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# A note on preserving the spark of a matrix

**Abstract.** Let $\mathcal{M}_{m\times n}(\mathbb{F})$ be the vector space of all $m\times n$ matrices over a field $\mathbb{F}$. In the case where $m \geq n$, $\operatorname{char}(\mathbb{F}) \neq 2$ and $\mathbb{F}$ has at least five elements, we give a complete characterization of linear maps $\Phi\colon \mathcal{M}_{m\times n}(\mathbb{F}) \to \mathcal{M}_{m\times n}(\mathbb{F})$ such that $\operatorname{spark}(\Phi(A)) = \operatorname{spark}(A)$ for any $A \in \mathcal{M}_{m\times n}(\mathbb{F})$.

## 1. Preliminaries and introduction

Throughout the text, $m$ and $n$ stand for positive integers, and $\mathbb{F}$ denotes a field. We define $\mathcal{M}_{m\times n}(\mathbb{F})$ to be the vector space of all $m \times n$ matrices over $\mathbb{F}$. The $m \times n$ zero matrix will be denoted by $O_{m\times n}$ and the $n$th full linear group over $\mathbb{F}$ by $\mathcal{GL}_n(\mathbb{F})$ (i.e., $\mathcal{GL}_n(\mathbb{F}) = \{V \in \mathcal{M}_{n\times n}(\mathbb{F}) : \det(V) \neq 0\}$). Finally, if $x_1, \ldots, x_n$ are the components of a (row or column) vector $x \in \mathbb{F}^n$, then the *Hamming weight* of $x$ is defined by

$$\|x\|_0 = \#\big\{j \in \{1, \ldots, n\} : \ x_j \neq 0\big\}.$$

In [3], Donoho and Elad introduced the concept of spark of a matrix into the mathematical theory of compressed sensing. Let us recall the definition.

Definition 1.1
*Suppose that $C_1, \ldots, C_n \in \mathcal{M}_{m\times 1}(\mathbb{F})$ are the columns of a matrix $A \in \mathcal{M}_{m\times n}(\mathbb{F})$. The spark of $A$ is defined to be the infimum of the set of all positive integers $\ell$ with the property that*

$$\exists\, j_1, \ldots, j_\ell \in \{1, \ldots, n\} : \begin{cases} j_1 < \ldots < j_\ell, \\ C_{j_1}, \ldots, C_{j_\ell} \text{ are linearly dependent.} \end{cases}$$

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The following facts about the spark are well known and easy to prove.

Proposition 1.2
*Let* $A \in \mathcal{M}_{m\times n}(\mathbb{F})$ *and* $U \in \mathcal{GL}_m(\mathbb{F})$. *Then*

(i) $\operatorname{spark}(A) \in \{1, \dots, n\} \cup \{+\infty\}$,

(ii) $\operatorname{spark}(A) = +\infty$ *if and only if* $\operatorname{rank}(A) = n$,

(iii) $\operatorname{spark}(A) = 1$ *if and only if* $A$ *has a zero column,*

(iv) $\operatorname{spark}(A) \leq \operatorname{rank}(A) + 1$ *whenever* $\operatorname{spark}(A) \neq +\infty$,

(v) $\operatorname{spark}(UA) = \operatorname{spark}(A)$.

When dealing with a reasonable map $f$ defined on $\mathcal{M}_{m\times n}(\mathbb{F})$, it is always of interest to know what linear endomorphisms $\Phi\colon \mathcal{M}_{m\times n}(\mathbb{F}) \to \mathcal{M}_{m\times n}(\mathbb{F})$ have the property that $f(\Phi(A)) = f(A)$ for any $A \in \mathcal{M}_{m\times n}(\mathbb{F})$. Such endomorphisms are called *linear preservers* of the map $f$. The theory of linear preserver problems dates back to 1890s (Frobenius' theorem on linear preservers of the determinant function) and still attracts the attention of many mathematicians. We refer to [2] for a nice overview of results.

This note provides some remarks on linear preservers of the function

$$\mathcal{M}_{m\times n}(\mathbb{F}) \ni A \mapsto \operatorname{spark}(A) \in \{1, \dots, n\} \cup \{+\infty\}.$$

We will need the following technical definition.

