

**Annales Universitatis Paedagogicae Cracoviensis  
Studia Mathematica XIII (2014)***Kamil Kular***Semiprime rings with nilpotent Lie ring of inner derivations**

**Abstract.** We give an elementary and self-contained proof of the theorem which says that for a semiprime ring commutativity, Lie-nilpotency, and nilpotency of the Lie ring of inner derivations are equivalent conditions.

**1. Preliminaries and introduction**

Throughout the text  $R$  stands for an associative ring (possibly without identity) and  $n$  for a positive integer. By  $Z(R)$  we denote the center of  $R$ . The ring  $R$  is said to be *semiprime*, if

$$\forall a \in R : aRa = \{0\} \implies a = 0.$$

Let us recall that  $R$  may be regarded as a Lie ring with the Lie multiplication defined by  $[x, y] = xy - yx$ . For  $n \geq 3$  and  $x_1, \dots, x_n \in R$  we define inductively

$$[x_1, \dots, x_n] = [[x_1, \dots, x_{n-1}], x_n].$$

The ring  $R$  is said to be *Lie-nilpotent*, if there exists an  $n$  such that  $[x_1, \dots, x_{n+1}] = 0$  for all  $x_1, \dots, x_{n+1} \in R$ . (Notice that  $R$  is Lie-nilpotent whenever it is commutative).

A map  $d: R \rightarrow R$  is called a derivation, if it is additive and satisfies the Leibniz rule

$$\forall x, y \in R : d(xy) = d(x)y + xd(y).$$

The set  $\text{Der}(R)$  of all derivations  $d: R \rightarrow R$  is a Lie ring under the pointwise addition and the Lie multiplication defined by

$$[d_1, d_2] = d_1 \circ d_2 - d_2 \circ d_1.$$

A set  $E \subseteq \text{Der}(R)$  is *abelian*, if  $[d_1, d_2] = 0$  for all  $d_1, d_2 \in E$ . For  $n \geq 3$  and  $d_1, \dots, d_n \in \text{Der}(R)$  we define  $[d_1, \dots, d_n]$  analogously as for elements of  $R$ . A Lie ring  $D$  of derivations on  $R$  (i.e.,  $D$  is a Lie subring of  $\text{Der}(R)$ ) is said to be nilpotent, if there exists a positive integer  $n$  such that  $[d_1, \dots, d_{n+1}] = 0$  for all  $d_1, \dots, d_{n+1} \in D$ . We define the nilpotency class of  $D$  as the infimum of the set

$$\{n \in \mathbb{N} \setminus \{0\} : [d_1, \dots, d_{n+1}] = 0 \text{ for all } d_1, \dots, d_{n+1} \in D\}.$$

Notice that the Lie ring  $D$  is abelian if and only if it is nilpotent of class 1.

Let  $a \in R$ . It is easy to see that  $\partial_a: R \ni x \mapsto [a, x] \in R$  is a derivation. This derivation is referred to as the *inner derivation* generated by  $a$ . One can prove that  $\text{IDer}(R) = \{\partial_a : a \in R\}$  is a Lie subring of  $\text{Der}(R)$  (cf. Proposition 2.2 in the sequel).

Commutativity of semiprime rings with derivations has been studied by several authors. In [3] Daif and Bell proved that a semiprime ring  $R$  is commutative whenever it admits a derivation  $d$  such that  $d([x, y]) = [x, y]$  for all  $x, y \in R$  or  $d([x, y]) = -[x, y]$  for all  $x, y \in R$ . A generalization of the Daif and Bell result can be found in [5] (see also [2] where some related results are presented). Argaç and Inceboz proved that a semiprime ring  $R$  is commutative whenever there exist a derivation  $d: R \rightarrow R$  and a positive integer  $n$  such that

$$(d(x)y + xd(y) + d(y)x + yd(x))^n = xy + yx$$

for all  $x, y \in R$  (see [1]).

