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On the gluing of hyperconvex metrics and diversities

Abstract. In this work we consider two hyperconvex diversities (or hyperconvex metric spaces) (X, δ_X) and (Y, δ_Y) with nonempty intersection and we wonder whether there is a natural way to glue them so that the new glued diversity (or metric space) remains being hyperconvex. We provide positive and negative answers in both situations.

1. Introduction and preliminaries

Diversity is a notion coming from the mid nineties motivated by phylogenetic studies and the geometry of metric trees [8]. Very recently diversities have been introduced in a general way as a kind of multi-way metrics in [3] also in relation to phylogenetic problems. The main motivation in [3] was to construct a notion of diversity tight span alike to the metric tight span introduced by Isbell in [9] for metric spaces, see also [4]. Metric tight spans of phylogenetic trees turned out to be a very useful tool in phylogenetics after [4].

Tight spans can be regarded as the minimal hyperconvex metric spaces where a given metric space can be isometrically embedded. These tight spans exist and are unique up to isometries for any metric space. In [3] the ideas of hyperconvex diversity and diversity tight spans were introduced in a successful way paralleling what it was already known for metric spaces. Further connections between metric spaces, metric tight spans, diversities and diversity tight spans were obtained in [3] for particular instances of diversities as the diameter and the phylogenetic diversities. The connection between hyperconvex diversities and the metric spaces that they induce have also been studied in [5]. In fact, one of the main problems that the authors met in [5], which was the lack of examples and constructions of diversities, is the motivation for the present work.

More precisely, to solve one of the main questions studied in [5] (see [5, Theorem 2.19]) required the construction *ad hoc* of an example of diversity through

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gluing techniques. It is the purpose of the current work to understand better the gluing construction from [5] and to make it as general as possible. In Section 2 we take up the problem for hyperconvex diversities and in Section 3 for hyperconvex metric spaces providing positive and negative results for both cases. In particular, the example needed in [5, Theorem 2.19] is shown in Section 2 as a particular one in a more general class.

Hyperconvex metric spaces were introduced by N. Aronszajn and P. Panitchpakdi in [1].

Definition 1.1
A metric space (X, d) is said to be *hyperconvex* if each family of closed balls $\bar{B}(x, r(x))$ with $x \in A \subset X$ such that

$$d(x, y) \leq r(x) + r(y), \qquad x, y \in A \tag{1}$$

has a nonempty intersection property

$$\bigcap_{x \in A} \bar{B}(x, r(x)) \neq \emptyset.$$

Hyperconvex metric spaces exhibit a large number of nice properties as being injective metric spaces and absolute nonexpansive retracts. The interested reader may check recent monographs as [6, 7] for these and more properties on hyperconvexity and tight spans.

Remark 1.2
We will assume that the set of centers A is actually X. It is easy to see that this does not bring any loss of generality.

For the purposes of [3] it was needed to introduce a general notion of diversity and to relate it to hyperconvexity.

Definition 1.3
Let X be a nonempty set and let $\langle X \rangle$ denote the family of all nonempty finite subsets of X. This set with a function $\delta \colon \langle X \rangle \to [0, \infty)$ is said to be *diversity* if:

(i) $\delta(A) = 0$ if and only if A is a singleton;

(ii) $\delta(A \cup C) \leq \delta(A \cup B) + \delta(B \cup C)$ for $A, B, C \in \langle X \rangle$.

For basic properties on diversities the reader should go to [3]. One of the most relevant of these properties is that any diversity function δ on a set X induces a natural distance on X given by $d(x, y) = \delta(\{x, y\})$. We will very often refer to this metric space as the induced metric space by the diversity.

The notion of hyperconvexity for diversities was also introduced in [3].

Definition 1.4
A diversity (X, δ) is said to be *hyperconvex* if for all $r \colon \langle X \rangle \to [0, \infty)$ such that

$$\delta\Big(\bigcup_{A \in \mathfrak{A}}\Big) \leq \sum_{A \in \mathfrak{A}} r(A)$$

for each finite subset $\mathfrak{A}$ of $\langle A\rangle$, there is $z \in X$ with

$$\delta(A \cup \{z\}) \leq r(A)$$

for all $A \in \langle X\rangle$.

