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# A note on the convergence of partial Szász-Mirakyan type operators

**Abstract.** In this paper we study approximation properties of partial modified Szasz-Mirakyan operators for functions from exponential weight spaces. We present some direct theorems giving the degree of approximation for these operators. The considered version of Szász-Mirakyan operators is more useful from the computational point of view.

## 1. Introduction

Let us denote by $C(\mathbb{R}_0)$ a set of all real-valued functions continuous on $\mathbb{R}_0 = [0;+\infty)$ and let $\mathbb{N}_0 = \mathbb{N} \cup \{0\}$. In paper [1] we investigated operators of Szász-Mirakyan type defined as follows

$$A_n^\nu(f;x) = \begin{cases} \sum_{k=0}^{\infty} p_{n,k}^\nu(x) f\Big(\frac{2k}{n}\Big), & x > 0; \\ f(0), & x = 0 \end{cases}$$

and

$$p_{n,k}^\nu(x) = \frac{1}{I_\nu(nx)} \frac{x^{2k+\nu}}{2^{2k+\nu} k!\,\Gamma(k+\nu+1)},$$

where $\Gamma$ is the gamma function and $I_\nu$ stands for the modified Bessel function, i.e.

$$I_\nu(z) = \sum_{k=0}^{\infty} \frac{z^{2k+\nu}}{2^{2k+\nu} k!\,\Gamma(k+\nu+1)}.$$

We studied the operators in polynomial weight spaces

$$C_p = \{f \in C(\mathbb{R}_0) : \ w_p f \text{ is uniformly continuous and bounded on } \mathbb{R}_0\},$$

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where $w_p$ was the polynomial weight function defined as follows

$$w_p(x) = \begin{cases} 1, & p = 0; \\ \dfrac{1}{1+x^p}, & p \in \mathbb{N} \end{cases}$$

for $x \in \mathbb{R}_0$. The space $C_p$ is a normed space with the norm

$$\|f\|_p = \sup_{x \in \mathbb{R}_0} w_p(x)|f(x)|.$$

In paper [4] a certain modification of the above operators was introduced

$$B_n^\nu(f, a_n; x) = \begin{cases} \displaystyle\sum_{k=0}^{[n(x+a_n)]} p_{n,k}^\nu(x) f\Big(\frac{2k}{n}\Big), & x > 0; \\ f(0), & x = 0 \end{cases}$$

for $f \in C_p$, where $(a_n)$ was a sequence of positive numbers such that

$$\lim_{n\to\infty} a_n \sqrt{n} = \infty \tag{1}$$

and $[n(x + a_n)]$ denotes the integral part of $n(x + a_n)$.

In the paper were studied approximation properties of these operators. Among others there was deduced that

$$\lim_{n\to\infty} \{B_n^\nu(f, a_n; x) - f(x)\} = 0$$

for every $f \in C_p$, uniformly on every interval $[x_1, x_2]$, $x_2 > x_1 \geq 0$. In that case the crucial assumption was that the sequence of positive numbers $(a_n)$ satisfied the condition (1).

Similar problems for Baskakov type operators were discussed in paper [5] and some generalization of truncated operators we can find in [6].

We shall drive analogous results for the modified Szász-Mirakyan operators for functions from exponential weight spaces

$$E_q = \{f \in C(\mathbb{R}_0) : \ v_q f \text{ is uniformly continuous and bounded on } \mathbb{R}_0\},$$

where $v_q$ is the exponential weight function defined as follows

$$v_q(x) = e^{-qx}, \qquad q \in \mathbb{R}_+$$

for $x \in \mathbb{R}_0$. The space $E_q$ is a normed space with the norm

$$\|f\|_q = \sup_{x\in\mathbb{R}_0} v_q(x)|f(x)|. \tag{2}$$

In this paper we will present a certain modification of the operator $B_n^\nu$. We shall apply the modification to prove the convergence of the operators in exponential weight spaces.

## 2. Main results

At the beginning of this section we will recall the definition of $\overline{A}_n^\nu$ and some preliminary results from papers [2] and [3], which we shall apply to prove main theorems.

If $\nu \in \mathbb{R}_0$ and $n \in \mathbb{N}$ we considered operators of Szász-Mirakyan type as follows

$$\overline{A}_n^\nu(f;x) = \begin{cases} \displaystyle\sum_{k=0}^{\infty} p_{n,k}^\nu(x) f\Big(\frac{2k}{n+q}\Big), & x > 0; \\ f(0), & x = 0 \end{cases}$$

for $f \in E_q$.