Definition 1.3
*Let* $U \in \mathcal{GL}_m(\mathbb{F})$ *and* $V \in \mathcal{GL}_n(\mathbb{F})$. *A map* $\Phi\colon \mathcal{M}_{m\times n}(\mathbb{F}) \to \mathcal{M}_{m\times n}(\mathbb{F})$ *is said to be* $(U, V)$*-standard, if either* $\Phi(A) = UAV$ *for all* $A \in \mathcal{M}_{m\times n}(\mathbb{F})$, *or* $m = n$ *and* $\Phi(A) = UA^{\mathrm{T}}V$ *for all* $A \in \mathcal{M}_{m\times n}(\mathbb{F})$.

Notice that the $(U, V)$-standard map is a linear automorphism of $\mathcal{M}_{m\times n}(\mathbb{F})$.

The note is based on the characterization of rank $k$ preservers given by Beasley and Laffey (see [1]), which we recall below.

Theorem 1.4
*Let* $k$ *be a positive integer such that* $k \leq \min\{m, n\}$. *Suppose that the field* $\mathbb{F}$ *has at least four elements. If a linear automorphism* $\Phi\colon \mathcal{M}_{m\times n}(\mathbb{F}) \to \mathcal{M}_{m\times n}(\mathbb{F})$ *satisfies the condition*

$$\forall\, A \in \mathcal{M}_{m\times n}(\mathbb{F}):\ \operatorname{rank}(A) = k \implies \operatorname{rank}(\Phi(A)) = k,$$

*then it is a* $(U, V)$*-standard map, for some* $U \in \mathcal{GL}_m(\mathbb{F})$ *and some* $V \in \mathcal{GL}_n(\mathbb{F})$.

## 2. Results

Our main purpose is to prove

Theorem 2.1
*If* $\mathbb{F}$ *has at least five elements,* $\mathrm{char}(\mathbb{F}) \neq 2$*, and* $m \geq n$*, then for a linear endomorphism* $\Phi\colon \mathcal{M}_{m\times n}(\mathbb{F}) \to \mathcal{M}_{m\times n}(\mathbb{F})$*, the following conditions are equivalent:*

(1) $\forall\, A \in \mathcal{M}_{m\times n}(\mathbb{F}):\ \mathrm{spark}(\Phi(A)) = \mathrm{spark}(A)$,

(2) *there exist a matrix* $U \in \mathcal{GL}_m(\mathbb{F})$*, a diagonal matrix* $D \in \mathcal{GL}_n(\mathbb{F})$*, and an* $n \times n$ *permutation matrix* $P$ *such that* $\forall\, A \in \mathcal{M}_{m\times n}(\mathbb{F}):\ \Phi(A) = UADP$.

The proof will use two simple propositions and a lemma. The propositions are of independent interest.

Proposition 2.2
*Let* $\Phi\colon \mathcal{M}_{m\times n}(\mathbb{F}) \to \mathcal{M}_{m\times n}(\mathbb{F})$ *be a linear map. Suppose that* $\mathrm{spark}(\Phi(A)) = \mathrm{spark}(A)$ *for any* $A \in \mathcal{M}_{m\times n}(\mathbb{F})$*. Then* $\Phi$ *is bijective.*

*Proof.* Pick a matrix $B \in \mathcal{M}_{m\times n}(\mathbb{F})$ such that $\Phi(B) = O_{m\times n}$. It is enough to show that $B = O_{m\times n}$. Since $\mathrm{spark}(B) = \mathrm{spark}(\Phi(B)) = 1$, the matrix $B$ has a zero column. Assume that $B$ has a nonzero column as well (and hence $n \geq 2$). Let $S \in \mathcal{M}_{m\times n}(\mathbb{F})$ be the matrix consisting of $n$ copies of this nonzero column. Then $\mathrm{spark}(S) = 2$ and

$$\mathrm{spark}(\Phi(S)) = \mathrm{spark}(\Phi(S-B)) = \mathrm{spark}(S-B) = 1,$$

a contradiction. Consequently, $B = O_{m\times n}$.

Proposition 2.3
*Suppose that* $\mathrm{char}(\mathbb{F}) \neq 2$*. Then, for a matrix* $V \in \mathcal{GL}_n(\mathbb{F})$*, the following conditions are equivalent:*

(1) $\forall\, A \in \mathcal{M}_{m\times n}(\mathbb{F}):\ \mathrm{spark}(AV) = \mathrm{spark}(A)$,

(2) *there exist a diagonal matrix* $D \in \mathcal{GL}_n(\mathbb{F})$ *and an* $n \times n$ *permutation matrix* $P$ *such that* $V = DP$.