The aim of the present note is to give an elementary and self-contained proof of

#### THEOREM 1.1

*Suppose that  $R$  is semiprime. Then the following conditions are equivalent:*

- (1) *the Lie ring  $\text{IDer}(R)$  is nilpotent,*
- (2)  *$R$  is commutative,*
- (3)  *$R$  is Lie-nilpotent.*

Even though similar results are known, the theorem seems to be not explicitly stated in the literature. We will present a nice application of this theorem to the study of abstract derivations in the next paper.

We refer the reader to [4] for terminology, definitions and basic facts in ring theory.

## 2. Some lemmas and useful facts

Before presenting the elementary proof of Theorem 1.1, we collect some lemmas and useful remarks. The first lemma is very well known. We include its proof for the sake of completeness.

#### LEMMA 2.1

*For any  $d \in \text{Der}(R)$  and  $x \in R$  we have  $[d, \partial_x] = \partial_{d(x)}$ .*

*Proof.* If  $a \in R$ , then

$$\begin{aligned} [d, \partial_x](a) &= d(\partial_x(a)) - \partial_x(d(a)) = d(xa - ax) - xd(a) + d(a)x \\ &= d(x)a + xd(a) - d(a)x - ad(x) - xd(a) + d(a)x \\ &= \partial_{d(x)}(a). \end{aligned}$$

Using the above lemma we immediately obtain

**PROPOSITION 2.2**

Suppose that  $n \geq 2$  and  $x_1, \dots, x_{n+1} \in R$ . Then

- (i)  $[\partial_{x_1}, \dots, \partial_{x_n}]$  is an inner derivation,
- (ii)  $[\partial_{x_1}, \dots, \partial_{x_{n+1}}] = \partial_{[\partial_{x_1}, \dots, \partial_{x_n}](x_{n+1})}$ .

The next two lemmas seem to be of separate interest.

**LEMMA 2.3**

Let  $x \in R$ . The following conditions are equivalent:

- (1)  $[\partial_x, \partial_y] = 0$  for all  $y \in R$  (i.e.,  $\partial_x$  is a central element of the Lie ring  $\text{IDer}(R)$ ),
- (2)  $\partial_x(y) \in Z(R)$  for all  $y \in R$ ,
- (3)  $\partial_a \circ \partial_x = 0$  for all  $a \in R$ .

*Proof.* By Lemma 2.1, condition (1) is satisfied if and only if

$$\forall y, a \in R : \partial_{\partial_x(y)}(a) = 0, \quad (1)$$

and this means exactly that  $\partial_x(y) \in Z(R)$  for any  $y \in R$ . On the other hand, it is obvious that (1) holds true if and only if

$$\forall y, a \in R : [a, \partial_x(y)] = 0,$$

which means exactly that  $\partial_a \circ \partial_x = 0$  for all  $a \in R$ .

**LEMMA 2.4**

Suppose that  $n \geq 2$ . The following conditions are equivalent:

- (1)  $[\partial_{x_1}, \dots, \partial_{x_{n+1}}] = 0$  for all  $x_1, \dots, x_{n+1} \in R$  (i.e., the Lie ring  $\text{IDer}(R)$  is nilpotent of class at most  $n$ ),
- (2)  $\partial_a \circ [\partial_{x_1}, \dots, \partial_{x_n}] = 0$  for all  $a, x_1, \dots, x_n \in R$ .

*Proof.* Implication (1)  $\Rightarrow$  (2) is an immediate consequence of Proposition 2.2 (i) and the previous lemma. So let us assume that (2) is satisfied. Then

$$\forall a, x_1, \dots, x_{n+1} \in R : \partial_a([\partial_{x_1}, \dots, \partial_{x_n}](x_{n+1})) = 0,$$

and hence  $[\partial_{x_1}, \dots, \partial_{x_n}](x_{n+1}) \in Z(R)$  for any  $x_1, \dots, x_{n+1} \in R$ . Therefore, by Proposition 2.2 (ii), we have

$$[\partial_{x_1}, \dots, \partial_{x_{n+1}}] = \partial_{[\partial_{x_1}, \dots, \partial_{x_n}](x_{n+1})} = 0$$

for all  $x_1, \dots, x_{n+1} \in R$ .

The last lemma will also play a crucial role in the sequel.

LEMMA 2.5

If  $\partial_y \circ \partial_x = 0$  for some  $x, y \in R$ , then  $(\partial_x(y))^2 = 0$ .

