

Annales Universitatis Paedagogicae Cracoviensis Studia Mathematica XIV (2015)

Lidia Skóra

Boundary value problem for the second order impulsive delay differential equations

Abstract. We present some existence and uniqueness result for a boundary value problem for functional differential equations of second order with impulses at fixed points.

1. Introduction

Impulsive differential equations describe processes that are subjected to abrupt changes in their state at fixed or variable times and present a natural framework for mathematical modelling of several real-world problems (see [6, 10]). In consequence, the study of impulsive differential equations is of great interest both for the theoretical and practical point of view.

The theory of impulsive differential equations and impulsive functional differential equations has been an important area of investigations in recent years. Among others the existence of solutions of the first and the second order impulsive functional differential equations by using the fixed point argument such as the Banach contraction principle, fixed point index theory and monotone iterative technique were discussed. We mention here the papers [1, 2, 3, 4, 5, 7, 8, 9, 12] and the references therein.

In the present paper we shall investigate the existence of the solutions of the boundary value problem for the second order delay differential systems with impulses at fixed points. The existence results for the boundary value problem for the second order delay differential equations of the above type without impulsive conditions have been studied in [11].

2. Preliminaries

In this paper we consider the following second order boundary value problem with impulses at fixed points

$$\begin{aligned}
x''(t) &= f(t, x_t), & t \in J' = J \setminus \{t_1, \dots, t_p\}, \\
\Delta x(t_k) &= I_{0k}(x(t_k), x'(t_k)), & k = 1, \dots, p, \\
\Delta x'(t_k) &= I_{1k}(x(t_k), x'(t_k)), & k = 1, \dots, p, \\
x_0 &= \phi, \\
x'(T) &= \beta x'(0), & \beta > 1,
\end{aligned} \tag{1}$$

where $J = [0, T]$, $T > 0$, $0 = t_0 < t_1 < \dots < t_p < t_{p+1} = T$, $f: J \times PC([-\tau, 0], \mathbb{R}^n) \rightarrow \mathbb{R}^n$ is given a function, $\phi \in PC([-\tau, 0], \mathbb{R}^n)$, $\tau > 0$,

$$\begin{aligned}
PC([-\tau, 0], \mathbb{R}^n) &= \{x: [-\tau, 0] \rightarrow \mathbb{R}^n : x(t^-) = x(t) \text{ for all } t \in (-\tau, 0], x(t^+) \\
&\text{exists for all } t \in [-\tau, 0], \text{ and } x(t^+) = x(t) \text{ for all} \\
&\text{but at most a finite number of points } t \in [-\tau, 0]\}.
\end{aligned}$$

For any function $x: [-\tau, T] \rightarrow \mathbb{R}^n$ and any $t \in J$, we let x_t denote the function $x_t: [-\tau, 0] \rightarrow \mathbb{R}^n$ defined by

$$x_t(s) = x(t + s), \quad s \in [-\tau, 0].$$

Here $x_t(\cdot)$ represents the history of the state from time $t - \tau$, up to the present time t . Condition $x_0 = \phi$ implies that $x(s) = \phi(s)$, $s \in [-\tau, 0]$. The supremum norm of $\phi \in PC([-\tau, 0], \mathbb{R}^n)$ is defined by

$$\|\phi\|_0 = \sup_{-\tau \leq s \leq 0} \|\phi(s)\|.$$

Let $L^1([-\tau, 0], \mathbb{R}^n)$ denote the Banach space of Lebesgue integrable functions $y: [-\tau, 0] \rightarrow \mathbb{R}^n$ with norm

$$\|y\|_{L^1} = \int_{-\tau}^0 \|y(t)\| dt.$$

Obviously $PC([-\tau, 0], \mathbb{R}^n) \subset L^1([-\tau, 0], \mathbb{R}^n)$ and for $\phi \in PC([-\tau, 0], \mathbb{R}^n)$,

$$\|\phi\|_{L^1} \leq \tau \|\phi\|_0.$$

$\Delta x(t_k)$, $\Delta x'(t_k)$ denote the jump of $x(t)$, $x'(t)$ at $t = t_k$, i.e.

$$\begin{aligned}
\Delta x(t_k) &= x(t_k^+) - x(t_k^-), \\
\Delta x'(t_k) &= x'(t_k^+) - x'(t_k^-),
\end{aligned}$$

where $x(t_k^+)$, $x'(t_k^+)$, $x(t_k^-)$, $x'(t_k^-)$ represent the right and left limits of $x(t)$, $x'(t)$ at $t = t_k$, respectively, $I_{0k}, I_{1k}: \mathbb{R}^n \times \mathbb{R}^n \rightarrow \mathbb{R}^n$.