*Proof.* Let $W = [w_{ij}] \in \mathcal{M}_{n\times n}(\mathbb{F})$ be such that

$$\exists\, \ell, p, q \in \{1, \ldots, n\}: \begin{cases} p \neq q, \\ w_{p\ell} \neq 0,\ w_{q\ell} \neq 0. \end{cases}$$

We will show that there is a matrix $B = [b_{kj}] \in \mathcal{M}_{m\times n}(\mathbb{F})$ with $\mathrm{spark}(B) = 2$ and $\mathrm{spark}(BW) = 1$.

Let $\lambda = w_{1\ell} + \ldots + w_{n\ell}$. Assume that $w_{r\ell} - \lambda \neq 0$ for some $r \in \{p, q\}$. Then it suffices to define

$$b_{kj} = \begin{cases} 1, & \text{if } k = 1 \text{ and } j \neq r, \\ 1 - \lambda w_{r\ell}^{-1}, & \text{if } k = 1 \text{ and } j = r, \\ 0, & \text{otherwise.} \end{cases}$$

Assume, therefore, that $w_{p\ell} - \lambda = 0 = w_{q\ell} - \lambda$. Then $\lambda + (\lambda - w_{p\ell} - w_{q\ell}) = 0$, and hence it suffices to define

$$b_{kj} = \begin{cases} 2, & \text{if } k = 1 \text{ and } j \notin \{p, q\}, \\ 1, & \text{if } k = 1 \text{ and } j \in \{p, q\}, \\ 0, & \text{otherwise.} \end{cases}$$

Thus, if a matrix $V \in \mathcal{GL}_n(\mathbb{F})$ satisfies condition (1), then each column of $V$ (and hence each row) has exactly one element different from 0. Condition (2) follows. Implication (2) $\Rightarrow$ (1) is obvious and holds true over an arbitrary field.

We denote by $\Sigma_n$ the set of all permutations of $\{1, \dots, n\}$.

Corollary 2.4
*Suppose that* $\operatorname{char}(\mathbb{F}) \neq 2$. *Then, for a linear endomorphism* $f\colon \mathcal{M}_{1\times n}(\mathbb{F}) \to \mathcal{M}_{1\times n}(\mathbb{F})$*, the following conditions are equivalent:*

(1) $\forall\, x \in \mathcal{M}_{1\times n}(\mathbb{F}):\ \|f(x)\|_0 = \|x\|_0$,

(2) $\exists\, \sigma \in \Sigma_n\, \exists\, a_1, \dots, a_n \in \mathbb{F} \setminus \{0\}\, \forall\, x = (x_1, \dots, x_n) \in \mathcal{M}_{1\times n}(\mathbb{F}):\ f(x) = (a_1 x_{\sigma(1)}, \dots, a_n x_{\sigma(n)})$,

(3) $\forall\, x \in \mathcal{M}_{1\times n}(\mathbb{F}):\ \operatorname{spark}(f(x)) = \operatorname{spark}(x)$.

*Proof.* If $n = 1$, then there is nothing to do. Assume that $n \geq 2$. Then $\operatorname{spark}(x) \in \{1, 2\}$ for all $x \in \mathcal{M}_{1\times n}(\mathbb{F})$. Moreover, $\operatorname{spark}(x) = 2$ for some $x \in \mathcal{M}_{1\times n}(\mathbb{F})$ if and only if $\|x\|_0 = n$. These two properties yield implication (1) $\Rightarrow$ (3). Let us proceed to (3) $\Rightarrow$ (2). If condition (3) is satisfied, then by Proposition 2.2, the endomorphism $f$ is bijective, and hence

$$\exists\, V \in \mathcal{GL}_n(\mathbb{F})\, \forall\, x \in \mathcal{M}_{1\times n}(\mathbb{F}):\ f(x) = xV.$$

Condition (2) now follows from Proposition 2.3. Implication (2) $\Rightarrow$ (1) is obvious.

The above equivalence (1) $\Leftrightarrow$ (2) is well known and can be easily proved over an arbitrary field, without involving the concept of spark. Implication (1) $\Rightarrow$ (3) also holds true over an arbitrary field.

