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Starlike functions of complex order involving q-hypergeometric functions with fixed point

Abstract. Recently Kanas and Ronning introduced the classes of starlike and convex functions, which are normalized with $f(\xi) = f'(\xi) - 1 = 0$, ξ ($|\xi| = d$) is a fixed point in the open disc $\mathbb{U} = \{z \in \mathbb{C} : |z| < 1\}$. In this paper we define a new subclass of starlike functions of complex order based on q-hypergeometric functions and continue to obtain coefficient estimates, extreme points, inclusion properties and neighbourhood results for the function class $\mathcal{TS}_\xi(\alpha, \beta, \gamma)$. Further, we obtain integral means inequalities for the function $f \in \mathcal{TS}_\xi(\alpha, \beta, \gamma)$.

1. Introduction

Let ξ ($|\xi| = d$) be a fixed point in the unit disc $\mathbb{U} := \{z \in \mathbb{C} : |z| < 1\}$. Denote by $\mathcal{A}(\xi)$ the class of functions which are regular and normalized by $f(\xi) = f'(\xi) - 1 = 0$ consisting of the functions of the form

$$f(z) = (z - \xi) + \sum_{n=2}^{\infty} a_n (z - \xi)^n, \qquad (z - \xi) \in \mathbb{U}. \tag{1}$$

Also denote by $\mathcal{S}_\xi = \{f \in \mathcal{A}(\xi) : f \text{ is univalent in } \mathbb{U}\}$, the subclass of $\mathcal{A}(\xi)$. Denote by $\mathcal{T}_\xi$ the subclass of $\mathcal{S}_\xi$ consisting of the functions of the form

$$f(z) = (z - \xi) - \sum_{n=2}^{\infty} a_n (z - \xi)^n, \qquad a_n \geq 0. \tag{2}$$

Note that $\mathcal{S}_0 = \mathcal{S}$ and $\mathcal{T}_0 = \mathcal{T}$ be the subclasses of $\mathcal{A} = \mathcal{A}(0)$ consisting of univalent functions in $\mathbb{U}$. By $\mathcal{S}^*_\xi(\beta)$ and $\mathcal{K}_\xi(\beta)$ respectively, we mean the classes of analytic

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functions that satisfy the analytic conditions

$$\Re\Big\{\frac{(z-\xi)f'(z)}{f(z)}\Big\} > \beta, \quad \Re\Big\{1+\frac{(z-\xi)f''(z)}{f'(z)}\Big\} > \beta \quad \text{and} \quad (z-\xi)\in\mathbb{U}$$

for $0 \le \beta < 1$ introduced and studied by Kanas and Ronning [9]. The class $\mathcal{S}^*_\xi(0)$ is defined by geometric property that the image of any circular arc centered at ξ is starlike with respect to $f(\xi)$ and the corresponding class $\mathcal{K}^*_\xi(0)$ is defined by the property that the image of any circular arc centered at ξ is convex. We observe that the definitions are somewhat similar to the ones introduced by Goodman in [8] for uniformly starlike and convex functions, except that in this case the point ξ is fixed. In particular, $\mathcal{K} = \mathcal{K}_0(0)$ and $\mathcal{S}^*_0 = \mathcal{S}^*(0)$ respectively, are the well-known standard classes of convex and starlike functions[10, 19].

We recall a generalized q-Taylors formula in fractional q-calculus and certain q-generating functions for q-hypergeometric functions studied more recently by Purohit and Raina [15] and further by Mohammed Aabed and Maslina Darus [1]. For complex parameters $a_1,\ldots,a_l$ and $b_1,\ldots,b_m$ $(b_j \ne 0,-1,\ldots;\ j=1,2,\ldots,m)$ the *q-hypergeometric function* ${}_l\Psi_m(z)$ is defined by

$$\begin{aligned} {}_l\Psi_m(a_1,\ldots,a_l;b_1,\ldots,b_m;q,z) \\ := \sum_{n=0}^{\infty}\frac{(a_1;q)_n\ldots(a_l;q)_n}{(b_1;q)_n\ldots(b_m;q)_n}\big[(-1)^n\ q^{\binom{n}{2}}\big]^{1+m-l}z^n \end{aligned} \tag{3}$$

with $\binom{n}{2} = \frac{n(n-1)}{2}$, where $q \ne 0$ when $l > m+1$ $(l,m \in \mathbb{N}_0 = \mathbb{N}\cup\{0\};\ z\in\mathbb{U})$.