In order to define the concept of solution for (1) we introduce the following sets of functions

$$PC[J, \mathbb{R}^n] = \{x: J \rightarrow \mathbb{R}^n : x \text{ is continuous at } t \neq t_k, \text{ left continuous at } t = t_k, \\ \text{and } x(t_k^+) \text{ exists, } k = 1, \dots, p\},$$

$$PC^1[J, \mathbb{R}^n] = \{x \in PC[J, \mathbb{R}^n] : x'(t) \text{ exists and is continuous at } t \neq t_k, \\ \text{and } x'(t_k^+), x'(t_k^-) \text{ exist for } k = 1, \dots, p\}.$$

We define $x'(t_k) = x'(t_k^-)$. Moreover, in (1) and in what follows, $x'(t_k)$ is understood as $x'(t_k^-)$. Note that for $x \in PC^1[J, \mathbb{R}^n]$, $x' \in PC[J, \mathbb{R}^n]$ and $PC[J, \mathbb{R}^n]$, $PC^1[J, \mathbb{R}^n]$ are Banach spaces with the norms

$$\|x\|_{PC} = \max\{e^{-t} \max\{\|x(s)\| : s \in [0, t]\}, t \in J\}, \\ \|x\|_{PC^1} = \max\{\|x\|_{PC}, \|x'\|_{PC}\}.$$

We shall prove an existence result for (1) by using the Banach contraction principle.

3. Auxiliary result

Let us start by defining what we mean by a solution of problem (1). Denote $C^* = PC^1([-\tau, T], \mathbb{R}^n) \cap C^2(J', \mathbb{R}^n)$.

DEFINITION 3.1

A function $x \in C^*$ is said to be a solution of (1) if x satisfies (1).

We need the following auxiliary lemma.

LEMMA 3.2

Assume that $f \in C(J \times L^1([-\tau, 0], \mathbb{R}^n), \mathbb{R}^n)$. Function $x \in C^*$ is a solution of (1) if and only if x is a solution of the following integral equation

$$x(t) = \begin{cases} \phi(t), & t \in [-\tau, 0], \\ \phi(0) + \frac{t}{\beta - 1} \left[\int_0^T f(s, x_s) ds + \sum_{k=1}^p I_{1k}(x(t_k), x'(t_k)) \right] \\ \quad + \int_0^t (t - s) f(s, x_s) ds \\ \quad + \sum_{0 < t_k < t} (I_{0k}(x(t_k), x'(t_k)) + (t - t_k) I_{1k}(x(t_k), x'(t_k))), & t \in J. \end{cases}$$

Proof. First we prove that the integrals $\int_0^T f(s, x_s) ds$, $\int_0^t (t - s) f(s, x_s) ds$ exist. Consider the function $x \in PC^1(J, \mathbb{R}^n)$ such that

$$x_t(s) = x(t + s), \quad s \in [-\tau, 0]$$

and

$$x(t+s) = \phi(t+s), \quad \text{if } t+s \leq 0.$$

We have

$$x_t \in PC([- \tau, 0], \mathbb{R}^n) \subseteq L^1([- \tau, 0], \mathbb{R}^n).$$

For any $t_0 \in J$, if $t \rightarrow t_0$ then

$$x_t(s) \rightarrow x_{t_0}(s) \quad \text{a.e. } s \in [- \tau, 0]$$

and

$$\lim_{t \rightarrow t_0} \|x_t - x_{t_0}\|_{L^1} = \lim_{t \rightarrow t_0} \int_{-\tau}^0 \|x(t+s) - x(t_0+s)\| ds = 0.$$

This implies that for any $x \in PC^1(J, \mathbb{R}^n)$, $f(t, x_t)$ is continuous on J except on a set of countable points. Then $f(t, x_t)$ is Lebesgue integrable on any bounded interval.

