y)$, $B(z, \xi, y) = f(z, \xi)$. Then (2.1a) becomes (3.4). Also (2.1b) is satisfied by (3.3), with $\Lambda = c_3$. (2.1c) and (2.1d) obviously hold, because for any $\Lambda_1 > 0$ we have $|A(z_1, \xi_1, y) - A(z_2, \xi_2, y)| = |a(y) - a(y)| = 0$. We have

$$|B(z_1, \xi_1, y)| = |f(z_1, \xi_1)| \le a_{w_+(z_0)}(z_1) \le \|a_{w_+(z_0)}\|_{L^\infty} \le \Lambda_1(1 + h(|y\|)|y|)$$

for all $y \in \mathbb{R}^N$, with $\Lambda_1 = \|a_{w_+(z_0)}\|_{L^\infty}$. For such A and B, u_0 also solves $\operatorname{div} A(z,u,\nabla u) + B(z,u,\nabla u) = 0$. As $1 < q < p$ we can easily check that $u_0 \in W^{1,H}(\Omega)$, namely

$$\begin{aligned}\int_\Omega H(|\nabla u_0|)\,\mathrm{d}z &= \int_\Omega \int_0^{|\nabla u_0|} h(t)\,\mathrm{d}t\,\mathrm{d}z \le \int_\Omega \Big(\frac{c_2}{q}|\nabla u_0|^q + \frac{c_2}{p}|\nabla u_0|^p\Big)\,\mathrm{d}z \\ &\le \frac{c_2}{q}|\Omega|_N + \Big(\frac{c_2}{q} + \frac{c_2}{p}\Big)\|\nabla u_0\|_p^p \\ &< \infty\end{aligned}$$

as $u_0 \in W^{1,p}(\Omega)$ (see (3.2)). So u_0 satisfies the assumptions of Theorem 2.3 with $M_0 = w_+(z_0)$, hence $u_0 \in C^{1,\beta}(\Omega)$ with some $\beta > 0$, depending on α, c_0, c_3, δ, N. Thus u_0 is a smooth solution of Problem (1.1). From the strong maximum principle (see Theorem 2.4), it follows that $u_0 \in \operatorname{int} C_+$. To prove that, in Theorem 2.4, we put $A := a_0$. Then the map $s \mapsto sA(s)$ is strictly increasing in $\mathbb{R}^+$ as G_0 is strictly convex (see Proposition 3.1). Also

$$L(s) = s^2 a_0(s) - \int_0^s t a_0(t)\,\mathrm{d}t = \int_0^s (t a_0(t) + t^2 a_0'(t))\,\mathrm{d}t$$

is a strictly increasing C^1-function. From (4.5) and (3.13) we have, that there exists some $\lambda_0 > 0$ such that for a.e. $z \in \Omega$

$$-\operatorname{div} a_0(|\nabla u_0(z)|)\nabla u_0(z) + \lambda_0 u_0(z)^{r-1} = f(z, u_0(z)) + \lambda_0 u_0(z)^{r-1} \ge 0.$$

Thus $u_0 \in C^{1,\beta}(\Omega)$ is a classical distribution solution of

$$\operatorname{div} a_0(|\nabla u_0(z)|)\nabla u_0 - \lambda_0 u_0^{r-1} \le 0.$$

For $y \in \mathbb{R}^N$ and $i, j = 1, \dots, N$ we have that

$$\frac{\partial a_j}{\partial y_i}(y) = \frac{\partial}{\partial y_i}(a_0(|y|)y_j) = a_0'(|y|)\frac{y_i y_j}{|y|} + \delta_{ij} a_0(|y|),$$

where δ_{ij} is the Kronecker delta. Thus from (3.4) we obtain for any $s \ge 0$

$$a_0'(s)s^4 + a_0(s)s^2 \ge c_1 s^p, \qquad s \ge 0. \tag{4.6}$$

As for $0 \le s < 1$ we have $a_0'(s)s^2 \ge a_0'(s)s^3$, it follows from (4.6) that

$$L(s) \ge c_1 \int_0^s t^{p-1}\,\mathrm{d}t = \frac{c_1}{p}s^p \ge \frac{c_1}{p}s^r, \qquad 0 \le s < 1.$$

We define $L_0(s) := \frac{c_1}{p}s^r$. We have that $L, L_0\colon [0,1) \to \mathbb{R}$ are strictly increasing, $L(0) = L_0(0) = 0$ and $L(s) \ge L_0(s)$ for any $0 \le s < 1$, so from Leoni ([4], p.6) we infer that there exists some constant $\delta_2 > 0$ such that

$$L^{-1}(s) \le L_0^{-1}(s), \qquad 0 \le s < \delta_2. \tag{4.7}$$

Setting $p(s) := \lambda_0 s^{r-1}$, from (4.7) we obtain for any $\epsilon < (\frac{r}{\lambda_0}\delta_2)^{\frac{1}{r}}$

$$\int_0^\epsilon \frac{1}{L^{-1}(P(s))}\,\mathrm{d}s \geq \int_0^\epsilon \frac{1}{L_0^{-1}(P(s))}\,\mathrm{d}s = \widehat{c}_8 \int_0^\epsilon \frac{1}{t}\,\mathrm{d}s = \infty,$$

where $\widehat{c}_8 > 0$ is a constant depending on λ_0, c_1, p, r. Thus we have shown that the assumptions of Theorem 2.4 are satisfied and we are allowed to apply the strong maximum principle as required.

Similarly, using the truncation

$$\widehat{f}_-(z,\xi) = \begin{cases} 0, & \text{if } \xi > 0, \\ f(z,\xi) + \xi^{p-1}, & \text{if } w_-(z) \leq \xi \leq 0, \\ f(z,w_-(z)) + \xi^{p-1} + \psi(\xi) - \psi(w_-(z)), & \text{if } \xi < w_-(z) \end{cases}$$

we prove the existence of the nontrivial negative smooth solution.

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