The q-shifted factorial is defined for $a,q\in\mathbb{C}$ as a product of n factors by

$$(a;q)_n = \begin{cases} 1, & n=0, \\ (1-a)(1-aq)\ldots(1-aq^{n-1}), & n\in\mathbb{N} \end{cases}$$

and in terms of basic analogue of the gamma function

$$(q^a;q)_n = \frac{\Gamma_q(a+n)(1-q)^n}{\Gamma_q(a)}, \qquad n>0. \tag{4}$$

It is interest to note that $\lim_{q\to1^-}\frac{(q^a;q)_n}{(1-q)^n} = (a)_n = a(a+1)\ldots(a+n-1)$ the familiar Pochhammer symbol.

Now for $z\in\mathbb{U}$, $0<|q|<1$ and $l=m+1$, the basic q-hypergeometric function defined in (3) takes the form

$${}_l\psi_m(a_1,\ldots,a_l;b_1,\ldots,b_m;q,z) = \sum_{n=0}^{\infty}\frac{(a_1;q)_n\ldots(a_l;q)_n}{(q;q)_n(b_1;q)_n\ldots(b_m,q)_n}z^n$$

which converges absolutely in the open unit disk $\mathbb{U}$. Let

$$\mathcal{I}(a_l,b_m;q;z) = z\,{}_l\psi_m(a_1,\ldots,a_l;b_1,\ldots,b_m;q,z) = \sum_{n=0}^{\infty}\Upsilon_n^{l,m}[a_1,q]z^{n+1},$$

where for convenience,

$$\Upsilon_n^{l,m}[a_1, q] = \frac{(a_1; q)_n \dots (a_l; q)_n}{(q; q)_n (b_1; q)_n \dots (b_m; q)_n}.$$

The operator $\mathcal{I}(a_l, b_m; q)f(z)$ was studied recently by Aabed and Darus [1].

In this paper we define a new linear operator for $(z - \xi) \in \mathbb{U}$, $|q| < 1$ and $l = m + 1$ as follows:

$$\begin{aligned}\mathcal{I}(a_l, b_m; q, z - \xi) &= (z - \xi)\, {}_l\psi_m(a_1, \dots, a_l; b_1, \dots, b_m; q, z - \xi) \\ &= \sum_{n=0}^{\infty} \Upsilon_n^{l,m}[a_1, q](z - \xi)^{n+1}.\end{aligned}$$

Using the above, we let

$$\mathcal{I}(a_l, b_m; q, z - \xi) * f(z) = \mathcal{I}_m^l f(z) = (z - \xi) + \sum_{n=2}^{\infty} \Upsilon_n^{l,m}[a_1, q] a_n (z - \xi)^n, \tag{5}$$

where

$$\Upsilon_m^l(n) = \Upsilon_n^{l,m}[a_1, q] = \frac{(a_1; q)_{n-1} \dots (a_l; q)_{n-1}}{(q; q)_{n-1} (b_1; q)_{n-1} \dots (b_m; q)_{n-1}}$$

unless otherwise stated.

For $a_i = q^{\alpha_i}$, $b_j = q^{\beta_j}$, $\alpha_i, \beta_j \in \mathbb{C}$, and $\beta_j \neq 0, -1, -2, \dots$, $(i = 1, \dots, l,\ j = 1, \dots, m)$ and $q \to 1$, we obtain the well-known Dziok-Srivastava linear operator [7, 6] (for $l = m + 1$). For $l = 1$, $m = 0$, $a_1 = q$, and further specializing the parameters, it gives many (well known and new) integral and differential operators introduced and studied in [4, 5, 10, 13, 16].

Making use of the operator $\mathcal{I}_m^l$ and motivated by the results discussed by Altintas et al. [2], (see [14] and references stated therein) and Aouf et al. [3], in this paper we introduce a new subclass $\mathcal{S}_\xi(\alpha, \beta, \gamma)$ of analytic functions of complex order associated with q-hypergeometric functions as given below.