LEMMA 1
*For all $\nu \in \mathbb{R}_0$, $n \in \mathbb{N}$ and $x \in \mathbb{R}_0$ we have*

$$\overline{A}_n^\nu(1;x) = 1.$$

LEMMA 2
*For all $q \in \mathbb{R}_+$ and $\nu \in \mathbb{R}_0$ there exists a positive constant $M(q,\nu)$ such that for each $n \in \mathbb{N}$ we have*

$$\Big\| \overline{A}_n^\nu\Big(\frac{1}{v_q};\cdot\Big)\Big\|_q \le M(q,\nu).$$

Applying the definition of $\overline{A}_n^\nu$ and (2) we get

LEMMA 3
*For all $q \in \mathbb{R}_+$ and $\nu \in \mathbb{R}_0$ there exists a positive constant $M(q,\nu)$ such that for each $n \in \mathbb{N}$ we have*

$$\big\|\overline{A}_n^\nu(f;\cdot)\big\|_q \le M(q,\nu)\|f\|_q.$$

Notice that operators $\overline{A}_n^\nu$ are bounded and transform the space $E_q$ into itself.

LEMMA 4
*For all $q \in \mathbb{R}_+$ and $\nu \in \mathbb{R}_0$ there exists a positive constant $M(q,\nu)$ such that for each $x \in \mathbb{R}_0$ and $n \in \mathbb{N}$ we have*

$$\big|\overline{A}_n^\nu((t-x)^2;x)\big| \le M(q,\nu)\frac{x(x+1)}{n}.$$

LEMMA 5
*For all $q \in \mathbb{R}_+$ and $\nu \in \mathbb{R}_0$ there exists a positive constant $M(q,\nu)$ such that for all $x \in \mathbb{R}_0$ and $n \in \mathbb{N}$ we have*

$$v_q(x)\overline{A}_n^\nu\Big(\frac{(t-x)^2}{v_q(t)};x\Big) \le M(q,\nu)\frac{x(x+1)}{n}.$$

Theorem 6
*If $\nu \in \mathbb{R}_0$ and $f \in E_q$ with some $q \in \mathbb{R}_+$, then for all $x \in \mathbb{R}_0$*

$$\lim_{n\to\infty} \left\{\overline{A}_n^{\nu}(f;x) - f(x)\right\} = 0.$$

*Moreover, the above convergence is uniform on every set $[x_1, x_2]$ with $0 \le x_1 < x_2$.*

*Proof.* Let $f \in E_q$ with some $q \in \mathbb{R}_+$. Pick $x \in \mathbb{R}_0$ and $\varepsilon > 0$. There exists a number $\delta$ such that

$$|f(t) - f(x)| < \frac{\varepsilon}{2} \tag{3}$$

for $|t - x| < \delta$, $t \in \mathbb{R}_0$. By linearity of $\overline{A}_n^{\nu}$ and Lemma 1 we get

$$\begin{aligned}
&\left|\overline{A}_n^{\nu}(f;x) - f(x)\right| \\
&\quad \le \overline{A}_n^{\nu}(|f - f(x)|;x) \\
&\quad = \sum_{|\frac{2k}{n+q}-x|<\delta} p_{n,k}^{\nu}(x)\left|f\Big(\frac{2k}{n+q}\Big) - f(x)\right| + \sum_{|\frac{2k}{n+q}-x|\ge\delta} p_{n,k}^{\nu}(x)\left|f\Big(\frac{2k}{n+q}\Big) - f(x)\right| \\
&\quad = I_1 + I_2.
\end{aligned}$$

Hence by (3) we obtain $I_1 < \frac{\varepsilon}{2}$. Further we get

$$I_2 \le \frac{\|f\|_q}{\delta}\,\overline{A}_n^{\nu}\Big(\frac{|t-x|}{v_q(t)};x\Big) + \frac{\|f\|_q}{\delta v_q(x)}\,\overline{A}_n^{\nu}(|t-x|;x).$$

Using the Hölder inequality and Lemmas 2–5 we have

$$\begin{aligned}
I_2 &\le \frac{\|f\|_q}{\delta}\Big[\overline{A}_n^{\nu}\Big(\frac{(t-x)^2}{v_q(t)};x\Big)\Big]^{\frac12}\Big[\overline{A}_n^{\nu}\Big(\frac{1}{v_q(t)};x\Big)\Big]^{\frac12} + \frac{\|f\|_q}{\delta v_q(x)}\Big[\overline{A}_n^{\nu}((t-x)^2;x)\Big]^{\frac12} \\
&\le \frac{\|f\|_q}{\delta}\Big[M(q,\nu)\frac{x(x+1)}{nv_q(x)}\Big]^{\frac12}\Big[M(q,\nu)\frac{1}{v_q(x)}\Big]^{\frac12} + \frac{\|f\|_q}{\delta v_q(x)}\Big[M(q,\nu)\frac{x(x+1)}{n}\Big]^{\frac12} \\
&< \frac{\varepsilon}{2}.
\end{aligned}$$

The above estimations imply the convergence in Theorem 6.