Connections between a hyperconvex diversity and its induced metric space have been studied in [3, 5]. In particular it was shown in [5] that the metric induced by a hyperconvex diversity need not be hyperconvex itself.

2. Results for structures with diversity

In [5] authors considered the construction of two metric trees equipped with different types of diversities and asked whether their sum may happen hyperconvex (look at the proof of Theorem 2.19 in [5]). The intersection of these two metric tree was a singleton and each metric trees was endowed with the natural phylogenetic or diameter diversity for metric trees given in [3] so both were hyperconvex. Then the new diversity was defined as next proposition (for a proof see [5]) describes for general diversities.

Proposition 2.1
Let X and Y be two sets such that $X \cap Y = \{\theta\}$ and (X, δ_X) and (Y, δ_Y) are diversities. Then $(X \cup Y, \delta)$ is diversity where

$$\delta(A) = \delta_X((A \cap X) \cup \{\theta\}) + \delta_Y((A \cap Y) \cup \{\theta\}),\ A \cap (X \setminus \{\theta\}) \neq \emptyset,\ A \cap (Y \setminus \{\theta\}) \neq \emptyset$$

and $\delta(A) = \delta_X(A)$ (or $\delta(A) = \delta_Y(A)$) for $A \subset X$ (or $A \subset Y$), respectively.

Proof. The proof of this proposition follows in a straightforward way after distinguishing 27 different cases. We choose not to show the proof in this paper but the interested reader may consult Theorem 2.19 in [5].

It is our purpose in this section to study further when given δ_X and δ_Y two hyperconvex diversities we can add them, in the spirit of the above proposition, so that the new diversity remains hyperconvex. In the sequel we drop subindices of diversities when no confusion arises. We begin with a positive result which contains the case studied in Theorem 2.19 in [5] as a particular case.

Theorem 2.2
Let (X, δ) and (Y, δ) be two hyperconvex diversities with $X \cap Y = \{\theta\}$. Then $(X \cup Y, \delta)$ with the function δ defined as in Proposition 2.1 is hyperconvex.

Proof. Let us consider a function $r\colon \langle X \cup Y\rangle \to [0, \infty)$ such that

$$\delta\left(\bigcup_{k=1}^{n} A_k\right) \leq \sum_{k=1}^{n} r(A_k) \tag{2}$$

for any finite collection of finite subsets A_k of $X \cup Y$, $k = 1, \dots, n$ with $n \in \mathbb{N}$.

If for each $Z \in \langle X \cup Y \rangle$ we have $r(Z) \geq \delta(Z \cup \{\theta\})$, then there is nothing to prove. Indeed, we may take $Z \in \langle X \cup Y \rangle$ and make $A = Z \cap X$ and $B = Z \cap Y$ then

$$\delta(A \cup B) = \delta(A \cup \{\theta\}) + \delta(B \cup \{\theta\}) \leq r(A) + r(B)$$

and we are done. If this is not the case then, without loss of generality, we may suppose that there is $A_0 \in \langle X \setminus \{\theta\} \rangle$ such that $r(A_0) < \delta(A_0 \cup \{\theta\})$. Then for all $B \in \langle Y \rangle$ it must be the case that $r(B) > \delta(B \cup \{\theta\})$, otherwise, from the definition of δ,

$$r(A_0) + r(B) < \delta(A_0 \cup \{\theta\}) + \delta(B \cup \{\theta\}) = \delta(A_0 \cup B),$$

which contradicts (2).

Now we define a function $\bar{r}$ on $\langle X \rangle$. For $A \in \langle X \rangle$ with $\theta \in A$ we define $\bar{r}$(A) in the following way: let $\bar{Y} = \langle Y \setminus \{\theta\} \rangle$, then

$$\bar{r}(A) = \min \Big\{ r(A), \inf_{B \in \bar{Y}} (r(A \cup B) - \delta(B \cup \{\theta\})), \\ \inf_{B \in \bar{Y}} (r((A \cup B) \setminus \{\theta\}) - \delta(B \cup \{\theta\})) \Big\}.$$

For $A \in \langle X \rangle$ with $\theta \notin A$ we define $\bar{r}(A) = r(A)$. We claim that $\bar{r}$ is well-defined and nonnegative because $r(B) > \delta(B \cup \{\theta\})$ for $B \in \langle Y \rangle$.