For $-1 \le \alpha < 1$, $\beta \ge 0$ and $\gamma \in \mathbb{C} \setminus \{0\}$, we let $\mathcal{S}_\xi(\alpha, \beta, \gamma)$ be the subclass of $\mathcal{A}(\xi)$ consisting of functions of the form (1) and satisfying the analytic criterion

$$\Re\Big(1 + \frac{1}{\gamma}\Big[\frac{(z - \xi)(\mathcal{I}_m^l f(z))'}{\mathcal{I}_m^l f(z)} - \alpha\Big]\Big) > \beta\Big|1 + \frac{1}{\gamma}\Big[\frac{(z - \xi)(\mathcal{I}_m^l f(z))'}{\mathcal{I}_m^l f(z)} - 1\Big]\Big|$$

for every $z \in \mathbb{U}$, where $\mathcal{I}_m^l f(z)$ is given by (5). We also let $\mathcal{TS}_\xi(\alpha, \beta, \gamma) = \mathcal{S}_\xi(\alpha, \beta, \gamma) \cap \mathcal{T}_\xi$.

Example 1

We note that $\mathcal{S}_\xi(1, 0, \gamma) \equiv \mathcal{S}_\xi^*(\gamma)$, the class of starlike functions of complex order γ $(\gamma \in \mathbb{C} \setminus \{0\})$, satisfying the following conditions

$$\frac{f(z)}{z - \xi} \neq 0 \qquad \text{and} \qquad \Re\Big(1 + \frac{1}{\gamma}\Big[\frac{(z - \xi)(\mathcal{I}_m^l f(z))'}{\mathcal{I}_m^l f(z)} - 1\Big]\Big) > 0.$$

Further,

$$\mathcal{S}_\xi^*((1-\delta)\cos\lambda\, e^{-i\lambda}) = S_\xi^*(\delta,\lambda), \qquad |\lambda| < \frac{\pi}{2}; \quad 0 \le \delta \le 1$$

and

$$\mathcal{S}_\xi^*(\cos\lambda\, e^{-i\lambda}) = \mathcal{S}_\xi^*(\lambda), \qquad |\lambda| < \frac{\pi}{2},$$

where $S_\xi^*(\delta,\lambda)$ denotes the subclass of λ-spiral-like function of order δ and $S_\xi^*(\lambda)$ denotes spiral-like functions with fixed point analogous to the classes introduced and investigated by Libera [11] and Spacek [18](Also see[21]), respectively.

The main object of this paper is to study some usual properties such as the coefficient bounds, extreme points, radii of close to convexity, starlikeness and convexity for the class $\mathcal{TS}_\xi(\alpha,\beta,\gamma)$. Further, we obtain neighborhood results and integral means inequalities for aforementioned class.

2. Coefficient bounds

In this section we obtain a necessary and sufficient condition for functions $f \in \mathcal{TS}_\xi(\alpha,\beta,\gamma)$.

Theorem 2.1

A necessary and sufficient condition for f of the form (2) to be in the class $\mathcal{TS}_\xi(\alpha,\beta,\gamma)$ is

$$\sum_{n=2}^{\infty}[(n+|\gamma|)(1-\beta)-(\alpha-\beta)](r+d)^{n-1}\Upsilon_m^l(n)a_n \le (1-\alpha)+|\gamma|(1-\beta), \quad (6)$$

where $-1 \le \alpha < 1$, $\beta \ge 0$ and $\gamma \in \mathbb{C}\setminus\{0\}$.

Proof. Assume that $f \in \mathcal{TS}_\xi(\alpha,\beta,\gamma)$, then

$$\Re\Big(1+\frac{1}{\gamma}\Big[\frac{(z-\xi)(\mathcal{I}_m^l f(z))'}{\mathcal{I}_m^l f(z)}-\alpha\Big]\Big) > \beta\Big|1+\frac{1}{\gamma}\Big[\frac{(z-\xi)(\mathcal{I}_m^l f(z))'}{\mathcal{I}_m^l f(z)}-1\Big]\Big|,$$

$$\Re\Big(1+\frac{1}{\gamma}\Big[\frac{(z-\xi)(1-\alpha)-\sum_{n=2}^{\infty}(n-\alpha)\Upsilon_m^l(n)a_n(z-\xi)^n}{(z-\xi)-\sum_{n=2}^{\infty}\Upsilon_m^l(n)a_n(z-\xi)^n}\Big]\Big)$$
$$> \beta\Big|1-\frac{1}{\gamma}\Big[\frac{\sum_{n=2}^{\infty}(n-1)\Upsilon_m^l(n)a_n(z-\xi)^n}{(z-\xi)-\sum_{n=2}^{\infty}\Upsilon_m^l(n)a_n(z-\xi)^n}\Big]\Big|.$$