Assume that the function $x \in C^*$ is a solution of (1). The function x can be written of the form

$$\begin{aligned} x(t) &= x(0) + tx'(0) + \int_0^t (t-s)x''(s) ds \\ &+ \sum_{0 < t_k < t} [(x(t_k^+) - x(t_k)) + (t-t_k)(x'(t_k^+) - x'(t_k))]. \end{aligned} \quad (2)$$

Differentiating (2), we get

$$x'(t) = x'(0) + \int_0^t x''(s) ds + \sum_{0 < t_k < t} (x'(t_k^+) - x'(t_k)).$$

Hence

$$x'(T) = x'(0) + \int_0^T x''(s) ds + \sum_{k=1}^p (x'(t_k^+) - x'(t_k)).$$

Using the boundary condition we obtain

$$x'(0) + \int_0^T x''(s) ds + \sum_{k=1}^p (x'(t_k^+) - x'(t_k)) = \beta x'(0).$$

Thus

$$x'(0) = \frac{1}{\beta - 1} \left[\int_0^T x''(s) ds + \sum_{k=1}^p (x'(t_k^+) - x'(t_k)) \right]. \quad (3)$$

Equation (2), together with (1) and (3) implies

$$\begin{aligned} x(t) &= \Phi(0) + \frac{t}{\beta - 1} \left[\int_0^T f(s, x_s) ds + \sum_{k=1}^p I_{1k}(x(t_k), x'(t_k)) \right] \\ &+ \int_0^t (t-s)f(s, x_s) ds \\ &+ \sum_{0 < t_k < t} (I_{0k}(x(t_k), x'(t_k)) + (t-t_k)I_{1k}(x(t_k), x'(t_k))). \end{aligned} \quad (4)$$

Conversely, if x is a solution of equation (4), then direct differentiation of (4) gives

$$\begin{aligned} x'(t) &= \frac{1}{\beta-1} \left[\int_0^T f(s, x_s) ds + \sum_{k=1}^p I_{1k}(x(t_k), x'(t_k)) \right] \\ &+ \int_0^t f(s, x_s) ds + \sum_{0 < t_k < t} I_{1k}(x(t_k), x'(t_k)), \quad t \in J' \end{aligned} \quad (5)$$

and

$$x''(t) = f(t, x_t), \quad t \in J'.$$

Obviously

$$\begin{aligned} \Delta x(t_k) &= I_{0k}(x(t_k), x'(t_k)), \quad k = 1, \dots, p, \\ \Delta x'(t_k) &= I_{1k}(x(t_k), x'(t_k)), \quad k = 1, \dots, p. \end{aligned}$$

From (5) we have

$$x'(0) = \frac{1}{\beta-1} \left[\int_0^T f(s, x_s) ds + \sum_{k=1}^p I_{1k}(x(t_k), x'(t_k)) \right]$$

and

$$\begin{aligned} x'(T) &= \frac{1}{\beta-1} \left[\int_0^T f(s, x_s) ds + \sum_{k=1}^p I_{1k}(x(t_k), x'(t_k)) \right] \\ &+ \int_0^T f(s, x_s) ds + \sum_{k=1}^p I_{1k}(x(t_k), x'(t_k)) \\ &= \frac{\beta}{\beta-1} \left[\int_0^T f(s, x_s) ds + \sum_{k=1}^p I_{1k}(x(t_k), x'(t_k)) \right], \end{aligned}$$

which gives

$$x'(T) = \beta x'(0).$$

4. Main result

We introduce the following assumptions on the functions appearing in the problem (1):

(H1) There exists a function $m \in C(J, \mathbb{R}^+)$ such that

$$\|f(t, x_t) - f(t, y_t)\| \leq m(t)e^{-t} \|x_t - y_t\|_{L^1}$$

for any $t \in J$ and $x, y \in PC^1(J, \mathbb{R}^n)$.

(H2) There exist nonnegative constants $c_{ik}, \tilde{c}_{ik}, i = 0, 1, k = 1, 2, \dots, p$, such that the functions $I_{ik}, i = 0, 1, k = 1, 2, \dots, p$ verify the following conditions

$$\begin{aligned} &\|I_{ik}(x(t_k), x'(t_k)) - I_{ik}(y(t_k), y'(t_k))\| \\ &\leq c_{ik}e^{-t_k} \|x(t_k) - y(t_k)\| + \tilde{c}_{ik}e^{-t_k} \|x'(t_k) - y'(t_k)\| \end{aligned}$$

for any $x, y \in PC^1(J, \mathbb{R}^n)$.

Denote

$$M = \int_0^T m(r) dr, \quad C_0 = \sum_{k=1}^p (c_{0k} + \tilde{c}_{0k}), \quad C_1 = \sum_{k=1}^p (c_{1k} + \tilde{c}_{1k}).$$

THEOREM 4.1

Assume that $f \in C(J \times L^1([-\tau, 0], \mathbb{R}^n), \mathbb{R}^n)$. If the assumptions (H1), (H2) hold and

$$\frac{\beta}{\beta - 1}(\tau M + C_1) + C_0 < 1$$

then the problem (1) has a unique solution $x \in C^*$.