In the space $E_q$ we define the following class of partial operators

$$\overline{B}_n^{\nu}(f,a_n;x) = \begin{cases} \displaystyle\sum_{k=0}^{[(n+q)(x+a_n)]} p_{n,k}^{\nu}(x) f\Big(\frac{2k}{n+q}\Big), & x > 0; \\ f(0), & x = 0, \end{cases}$$

where we replace the infinite summing in $\overline{A}_n^{\nu}$ by the finite one and we still have the assumption (1).

Theorem 7
*If $\nu \in \mathbb{R}_0$ and $f \in E_q$ with some $q \in \mathbb{R}_+$ then for all $x \in \mathbb{R}_0$*

$$\lim_{n\to\infty} \left\{\overline{B}_n^{\nu}(f,a_n;x) - f(x)\right\} = 0.$$

*Moreover, the above convergence is uniform on every interval $[x_1, x_2]$, $x_2 > x_1 \ge 0$.*

*Proof.* Let $f \in E_q$ with some $q \in \mathbb{R}_+$. From the definitions of the operators $\overline{A}_n^\nu$ and $\overline{B}_n^\nu$ we get

$$\begin{aligned}
&\overline{B}_n^\nu(f, a_n; x) - f(x) \\
&\quad = \sum_{k=0}^{[(n+q)(x+a_n)]} p_{n,k}^\nu(x) f\Big(\frac{2k}{n+q}\Big) - f(x) \\
&\quad = \sum_{k=0}^{\infty} p_{n,k}^\nu(x) f\Big(\frac{2k}{n+q}\Big) - f(x) - \sum_{k=[(n+q)(x+a_n)]+1}^{\infty} p_{n,k}^\nu(x) f\Big(\frac{2k}{n+q}\Big) \\
&\quad = \overline{A}_n^\nu(f; x) - f(x) - R_n^\nu(f, a_n; x)
\end{aligned}$$

for $x \in \mathbb{R}_0$ and $n \in \mathbb{N}$. Observe that

$$\Big|\frac{2k}{n+q} - x\Big| \geq a_n \qquad \text{if} \qquad k \geq [(n+q)(x+a_n)] + 1.$$

Analogously as in the previous proof we can write the following estimation

$$\begin{aligned}
\big|R_n^\nu(f, a_n; x)\big| &\leq \sum_{k=[(n+q)(x+a_n)]+1}^{\infty} p_{n,k}^\nu(x) \Big|f\Big(\frac{2k}{n+q}\Big)\Big| \\
&\leq \sum_{|\frac{2k}{n+q} - x| \geq a_n} p_{n,k}^\nu(x) \Big|f\Big(\frac{2k}{n+q}\Big)\Big| \leq \frac{\|f\|_q}{a_n} \sum_{k=0}^{\infty} p_{n,k}^\nu(x) \frac{|\frac{2k}{n+q} - x|}{v_q(\frac{2k}{n+q})} \\
&= \frac{\|f\|_q}{a_n} \overline{A}_n^\nu\Big(\frac{|t-x|}{v_q(t)}; x\Big).
\end{aligned}$$

The Hölder inequality, Lemmas 2 and 5 imply

$$\begin{aligned}
\big|R_n^\nu(f, a_n; x)\big| &\leq \frac{\|f\|_q}{a_n} \Big[\overline{A}_n^\nu\Big(\frac{(t-x)^2}{v_q(t)}; x\Big)\Big]^{\frac{1}{2}} \Big[\overline{A}_n^\nu\Big(\frac{1}{v_q(t)}; x\Big)\Big]^{\frac{1}{2}} \\
&\leq M(q, \nu) \frac{\|f\|_q}{a_n} \frac{\sqrt{x(x+1)}}{\sqrt{n}} \frac{1}{v_q(x)}.
\end{aligned}$$

In view of (1) we obtain the required result.

Notice that the same simplified method we can use in polynomial weight spaces $C_p$ to estimate the reminder of the series $A_n^\nu$, which was considered in [4].

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