On choosing the values of $(z-\xi)$ on the positive real axis, where $0 < |z-\xi| \le r+d < 1$, we have

$$\Big\{1+\frac{1}{|\gamma|}\Big(\frac{(1-\alpha)-\sum_{n=2}^{\infty}(n-\alpha)\Upsilon_m^l(n)a_n(r+d)^{n-1}}{1-\sum_{n=2}^{\infty}\Upsilon_m^l(n)a_n(r+d)^{n-1}}\Big)\Big\}$$
$$> \beta\Big\{1-\frac{1}{|\gamma|}\Big(\frac{\sum_{n=2}^{\infty}(n-1)\Upsilon_m^l(n)a_n(r+d)^{n-1}}{1-\sum_{n=2}^{\infty}\Upsilon_m^l(n)a_n(r+d)^{n-1}}\Big)\Big\}.$$

The simple computation leads the desired inequality

$$\sum_{n=2}^{\infty}[(n+|\gamma|)(1-\beta)-(\alpha-\beta)]\Upsilon_m^l(n)a_n(r+d)^{n-1}\le(1-\alpha)+|\gamma|(1-\beta).$$

Conversely, suppose that (6) is true for $(z-\xi)\in\mathbb{U}$, then

$$\Re\Big(1+\frac{1}{\gamma}\Big[\frac{(z-\xi)(\mathcal{I}_m^lf(z))'}{\mathcal{I}_m^lf(z)}-\alpha\Big]\Big)-\beta\Big|1+\frac{1}{\gamma}\Big[\frac{(z-\xi)(\mathcal{I}_m^lf(z))'}{\mathcal{I}_m^lf(z)}-1\Big]\Big|>0.$$

If

$$\begin{aligned}1+\frac{1}{|\gamma|}\Big(&\frac{(1-\alpha)-\sum_{n=2}^{\infty}(n-\alpha)\Upsilon_m^l(n)a_n|z-\xi|^{n-1}}{1-\sum_{n=2}^{\infty}\Upsilon_m^l(n)a_n|z-\xi|^{n-1}}\Big)\\&-\beta\Big[1-\frac{1}{|\gamma|}\Big(\frac{\sum_{n=2}^{\infty}(n-1)\Upsilon_m^l(n)a_n|z-\xi|^{n-1}}{1-\sum_{n=2}^{\infty}\Upsilon_m^l(n)a_n|z-\xi|^{n-1}}\Big)\Big]\ge0.\end{aligned}$$

That is if

$$\sum_{n=2}^{\infty}[(n+|\gamma|)(1-\beta)-(\alpha-\beta)]\Upsilon_m^l(n)a_n(r+d)^{n-1}\le(1-\alpha)+|\gamma|(1-\beta),$$

which completes the proof.

Corollary 2.2

Let the function f defined by (2) belongs $\mathcal{TS}_\xi(\alpha,\beta,\gamma)$. Then

$$a_n\le\frac{[(1-\alpha)+|\gamma|(1-\beta)]}{[(n+|\gamma|)(1-\beta)-(\alpha-\beta)]\Upsilon_m^l(n)(r+d)^{n-1}},$$

$n\ge2$, $-1\le\alpha<1$, $\beta\ge0$ and $\gamma\in\mathbb{C}\setminus\{0\}$, with equality for

$$f(z)=(z-\xi)-\frac{[(1-\alpha)+|\gamma|(1-\beta)]}{[(n+|\gamma|)(1-\beta)-(\alpha-\beta)]\Upsilon_m^l(n)}(z-\xi)^n.$$

For the sake of brevity we let

$$\Theta_d(n,\alpha,\beta,\gamma)=[(n+|\gamma|)(1-\beta)-(\alpha-\beta)](r+d)^{n-1},$$

$$\Theta_d(2,\alpha,\beta,\gamma)=[(2-\alpha-\beta)+|\gamma|(1-\beta)](r+d) \tag{7}$$

throughout our study.

In the next theorem we state extreme points for the functions of the class $\mathcal{TS}_\xi(\alpha,\beta,\gamma)$.