We will further prove that

$$\delta\left(\bigcup_{k=1}^{n} A_k\right) \leq \sum_{k=1}^{n} \bar{r}(A_k) \tag{3}$$

for all finite collection of finite subsets A_k, $k = 1, \ldots, n$, of X.

Let us divide this finite family A_k, $k = 1, \ldots, n$ into three groups. First let us take B_i, $i = 1, \ldots, n_1$ such that $\theta \notin B_i$. Hence $r(B_i) = \bar{r}(B_i)$. As the second group let us take C_j, $j = 1, \ldots, n_2$ such that $\theta \in C_j$ but $\bar{r}(C_j) = r(C_j)$. And let D_k, $k = 1, \ldots, n_3$ (and clearly $n_1 + n_2 + n_3 = n$) be the rest of subsets for which $\bar{r}$ properly comes from one of the infimum expressions.

First let us suppose that $n_2 \geq 1$. For each D_k one may find $E_k \in \bar{Y}$ such that one of the following inequality holds:

$$\delta(D_k \cup E_k) \leq r(D_k \cup E_k) \leq \bar{r}(D_k) + \delta(E_k \cup \{\theta\}) + \varepsilon,$$

$$\delta(D_k \cup E_k) = \delta((D_k \setminus \{\theta\}) \cup E_k) \leq r((D_k \setminus \{\theta\}) \cup E_k) \leq \bar{r}(D_k) + \delta(E_k \cup \{\theta\}) + \varepsilon.$$

Since

$$\theta \in \bigcap_{j=1}^{n_2} C_j \cap \bigcap_{k=1}^{n_3} D_k,$$

we have

$$\begin{aligned}
&\delta\Big(\bigcup_{i=1}^{n_1} B_i \cup \bigcup_{j=1}^{n_2} C_j \cup \bigcup_{k=1}^{n_3} D_k\Big)\\
&\qquad\le \delta\Big(\bigcup_{i=1}^{n_1} B_i \cup \bigcup_{j=1}^{n_2} C_j\Big) + \sum_{k=1}^{n_3} \delta(D_k)\\
&\qquad\le \sum_{i=1}^{n_1} r(B_i) + \sum_{j=1}^{n_2} r(C_j) + \sum_{k=1}^{n_3}(\bar r(D_k) + \delta(E_k \cup \{\theta\}) + \varepsilon - \delta(E_k \cup \{\theta\}))\\
&\qquad= \sum_{i=1}^{n_1} \bar r(B_i) + \sum_{j=1}^{n_2} \bar r(C_j) + \sum_{k=1}^{n_3} \bar r(D_k) + n_3\varepsilon.
\end{aligned}$$

Since ε was arbitrary, this finishes the proof of (3) in the case of $n_2 \ge 1$.

Now let us consider the case of $n_2 = 0$. Then we have

$$\theta \in \bigcap_{k=1}^{n_3} D_k$$

and it yields

$$\begin{aligned}
&\delta\Big(\bigcup_{i=1}^{n_1} B_i \cup \bigcup_{k=1}^{n_3} D_k\Big)\\
&\qquad\le \delta\Big(\bigcup_{i=1}^{n_1} B_i \cup D_1\Big) + \sum_{k=2}^{n_3} \delta(D_k)\\
&\qquad= \delta\Big(\bigcup_{i=1}^{n_1} B_i \cup D_1 \cup E_1\Big) - \delta(E_1 \cup \{\theta\}) + \sum_{k=2}^{n_3} \delta(D_k \cup E_k) - \delta(E_k \cup \{\theta\})\\
&\qquad\le \sum_{i=1}^{n_1} r(B_i) + \bar r(D_1) + \delta(E_1 \cup \{\theta\}) + \varepsilon - \delta(E_1 \cup \{\theta\})\\
&\qquad\quad + \sum_{k=2}^{n_3}(\bar r(D_k) + \delta(E_k \cup \{\theta\}) + \varepsilon - \delta(E_k \cup \{\theta\}))\\
&\qquad\le \sum_{i=1}^{n_1} \bar r(B_i) + \sum_{k=1}^{n_3} \bar r(D_k) + n_3\varepsilon,
\end{aligned}$$

which finishes the proof of (3) also for this case.