*Proof.*

$$\begin{aligned} 0 &= (\partial_y \circ \partial_x)(-xy) = \partial_y(-\partial_x(xy)) = -\partial_y(\partial_x(x)y + x\partial_x(y)) \\ &= -\partial_y(x\partial_x(y)) = -\partial_y(x)\partial_x(y) - x\partial_y(\partial_x(y)) = -\partial_y(x)\partial_x(y) \\ &= (\partial_x(y))^2. \end{aligned}$$

### 3. Main results

Let us begin with a quite simple theorem.

THEOREM 3.1

Suppose that  $R$  is semiprime and the Lie ring  $\text{IDer}(R)$  is abelian. Then  $R$  is commutative.

*Proof.* Pick arbitrary elements  $x, y \in R$ . It follows from Lemma 2.3 that  $\partial_x(y) \in Z(R)$ . Moreover, combining Lemma 2.3 and Lemma 2.5 yields  $(\partial_x(y))^2 = 0$ . Consequently,  $\partial_x(y)R\partial_x(y) = (\partial_x(y))^2R = \{0\}$ . Since  $R$  is semiprime, we obtain  $\partial_x(y) = 0$ , and hence  $xy = yx$ .

The following result is the technical heart of the note.

THEOREM 3.2

Let  $n \geq 2$ . Suppose that  $R$  is semiprime and

$$\forall x_1, \dots, x_{n+1} \in R: [\partial_{x_1}, \dots, \partial_{x_{n+1}}] = 0.$$

Then the Lie ring  $\text{IDer}(R)$  is nilpotent of class at most  $n - 1$ .

*Proof.* Lemma 2.4 implies  $\partial_a \circ [\partial_{x_1}, \dots, \partial_{x_n}] = 0$  for all  $a, x_1, \dots, x_n \in R$ . Define

$$x = \begin{cases} [\partial_{x_1}, \dots, \partial_{x_{n-1}}](x_n), & \text{if } n \geq 3, \\ \partial_{x_1}(x_2), & \text{if } n = 2. \end{cases}$$

By Lemma 2.1 and Proposition 2.2 (ii), we have  $[\partial_{x_1}, \dots, \partial_{x_n}] = \partial_x$ . Thus  $\partial_a \circ \partial_x = 0$  for any  $a \in R$ . It follows now from Lemmas 2.3 and 2.5 that

$$\forall a \in R: \begin{cases} \partial_x(a) \in Z(R), \\ (\partial_x(a))^2 = 0. \end{cases}$$

The semiprimeness therefore yields  $\partial_x(a) = 0$  for all  $a \in R$ . Consequently,  $[\partial_{x_1}, \dots, \partial_{x_n}] = \partial_x = 0$  for all  $x_1, \dots, x_n \in R$ , which means that  $\text{IDer}(R)$  is nilpotent of class at most  $n - 1$ .

*Proof of Theorem 1.1.* If the Lie ring  $\text{IDer}(R)$  is nilpotent, then by Theorem 3.2 it is abelian, and hence Theorem 3.1 implies that  $R$  is commutative. Implication (2)  $\Rightarrow$  (3) is obvious. Finally, if  $n \geq 2$ , then by Lemma 2.1 and Proposition 2.2 (ii) we have

$$\forall x_1, \dots, x_{n+1} \in R: [\partial_{x_1}, \dots, \partial_{x_n}](x_{n+1}) = [x_1, \dots, x_{n+1}].$$

This proves implication (3)  $\Rightarrow$  (1).

Let us conclude by a natural example of a Lie-nilpotent ring which is not commutative.

#### EXAMPLE

Consider a field  $\mathbb{F}$  and the matrix ring

$$\mathcal{R} = \left\{ \begin{pmatrix} 0 & a & b \\ 0 & 0 & c \\ 0 & 0 & 0 \end{pmatrix} : a, b, c \in \mathbb{F} \right\}.$$

The non-commutativity of  $\mathcal{R}$  is obvious. The ring is Lie-nilpotent because  $[A, B, C] = 0$  for any  $A, B, C \in \mathcal{R}$ .

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