Proof. We transform the problem (1) into a fixed point problem. For $x \in PC^1(J, \mathbb{R}^n)$, let

$$\begin{aligned} (Ax)(t) &= \phi(0) + \frac{t}{\beta - 1} \left[\int_0^T f(s, x_s) ds + \sum_{k=1}^p I_{1k}(x(t_k), x'(t_k)) \right] \\ &\quad + \int_0^t (t - s) f(s, x_s) ds \\ &\quad + \sum_{0 < t_k < t} (I_{0k}(x(t_k), x'(t_k)) + (t - t_k) I_{1k}(x(t_k), x'(t_k))), \end{aligned} \quad (6)$$

where $t \in J$, $x_s(r) = x(s + r) = \phi(s + r)$ for $s + r \leq 0$. Differentiation of (6) gives

$$\begin{aligned} (Ax)'(t) &= \frac{1}{\beta - 1} \left[\int_0^T f(s, x_s) ds + \sum_{k=1}^p I_{1k}(x(t_k), x'(t_k)) \right] \\ &\quad + \int_0^t f(s, x_s) ds + \sum_{0 < t_k < t} I_{1k}(x(t_k), x'(t_k)), \quad t \in J. \end{aligned}$$

For $x, y \in PC^1(J, \mathbb{R}^n)$ we have

$$\begin{aligned} \|(Ax)(t) - (Ay)(t)\| &\leq \frac{\beta t}{\beta - 1} \left[\int_0^T \|f(s, x_s) - f(s, y_s)\| ds \right. \\ &\quad \left. + \sum_{k=1}^p \|I_{1k}(x(t_k), x'(t_k)) - I_{1k}(y(t_k), y'(t_k))\| \right] \\ &\quad + \sum_{0 < t_k < t} \|I_{0k}(x(t_k), x'(t_k)) - I_{0k}(y(t_k), y'(t_k))\| \end{aligned}$$

and

$$\begin{aligned} \|(Ax)'(t) - (Ay)'(t)\| &\leq \frac{\beta}{\beta - 1} \left[\int_0^T \|f(s, x_s) - f(s, y_s)\| ds \right. \\ &\quad \left. + \sum_{k=1}^p \|I_{1k}(x(t_k), x'(t_k)) - I_{1k}(y(t_k), y'(t_k))\| \right], \end{aligned}$$

for $t \in J$.

This, together with the assumptions (H1), (H2) gives

$$\begin{aligned} \|(Ax)(t) - (Ay)(t)\| &\leq \frac{\beta t}{\beta - 1} \left[\int_0^T \tau m(s) e^{-s} \|x_s - y_s\|_0 ds \right. \\ &\quad \left. + \sum_{k=1}^p e^{-t_k} (c_{1k} \|x(t_k) - y(t_k)\| + \tilde{c}_{1k} \|x'(t_k) - y'(t_k)\|) \right] \\ &\quad + \sum_{0 < t_k < t} e^{-t_k} (c_{0k} \|x(t_k) - y(t_k)\| + \tilde{c}_{0k} \|x'(t_k) - y'(t_k)\|) \end{aligned}$$

and

$$\begin{aligned} \|(Ax)'(t) - (Ay)'(t)\| &\leq \frac{\beta}{\beta - 1} \left[\int_0^T \tau m(s) e^{-s} \|x_s - y_s\|_0 ds \right. \\ &\quad \left. + \sum_{k=1}^p e^{-t_k} (c_{1k} \|x(t_k) - y(t_k)\| + \tilde{c}_{1k} \|x'(t_k) - y'(t_k)\|) \right]. \end{aligned}$$

Notice that if $s \in [0, \tau]$, then

$$\begin{aligned} \|x_s - y_s\|_0 &= \sup_{r \in [-\tau, 0]} \|x(s+r) - y(s+r)\| \\ &= \max\{\|x(s+r) - y(s+r)\|, r \in [-s, 0]\} \\ &= \max\{\|x(r) - y(r)\|, r \in [0, s]\}. \end{aligned}$$

If $s \in (\tau, T]$,

$$\begin{aligned} \|x_s - y_s\|_0 &= \sup_{r \in [-\tau, 0]} \|x(s+r) - y(s+r)\| \\ &= \max\{\|x(r) - y(r)\|, r \in [s-\tau, s]\} \\ &\leq \max\{\|x(r) - y(r)\|, r \in [0, s]\}. \end{aligned}$$