Now let us notice that X is hyperconvex (with respect to (X, δ)). So there is a point $x_0 \in X$ such that

$$\bar r(A) \ge \delta(A \cup \{x_0\}) \tag{4}$$

for all $A \in \langle X\rangle$.

Now, if $A \in \langle X\rangle$, then $\bar r(A) \le r(A)$ and so $r(A) \ge \delta(A \cup \{x_0\})$ too. Next let us take B being a finite subset of $Y \setminus \{\theta\}$. Then

$$r(B) \ge \bar r(\{\theta\}) + \delta(B \cup \{\theta\}) = (*)$$

and on account of (4):

$$(*) \geq \delta(\{\theta, x_0\}) + \delta(B \cup \{\theta\}) = \delta(B \cup \{x_0\}).$$

Now let us take B being a finite subset of $(X \cup Y) \setminus \{\theta\}$. Then B is a sum of $B' \subset Y$ and $A \subset X$. Repeating previous considerations for $A \cup \{\theta\}$ instead of a singleton $\{\theta\}$:

$$r(B) = r(A \cup B') \geq \delta(A \cup \{\theta, x_0\}) + \delta(B' \cup \{\theta\}) = \delta(B \cup \{x_0\}).$$

Next we consider $B \in \langle X \cup Y \rangle$ which contains θ. Then $B = A \cup B'$, where $A \subset X$ (and $\theta \in A$) and $B' \subset Y \setminus \{\theta\}$. Therefore,

$$\begin{aligned} r(B) &= r(A \cup B') \geq \bar{r}(A) + \delta(B' \cup \{\theta\}) \\ &\geq \delta(A \cup \{x_0\}) + \delta(B' \cup \{\theta\}) = \delta(A \cup B' \cup \{x_0\}) \\ &= \delta(B \cup \{x_0\}). \end{aligned}$$

Clearly, if $B \subset Y$ contains θ the set A is equal to a singleton $\{\theta\}$.

Finally we have shown that there is $x_0 \in X$ such that for each $C \in \langle X \cup Y \rangle$ we have

$$r(C) \geq \delta(C \cup \{x_0\}),$$

and $X \cup Y$ is also hyperconvex.

We wonder next about what happens if the intersection of two hyperconvex diversities is larger than a singleton. In particular, we consider the case where the intersection is a metric segment $[a, b]$. A natural approach here would be to follow the gluing of metric spaces technique. One very natural way to glue two metric spaces (X, d) and (Y, d) (we denote respective metrics the same) with a metric segment $[a, b]$ as intersection is to define d on $X \cup Y$ by:

$$d(x, y) = \min_{c \in [a,b]} [d(x, c) + d(c, y)], \qquad x \in X \setminus [a, b],\ y \in Y \setminus [a, b], \tag{5}$$

(we will regard this distance function from now on as the *gluing metric*. Compare [2, Definition I.5.23 and Lemma I.5.24]).

Consider now (X, δ_X) and (Y, δ_Y) two given diversities with $X \cap Y$ being a metric segment. Assume both diversities coincide on this intersection segment. Then, following (5), a natural option to glue δ_X and δ_Y would be to define δ on $\langle X \cup Y \rangle$ by: if either $A \subseteq X$ or $A \subseteq Y$, $\delta(A)$ is the corresponding one; if A is not as above but $A \cap [a, b] = \emptyset$, then

$$\delta(A) = \min_{p \in [a,b]} [\delta_X((A \setminus Y) \cup \{p\}) + \delta_Y((A \setminus X) \cup \{p\})];$$

otherwise,

$$\delta(A) = \min_{\substack{P_1, P_2 \in \langle [a,b] \rangle, \\ p \in [a,b]}} [\delta_X((A \setminus Y) \cup P_1 \cup \{p\}) + \delta_Y((A \setminus X) \cup P_2 \cup \{p\})]$$

for disjoint $P_1, P_2 \in \langle [a, b] \rangle$ such that $P_1 \cup P_2 = A \cap [a, b]$ and for each $p \in P$. We close this section with an example showing that such a function does not define a diversity in general.

