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À la recherche de la preuve perdue: a simple proof of the Ficken theorem

Abstract. A new proof of the Ficken criterion is given together with a comment concerning the known proofs and related results.

1. Introduction

There are many characterizations of an inner product norm (in short: IP-norm), i.e. a norm given by an inner product via[1] $\|x\|^2 = \langle x, x\rangle$, $x \in X$, and since publishing the book [1] the topic seems to be exploited. Over 350 characterizations of an IP-norm contained therein make it a challenge to invent anything noteworthy in the subject. However, below we try to justify our effort put into dealing with this.

The parallelogram law for the norm

$$\|x + y\|^2 + \|x - y\|^2 = 2(\|x\|^2 + \|y\|^2), \qquad x, y \in X,$$

is one of the simple characterizations of an IP-norm known as the Jordan–von Neumann theorem. Though the proof is elementary, considering this criterion one may get the feel of the potential behind the question of distinguishing norms that happen to come from an inner product. So we have

$$\|x + y\|^2 = F(\|x - y\|, \|x\|, \|y\|), \quad \text{where } F(s, t, u) = 2(t^2 + u^2) - s^2.$$

Surprisingly, if the above holds with just **any** function F with x and y arbitrary, then the norm turns out to be an IP-norm [9].

We now state the criterion we want to discuss in detail.

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[1]In our case there is no need to distinguish between real and complex case since it can be easily shown that a complex norm if induced by a real inner product is necesarily induced by a complex one.

THEOREM 1 (FICKEN [7])
The norm $\|\cdot\|$ defined on the linear space X is induced by an inner product if and only if

$$\|ax + by\| = \|bx + ay\|, \qquad a, b \in \mathbb{R},$$

whenever $x, y \in X$ are such that $\|x\| = \|y\|$.

The proof given in [7] is elementary, almost four page long and incorrect. There is no mistake, but a flaw in the reasoning: one cannot see any proper way of obtaining similar conditions on page 364. Luckily, since [7] there have appeared correct proofs of the criterion, e.g. in [5] and [1] but none of them can be described as elementary and straightforward. The proof of the Ficken criterion was given also in [8] but unfortunately it is essentially the original Ficken's proof. As noted by Lorch in [9]

A very brief proof of it has been given by von Neumann and Lorch,

where 'it' stands for the Ficken criterion. No reference is given which leaves the reader a bit frustrated. Being among these frustrated readers the author of the present paper has decided to interfere. That is why we dare to give the proof in Section 2 which can be thought of as elementary. The proof goes along the ideas presented in [5] but omitting somewhat fuzzy geometric considerations which in our approach are replaced by the matrices defining isometries.

Discussing the Ficken criterion it is worthwhile to mention some associated Lorch's results [9] which are stronger versions of the original one. Let us formulate two of them,

1. there exists $\gamma \in \mathbb{R} \setminus \{-1, 0, 1\}$ such that $\|x + \gamma y\| = \|\gamma x + y\|$, whenever $\|x\| = \|y\|$,

2. $\|\alpha x + \alpha^{-1} y\| \geqslant \|x + y\|$, whenever $\alpha \in \mathbb{R} \setminus \{0\}$ and $\|x\| = \|y\|$,

each of which implies that the norm is an IP-norm. Most statements from [9] depend on the Ficken criterion. All these results can be seen as fancy characterizations of an ellipse among all centrally symmetric convex and closed neighbourhoods of origin of $\mathbb{R}^2$ (with center at origin). This is somewhat in the taste of [2] where we find a highly non-trivial characterization of a circle.

Our proof will make use of the following fact.

LEMMA 2
If the norm $\|\cdot\|$ satisfies the Ficken condition on X, then there are no segments contained in $\{x \in X : \|x\| = 1\}$.

Proof. This follows from Lemma 1 in [7] which states that $x = y$, whenever $x, y \in X$ are such that $\|x\| = \|y\| = \|\frac{1}{2}(x + y)\|$. For the readers' convenience we outline the elementary proof of this fact. We show that

$$\|(2n - 1)(x - y) - 2y\| = 2\|x\| \tag{1}$$

for all integer $n \geqslant 0$. The case $n = 0$ being trivial we turn to the induction step assuming the hypothesis for $n \geqslant 0$. Then the vectors $(2n - 1)(x - y) - 2y$ and

$2y$ have the same norm, and it is easily seen that the Ficken condition applied to these vectors with $a = 2$ and $b = 2n + 1$ does the job. In turn, the triangle inequality applied to (1) leads to

$$\|x - y\| \leqslant \frac{2\|x\| + 2\|y\|}{2n - 1}, \qquad n \geqslant 1,$$

which implies that $x = y$ after letting $n \to \infty$.