Therefore

$$\begin{aligned} \|(Ax)(t) - (Ay)(t)\| &\leq \frac{\beta t}{\beta - 1} \left[\int_0^T \tau m(s) e^{-s} \max_{r \in [0, s]} \|x(r) - y(r)\| ds \right. \\ &\quad \left. + \sum_{k=1}^p e^{-t_k} (c_{1k} \max_{r \in [0, t_k]} \|x(r) - y(r)\| + \tilde{c}_{1k} \max_{r \in [0, t_k]} \|x'(r) - y'(r)\|) \right] \\ &\quad + \sum_{0 < t_k < t} e^{-t_k} (c_{0k} \max_{r \in [0, t_k]} \|x(r) - y(r)\| + \tilde{c}_{0k} \max_{r \in [0, t_k]} \|x'(r) - y'(r)\|) \end{aligned}$$

and

$$\begin{aligned} \|(Ax)'(t) - (Ay)'(t)\| &\leq \frac{\beta}{\beta - 1} \left[\int_0^T \tau m(s) e^{-s} \max_{r \in [0, s]} \|x(r) - y(r)\| ds \right. \\ &\quad \left. + \sum_{k=1}^p e^{-t_k} (c_{1k} \max_{r \in [0, t_k]} \|x(r) - y(r)\| + \tilde{c}_{1k} \max_{r \in [0, t_k]} \|x'(r) - y'(r)\|) \right] \end{aligned}$$

for $t \in J$.

This implies that

$$\|(Ax)(t) - (Ay)(t)\| \leq \frac{\beta t}{\beta - 1}(\tau M + C_1)\|x - y\|_{PC^1} + C_0\|x - y\|_{PC^1}$$

and

$$\|(Ax)'(t) - (Ay)'(t)\| \leq \frac{\beta}{\beta - 1}(\tau M + C_1)\|x - y\|_{PC^1}, \quad t \in J.$$

In consequence

$$\max_{s \in [0, t]} \|(Ax)(s) - (Ay)(s)\| \leq \left(\frac{\beta t}{\beta - 1}(\tau M + C_1) + C_0 \right) \|x - y\|_{PC^1}$$

and

$$\max_{s \in [0, t]} \|(Ax)'(s) - (Ay)'(s)\| \leq \frac{\beta}{\beta - 1}(\tau M + C_1)\|x - y\|_{PC^1}, \quad t \in J.$$

Then

$$e^{-t} \max_{s \in [0, t]} \|(Ax)(s) - (Ay)(s)\| \leq \left(\frac{\beta}{\beta - 1}(\tau M + C_1) + C_0 \right) \|x - y\|_{PC^1}$$

and

$$e^{-t} \max_{s \in [0, t]} \|(Ax)'(s) - (Ay)'(s)\| \leq \frac{\beta}{\beta - 1}(\tau M + C_1)\|x - y\|_{PC^1}, \quad t \in J.$$

Hence we have the following estimate

$$\|Ax - Ay\|_{PC^1} \leq \alpha \|x - y\|_{PC^1}$$

with $\alpha = \frac{\beta}{\beta - 1}(\tau M + C_1) + C_0$. Thus A is a contractive operator and by Banach fixed point theorem, A has a unique fixed point $x \in PC^1(J, \mathbb{R}^n)$. The proof is complete.

When the impulsive functions are constants the boundary value problem (1) is of the form

$$\begin{aligned} x''(t) &= f(t, x_t), & t \in J', \\ \Delta x(t_k) &= \mu_{0k}, & k = 1, \dots, p, \\ \Delta x'(t_k) &= \mu_{1k}, & k = 1, \dots, p, \\ x_0 &= \phi, \\ x'(T) &= \beta x'(0), & \beta > 1, \end{aligned} \tag{7}$$

with $\mu_{0k}, \mu_{1k} \in \mathbb{R}^n$, $k = 1, \dots, p$.

As a consequence of the previous theorem, we have the following result.

COROLLARY 4.2

Assume that $f \in C(J \times L^1([-\tau, 0], \mathbb{R}^n), \mathbb{R}^n)$. If the assumption (H1) holds and

$$\int_0^T m(r) dr < \frac{\beta - 1}{\beta \tau}$$

then the problem (7) has a unique solution $u \in C^*$.

When $\mu_{ik} = 0$, $i = 0, 1$, $k = 1, \dots, p$ we obtain existence result for the boundary problem for second order delay differential equation without impulses under different assumptions than in [11].

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Institute of Mathematics
Cracow Technical University
Warszawska 24
31-155 Kraków
Poland
E-mail: lskora@usk.pk.edu.pl

Received: June 5, 2014; final version: October 20, 2014;
available online: January 13, 2015.