Example 2.3
Consider the points $a = (0, 0)$ and $b = (0, 2)$ in $\mathbb{R}^2$. Let X be a metric segment joining points x_1 and x_2 in $\mathbb{R}^2$ with length 4 and such that

$$d(x_1, x_2) = d(x_1, a) + d(a, b) + d(b, x_2) = 1 + 2 + 1.$$

Then X being a metric tree is obviously hyperconvex with respect to the diameter diversity (see [3] for details).

As the set Y let us consider the rectangle of $\mathbb{R}^2$:

$$Y = \{(x, y) \in \mathbb{R}^2 | \ x \in [0, 1] \wedge y \in [0, 2]\}$$

with the distance induced by the maximum norm. Y is an admissible subset of a hyperconvex space $(\mathbb{R}^2, d_{\max})$ (more on this kind of subsets the reader may find in [6, Section 5.1] or [7, Section 3]), so it is hyperconvex with respect to the metric and the diameter diversity. Then $[a, b] = \{(0, x) \colon x \in [0, 2]\} \subset Y$ is a metric segment being the intersection $X \cap Y$.

As a subset A one may consider $\{x_1, y\}$ while $B = \{x_2, y\}$, where $y = (1, 1) \in Y$. Hence

$$\delta(A) = d(x_1, a) + d(a, y) = 2,$$
$$\delta(B) = d(x_2, b) + d(b, y) = 2.$$

At the same time $A \cap B = \{y\}$ and

$$\delta(A \cup B) = \delta(\{x_1, x_2, y\}) = d(x_1, x_2) + d(a, y) = 5$$

so the condition (ii) of Definition 1.2 does not hold, and $(X \cup Y, \delta)$ is not a diversity.

3. Results for metric spaces

In this section we take up the same problem as above but for hyperconvex metric. That is, we want to study when the gluing of two hyperconvex metric is hyperconvex. It is trivial to check that two $\mathbb{R}$-trees glued through a common set by the metric given by (5) is again a metric tree and so hyperconvex. In our first approach to this problem we will consider the case where the two hyperconvex metric spaces X and Y intersect at exactly a metric segment $[a, b]$ which happens to be unique for both spaces, that is, there is no other metric segment connecting a and b neither in A nor B. We begin with the following technical lemma.

Lemma 3.1
Let X be a hyperconvex space such that $[a, b]$ is the unique metric segment joining two points a and b. Then for each $x \in X$ there is $c_x \in [a, b]$, which coincides with the metric projection of x in $[a, b]$, which is a gate from x to $[a, b]$, that is,

$$d(x, c) = d(x, c_x) + d(c_x, c) \tag{6}$$

for any $c \in [a, b]$.

Proof. Existence of the metric projection, that is, the point of minimal distance from $[a,b]$ to x, directly follows from the compactness of $[a,b]$. Take c_x as such a point if we show that it is a gate then unicity follows. Let us suppose next that there is $c \in [a,b]$ for which (6) does not hold. Then $d(x,c) < d(x,c_x) + d(c_x,c)$ and, from linear systems of equations, we can conclude that there are three positive numbers α, β, γ such that

$$\begin{cases} \alpha + \beta = d(x,c), \\ \alpha + \gamma = d(x,c_x), \\ \beta + \gamma = d(c_x,c). \end{cases}$$

Since X is hyperconvex, there is $c' \in \bar{B}(x,\alpha) \cap \bar{B}(c,\beta) \cap \bar{B}(c_x,\gamma)$. From the uniqueness of $[a,b]$ we have $c' \in [c,c_x] \subset [a,b]$ and $d(c',x) = \alpha < d(x,c_x)$, a contradiction.