REMARK 3

Following [5] we may produce yet another but less elementary argument showing the above lemma. Observe that for any two points $x, y \in X$ such that $\|x\| = \|y\| = 1$ there is a linear isometry of $X_0 \overset{\text{def}}{=} \text{lin}\{x, y\}$, the linear span of x and y, with respect to the norm $\| \cdot \|_1$ interchanging x and y. Indeed, if x, y are linearly independent, then $ax + by \mapsto by + ax$ is the isometry obtained via the Ficken condition. As a consequence we see that there are no segments contained in $C = \{v \in X_0 : \|v\| = 1\}$, since otherwise every point of C would lie on an open segment contained in C, which cannot happen as the set of extreme points of the closed unit ball cannot be empty (cf. the Carathéodory theorem [3]).

2. Proof of the Ficken theorem

The reader can easily check the necessity of the Ficken criterion, so we restrict our attention to the sufficiency. Let us begin with the case of a norm in $\mathbb{R}^2$. Assume that a norm $\| \cdot \|_1$ in $\mathbb{R}^2$ satisfies the Ficken condition, $\|(0, 1)\|_1 = \|(1, 0)\|_1 = 1$ and for every $(u, v) \in \mathbb{R}^2$ the inequality $\|(u, v)\|_1 \leqslant 1$ implies $|v| \leqslant 1$. We intend to show that $C_1 = \{x \in \mathbb{R}^2 : \|x\|_1 = 1\}$ is a unit circle with respect to the Euclidean norm.

Let $\|(u, v)\|_1 = 1$, $u \neq 0$, $v \neq 0$. The linear mappings interchanging $(0, 1)$ with (u, v) and $(1, 0)$ with (u, v) are given by matrices

$$A = \begin{bmatrix} -v & u \\ \frac{1}{u}(1 - v^2) & v \end{bmatrix} \quad \text{and} \quad B = \begin{bmatrix} u & \frac{1}{v}(1 - u^2) \\ v & -u \end{bmatrix} \quad \text{(respectively)}.$$

Note that they have to be isometries with respect to $\|\cdot\|_1$ as elucidated in Remark 3. Now,

$$BA \begin{bmatrix} 1 \\ 0 \end{bmatrix} = \begin{bmatrix} -uv + \frac{1}{uv}(1 - u^2)(1 - v^2) \\ -1 \end{bmatrix}.$$

The 'no-segment' property guaranteed by Lemma 2 allows us to deduce that $(0, -1)$ is the only point of C_1 with the second coordinate equal to -1. Hence the vector obtained above must have first coordinate equal to 0, which equivalently reads as $u^2 + v^2 = 1$. Thus C_1 is a unit circle.

We now consider a two-dimensional real linear space X equipped with the norm $\| \cdot \|$ satisfying the Ficken condition. Let $C = \{h \in X : \|h\| = 1\}$ and $f \in C$. By the Darboux property of $g \mapsto \|f + g\| - \|f - g\|$ we find $g \in C$ such that $\|f + g\| = \|f - g\|$. As a consequence, f and g are linearly independent and $\|\gamma f + g\| = \| - \gamma f + g\|$, $\gamma \in \mathbb{R}$, which is an easy reformulation of the Ficken

condition. The function $\gamma \mapsto \|\gamma f + g\|$ is convex and even, so it follows that it attains its global minimum equal to 1 at $\gamma = 0$. Thus

$$\|uf + vg\| \geqslant |v|, \qquad u, v \in \mathbb{R}.$$

It is a matter of routine to verify that the norm $\|(u, v)\|_1 \overset{\text{def}}{=} \|uf + vg\|$ satisfies the requirements imposed on the norm in the case of $\mathbb{R}^2$ above. Hence $\|uf + vg\|^2 = u^2 + v^2$ and the function

$$\varphi(u_1 f + v_1 g, u_2 f + v_2 g) \overset{\text{def}}{=} u_1 u_2 + v_1 v_2, \qquad u_1, u_2, v_1, v_2 \in \mathbb{R},$$

is a real inner product inducing the norm $\|\cdot\|$. The case of an arbitrary real normed space X follows from the observation that a normed space is an inner product space if and only if its every two-dimensional subspace is an inner product space which can be derived from the Jordan-von Neumann theorem mentioned in Section 1. In turn the case of a complex normed space can be dealt with by means of the Footnote 1.

3. An application of the Ficken criterion

In a normed space X one can consider *the Apollonius sets*

$$S_t(x, y) = \{u \in X : \|x - u\| = t\|y - u\|\},$$

where $x, y \in X$, $x \neq y$, $t \in (0, \infty) \setminus \{1\}$. These are obvious counterparts of the Apollonius circles. The reader may draw the Apollonius set $\|x - a\| = 2\|x - b\|$ in $\mathbb{R}^2$ with respect to the ℓ^1-norm with $a = (0, 0)$ and $b = (3, 1)$ to see that in general an Apollonius set does not have to be the boundary of a convex set. It can be readily verified that every $S_t(x, y)$ is a sphere in X provided the norm $\|\cdot\|$ is an IP-norm. The converse is also true which was shown in [4] and the proof was based on the Ficken criterion. The main argument is reduced to the planar geometry, but the objects considered in the proof are hard to visualize because they are used to reach the contradiction.

Acknowledgement

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