Example 2.5
*The linear endomorphism* $g\colon \mathcal{M}_{1\times 3}(\mathbb{Z}_2) \ni (x_1, x_2, x_3) \mapsto (x_1, x_1 + x_2 + x_3, x_3) \in \mathcal{M}_{1\times 3}(\mathbb{Z}_2)$ *satisfies the condition*

$$\forall\, x \in \mathcal{M}_{1\times 3}(\mathbb{Z}_2):\ \operatorname{spark}(g(x)) = \operatorname{spark}(x).$$

*However, $g$ is not a "Hamming isometry".*

Notice that a linear endomorphism $h\colon \mathcal{M}_{n\times 1}(\mathbb{F}) \to \mathcal{M}_{n\times 1}(\mathbb{F})$ is a preserver of the spark if and only if $h$ is bijective.

Let us return to the main purpose of the note.

LEMMA 2.6
*If $n \geq 2$, then no matrix $V \in \mathcal{GL}_n(\mathbb{F})$ has the property that* $\operatorname{spark}(A^{\mathrm{T}}V) = \operatorname{spark}(A)$ *for all $A \in \mathcal{M}_{n\times n}(\mathbb{F})$.*

*Proof.* Assume that $n \geq 2$ and pick a matrix $V \in \mathcal{GL}_n(\mathbb{F})$. Let $C \in \mathcal{M}_{n\times 1}(\mathbb{F})$ be a nonzero column such that every element of $C^{\mathrm{T}}V$ is different from 0. Define $S \in \mathcal{M}_{n\times n}(\mathbb{F})$ to be the matrix whose first column coincides with $C$ and any other column coincides with $O_{n\times 1}$. Then $\operatorname{spark}(S) = 1$ and $\operatorname{spark}(S^{\mathrm{T}}V) = 2$.

*Proof of Theorem 2.1.* Implication (2) $\Rightarrow$ (1) is obvious (cf. Proposition 1.2; the implication holds true over an arbitrary field and even if $m < n$). Assume that $\Phi$ satisfies condition (1). Then it follows from Proposition 2.2 that $\Phi$ is bijective. Moreover, if $\operatorname{rank}(A) = n$ for a matrix $A \in \mathcal{M}_{m\times n}(\mathbb{F})$, then $\operatorname{spark}(\Phi(A)) = \operatorname{spark}(A) = +\infty$, and hence $\operatorname{rank}(\Phi(A)) = n$. Theorem 1.4 yields therefore that $\Phi$ is a $(U, V)$-standard map for some $U \in \mathcal{GL}_m(\mathbb{F})$ and some $V \in \mathcal{GL}_n(\mathbb{F})$. Suppose, for a moment, that $m = n \geq 2$ and

$$\forall\, A \in \mathcal{M}_{m\times n}(\mathbb{F}) : \ \Phi(A) = UA^{\mathrm{T}}V.$$

Then $\operatorname{spark}(A^{\mathrm{T}}V) = \operatorname{spark}(\Phi(A)) = \operatorname{spark}(A)$ for any $A \in \mathcal{M}_{m\times n}(\mathbb{F})$, which contradicts Lemma 2.6. Consequently, $\Phi(A) = UAV$ for all $A \in \mathcal{M}_{m\times n}(\mathbb{F})$. This implies that for an arbitrary $A \in \mathcal{M}_{m\times n}(\mathbb{F})$, we have $\operatorname{spark}(AV) = \operatorname{spark}(UAV) = \operatorname{spark}(A)$. Thus, by Proposition 2.3, there exist a diagonal matrix $D \in \mathcal{GL}_n(\mathbb{F})$ and an $n \times n$ permutation matrix $P$ such that $V = DP$. The proof is complete.

## References


[1] L.B. Beasley, T.J. Laffey, *Linear operators on matrices: the invariance of rank-k matrices*, Linear Algebra Appl. **133** (1990), 175–184. Cited on 64.

[2] L.B. Beasley (ed.), C.-K. Li (ed.), M.H. Lim (ed.), R. Loewy (ed.), B.R. McDonald (ed.), S. Pierce (ed.), *A survey of linear preserver problems*, Linear Multilinear Algebra **33** (1992) no. 1-2, Gordon and Breach Science Publishers, Yverdon, 1992, pp. 1–129. Cited on 64.

[3] D.L. Donoho, M. Elad, *Optimally sparse representation in general (nonorthogonal) dictionaries via $\ell^1$ minimization*, Proc. Natl. Acad. Sci. USA **100** (2003) no. 5, 2197–2202. Cited on 63.



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