Corollary 3.2
Let X and Y be hyperconvex metric spaces with $[a,b] = X \cap Y$ and such that a and b are connected by a unique metric segment in both spaces. Consider d the gluing metric given by (5) *on $X \cup Y$. Then for $x \in X$ and $y \in Y$ there are respective c_x and c_y as in Lemma 3.1 such that*

$$d(x,y) = d(x,c_x) + d(c_x,c_y) + d(c_y,y).$$

Next we give our positive result in this direction.

Theorem 3.3
Let X and Y be hyperconvex metric spaces such that $X \cap Y = [a,b]$ and a and b are connected by a unique metric segment in both spaces, then $X \cup Y$ endowed with the gluing metric given by (5) *is hyperconvex too.*

Proof. Consider a family of closed balls $\{\bar{B}(z,r(z))\colon z \in X \cup Y\}$ such that $d(z_1,z_2) \le r(z_1) + r(z_2)$. One may assume that centers of balls do not repeat themselves. For each $z \in X \cup Y$ we define c_z in the same way as in Lemma 3.1. We consider the following three cases:

1. For each $z \in X \cup Y$ we have that $d(z,c_z) \le r(z)$ and each pair of balls $\bar{B}(c_z, r(z) - d(z,c_z))$ and $\bar{B}(c_{\bar{z}}, r(\bar{z}) - d(\bar{z},c_{\bar{z}})$ has a common point in $[a,b]$. Clearly,

   $$\bar{B}(c_z, r(z) - d(z,c_z)) \cap [a,b] \subset \bar{B}(z,r(z)), \qquad z \in X \cup Y.$$

   Let us notice that in this new family

   $$\{\bar{B}(c_z, r(z) - d(z,c_z)) \cap [a,b] : \ z \in X \cup Y\}$$

   points c_z may repeat.

   Since $[a,b]$ is hyperconvex, for this new family of balls we get a point $\bar{c} \in [a,b]$ such that $d(\bar{c},c_z) \le r(z) - d(z,c_z)$, so

   $$d(\bar{c},z) \le d(\bar{c},c_z) + d(c_z,z) \le r(z).$$

2. For each $z \in X \cup Y$ we have that $d(z, c_z) \leq r(z)$ but there is a pair of points u, w for which
$$d(v, c_v) + d(c_v, c_w) + d(c_w, w) > r(v) + r(w).$$
First let us notice that both points v and w must belong to one set X or Y. Otherwise, on account of Corollary 3.2, we have
$$d(v, c_v) + d(c_v, c_w) + d(c_w, w) = d(v, w) \leq r(v) + r(w),$$
which contradicts our assumption.
We may assume that all these points belong to X and fix one pair v, w of them. For this pair one may find $p, q \in [a, b]$ in such a way that
$$d(c_v, c_w) = d(c_v, p) + d(p, q) + d(q, c_w)$$
and
$$r(v) = d(v, c_v) + d(c_v, p),$$
$$r(w) = d(w, c_w) + d(c_w, q).$$
Now if $y \in Y \setminus [a, b]$, again on account of Corollary 3.2, we have that $p, q \in \bar{B}(y, r(y))$, so also $p, q \in \bar{B}(c_y, r(y) - d(y, c_y))$ and the family of balls $\{\bar{B}(c_y, r(y) - d(y, c_y))\colon y \in Y \setminus [a, b]\}$ has nonempty intersection. Next let us consider two balls $\bar{B}(c_y, r(y) - d(y, c_y))$ and $\bar{B}(x, r(x))$, where $x \in X$. Then
$$d(x, y) = d(x, c_x) + d(c_x, c_y) + d(c_y, y) \leq r(x) + r(y)$$
what implies
$$d(x, c_y) \leq r(x) + r(y) - d(c_y, y)$$
and this pair of balls also has nonempty intersection.
Now hyperconvexity of X implies that the collection of balls $\{\bar{B}(y, r(y)),\ y \in Y\}$ and $\{\bar{B}(x, r(x)),\ x \in X\}$ has a common point.
3. There is $z \in X \cup Y \setminus [a, b]$ for which $d(z, c_z) > r(z)$. Again, one may suppose that all these points belong to one set, say X. Then, on account of Corollary 3.2, for each $y \in Y \setminus [a, b]$ the point c_z must belong to $\bar{B}(c_y, r(y) - d(y, c_y))$ and taking these families of balls instead of $\{\bar{B}(y, r(y)),\ y \in Y\}$ we apply the hyperconvexity of X to obtain a nonempty intersection similarly as it was done in the previous case.

So the family $\{\bar{B}(z, r(z)) :\ z \in X \cup Y\}$ has a nonempty intersection and the sum $X \cup Y$ is hyperconvex with respect to the metric defined by (5).

Next we give an example showing that the conditions on uniqueness of the metric segment $[a, b]$ in Theorem 3.3 does not leave very much room for improvement. More precisely, this examples provides two hyperconvex metric spaces intersecting in a metric segment which is unique for one of these spaces but the gluing metric is not hyperconvex.

Example 3.4
Let us consider four points $\{x_i :\ i = 1, 2, 3, 4\}$ with the distance:

$$\begin{aligned} d(x_1, x_2) = 3,\ \ d(x_1, x_3) = 2,\ \ d(x_1, x_4) = 3, \\ d(x_2, x_3) = 3,\ \ d(x_2, x_4) = 2, \\ d(x_3, x_4) = 2. \end{aligned}$$

And let X be a tight span of this metric space (see [3, 9] or [7, Section 8] for precise definition of tight spans), then X is hyperconvex itself. First we want to show that the metric segment $[x_1, x_2]$ is unique in X. Let p belong to $[x_1, x_2]$. Then p may be considered as a minimal function such that

$$p(x_i) + p(x_j) \geq d(x_i, x_j), \qquad i, j \in \{1, 2, 3, 4\}$$

and moreover,

$$p(x_1) + p(x_2) = 3.$$

We consider the following three cases:

1. $p(x_1) \in [0, 1]$. Then $p(x_2) \geq 2$ and

$$\begin{aligned} p(x_3) \geq 2 - p(x_1) \geq 1 \geq 3 - p(x_2), \\ p(x_4) \geq 3 - p(x_1) \geq 2 \geq 2 - p(x_2), \\ p(x_3) + p(x_4) \geq 2, \end{aligned}$$

what implies

$$\begin{aligned} p(x_3) = 2 - p(x_1), \\ p(x_4) = 3 - p(x_1) \end{aligned}$$

and p is uniquely determined.

2. $p(x_1) \in (1, 2)$. Then

$$\begin{aligned} p(x_3) \geq 2 - p(x_1) \leq 1 \quad \wedge \quad p(x_3) \geq 3 - p(x_2) \geq 1, \\ p(x_4) \geq 2 - p(x_2) \leq 1 \quad \wedge \quad p(x_4) \geq 3 - p(x_1) \geq 1, \\ p(x_3) + p(x_4) \geq 2 \implies \begin{cases} p(x_3) = 3 - p(x_2), \\ p(x_4) = 3 - p(x_1). \end{cases} \end{aligned}$$

3. $p(x_1) \in [2, 3]$. This case is similar to the first one and hence

$$\begin{aligned} p(x_3) = 3 - p(x_2), \\ p(x_4) = 2 - p(x_2). \end{aligned}$$

So $[x_1, x_2]$ is unique as a subset of X. Moreover, if one takes a projection of points on the metric segment $[x_1, x_2]$, we have that

$$\begin{aligned} P_{[x_1,x_2]}(x_3) = y_3, \\ P_{[x_1,x_2]}(x_4) = y_4, \end{aligned}$$

where $d(x_1, y_3) = d(y_3, y_4) = d(y_4, x_2) = 1$.

As the set Y let us consider a rectangle of $\mathbb{R}^2$:

$$Y = \{(x, y) \in \mathbb{R}^2 |\ x \in [0, 3]\ \wedge\ y \in [-2, 2]\}$$

with the metric induced by the maximum norm and let $x_1 = (0, 0)$, $x_2 = (3, 0)$. Hence the metric segment $[x_1, x_2]$ is an intersection $X \cap Y$. Y being a subset of a hyperconvex space $(\mathbb{R}^2, d_{\max})$ is hyperconvex. For any pair (x, y) of points such that $x \in X \setminus [x_1, x_2]$ and $y \in Y$ we define

$$d(x, y) = \min_{p \in [x_1, x_2]} (d(x, p) + d(p, y)).$$

But let us notice that at the same time for all $p \in [x_1, x_2]$ we have

$$d(x_3, p) = d(x_3, y_3) + d(y_3, p).$$

Similar results may be obtained for x_4 (and y_4).

Now let us consider $\bar{y} = (1.5, 0.5)$ (then $\bar{y} \in Y \setminus X$) and the function $r\colon X \cup Y \to [0, 6]$ defined in the following way:

$$r(z) = \begin{cases} 1, & z \in \{x_3, x_4\}, \\ 0.5, & z = \bar{y}, \\ 6, & \text{otherwise}. \end{cases}$$

Clearly, $\operatorname{diam}(X \cup Y) \leq 6$ and

$$d(x_3, x_4) = 2 = r(x_3) + r(x_4),$$
$$d(x_i, \bar{y}) = 1.5 \leq r(x_i) + r(\bar{y}),$$

so r satisfies conditions (1). But at the same time there is no point in $X \cup Y$ which belongs to $\bar{B}(x_3, r(x_3)) \cap \bar{B}(x_4, r(x_4)) \cap \bar{B}(\bar{y}, r(\bar{y}))$. Indeed, let us suppose that z is such a point. Then $z \in Y$ because $d(z, \bar{y}) \leq 0.5$. At the same time $d(z, x_i) \leq 1$, so $z = y_i$ for $i = \{3, 4\}$, a contradiction. Therefore $X \cup Y$ is not hyperconvex.

Remark 3.5
One may wonder if there are examples for Theorem 3.3 that are not $\mathbb{R}$-trees. That is, hyperconvex spaces for which at least there are two points joined by a unique segments. Such examples are easy to build by directly gluing $\mathbb{R}$-trees to hyperconvex spaces through a given point although, however, a more interesting example may be space X in Example 3.4.

We close this work with a question.

Question 3.6
In Example 3.4 the metric segment was not externally hyperconvex with respect X (see [1, 7] for definitions and properties). So it remains as an open question whether the sum of two spaces such that their intersection is an externally hyperconvex metric segment with respect to both of them is also a hyperconvex space endowed with the gluing metric.

References

[1] N. Aronszajn, P. Panitchpakdi, *Extensions of uniformly continuous transformations and hyperconvex metric spaces*, Pacific J. Math. **6** (1956), 405–439. Cited on 66 and 75.

[2] M. Bridson, A. Haefliger, *Metric Spaces of Non-Positive Curvature*, Grundlehren der Mathematischen Wissenschaften 319, Springer-Verlag, Berlin, 1999. Cited on 70.

[3] D. Bryant, P.F. Tupper, *Hyperconvexity and tight-span theory for diversities*, Adv. Math. **231** (2012), no. 6, 3172–3198. Cited on 65, 66, 67, 71 and 74.

[4] A.W.M. Dress, *Trees, tight extensions of metric spaces, and the cohomological dimension of certain groups: a note on combinatorial properties of metric spaces*, Adv. in Math. **53** (1984), no. 3, 321–402. Cited on 65.

[5] R. Espínola, B. Piątek, *Diversities, hyperconvexity and fixed points*, Nonlinear Anal. **95** (2014), 229–245. Cited on 65, 66 and 67.

[6] R. Espínola, A. Fernández León, *Fixed Point Theory in Hyperconvex Metric Spaces*, Topics in Fixed Point Theory, 101–158, Springer, Berlin, 2013. Cited on 66 and 71.

[7] R. Espínola, M.A. Khamsi, *Introduction to hyperconvex spaces*, Handbook of metric fixed point theory, 391–435, Kluwer Acad. Publ., Dordrecht, 2001. Cited on 66, 71, 74 and 75.

[8] D. Faith, *Conservation evaluation and phylogenetic diversity*, Biol. Conserv. **61** (1992), 1–10. Cited on 65.

[9] J.R Isbell, *Injective envelopes of Banach spaces are rigidly attached*, Bull. Amer. Math. Soc. **70** (1964), 727–729. Cited on 65 and 74.

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