Theorem 2.3 (Extreme points)

Let

$$\begin{aligned}&f_1(z)=(z-\xi),\\&f_n(x)=(z-\xi)-\frac{[(1-\alpha)+|\gamma|(1-\beta)]}{[(n+|\gamma|)(1-\beta)-(\alpha-\beta)]\Upsilon_m^l(n)}(z-\xi)^n,\ n=2,3,\ldots.\end{aligned} \tag{8}$$

Then $f\in\mathcal{TS}_\xi(\alpha,\beta,\gamma)$ if and only if f can be expressed in the form $f(z)=\sum_{n=1}^{\infty}\omega_nf_n(z)$, where $\omega_n\ge0$ and $\sum_{n=1}^{\infty}\omega_n=1$.

The proof of the Theorem 2.3 follows on lines similar to the proof of the theorem on extreme points given in Silverman [19].

3. Close-to-convexity, starlikeness and convexity

In this section we obtain the radii of close-to-convexity, starlikeness and convexity for the class $\mathcal{TS}_\xi(\alpha,\beta,\gamma)$.

Theorem 3.1
Let $f\in\mathcal{TS}_\xi(\alpha,\beta,\gamma)$. Then f is close-to-convex of order δ $(0\le\delta<1)$ in the disc $|z-\xi|<R_1$, that is $\Re(f'(z))>\delta$, where

$$R_1=\inf_{n\ge 2}\Big[\frac{(1-\delta)\Theta_d(n,\alpha,\beta,\gamma)}{n[(1-\alpha)+|\gamma|(1-\beta)]}\Upsilon^l_m(n)\Big]^{\frac{1}{n-1}}.$$

Proof. Given $f\in\mathcal{T}_\xi$ and f is close-to-convex of order δ, we have

$$|f'(z)-1|<1-\delta. \tag{9}$$

For the left hand side of (9) we have

$$|f'(z)-1|\le\sum_{n=2}^{\infty}na_nR_1^{n-1}.$$

The last expression is less than $1-\delta$ if

$$\sum_{n=2}^{\infty}\frac{n}{1-\delta}a_nR_1^{n-1}<1.$$

Using the fact, that $f\in\mathcal{TS}_\xi(\alpha,\beta,\gamma)$ if and only if

$$\sum_{n=2}^{\infty}\frac{\Theta_d(n,\alpha,\beta,\gamma)}{(1-\alpha)+|\gamma|(1-\beta)}\Upsilon^l_m(n)a_n<1.$$

We can say (9) is true if

$$\frac{n}{1-\delta}R_1^{n-1}\le\frac{\Theta_d(n,\alpha,\beta,\gamma)}{(1-\alpha)+|\gamma|(1-\beta)}\Upsilon^l_m(n).$$

Or equivalently,

$$R_1\le\Big[\frac{(1-\delta)\Theta_d(n,\alpha,\beta,\gamma)}{n[(1-\alpha)+|\gamma|(1-\beta)]}\Upsilon^l_m(n)\Big]^{\frac{1}{n-1}}.$$

Which completes the proof.

Theorem 3.2
Let $f\in\mathcal{TS}_\xi(\alpha,\beta,\gamma)$. Then

1. *f is starlike of order δ $(0\le\delta<1)$ in the disc $|z-\xi|<R_2$; that is, $\Re(\frac{(z-\xi)f'(z)}{f(z)})>\delta$, where*

$$R_2=\inf_{n\ge 2}\Big\{\frac{(1-\delta)}{(n-\delta)}\frac{\Theta_d(n,\alpha,\beta,\gamma)}{[(1-\alpha)+|\gamma|(1-\beta)]}\Upsilon^l_m(n)\Big\}^{\frac{1}{n-1}},$$

2. *f is convex of order δ $(0 \leq \delta < 1)$ in the unit disc $|z-\xi| < R_3$, that is $\Re(1+\frac{(z-\xi)f''(z)}{f'(z)}) > \delta$, where*

$$R_3 = \inf_{n\geq 2}\left\{\frac{(1-\delta)}{n(n-\delta)}\frac{\Theta_d(n,\alpha,\beta,\gamma)}{[(1-\alpha)+|\gamma|(1-\beta)]}\Upsilon^l_m(n)\right\}^{\frac{1}{n-1}}.$$

These results are sharp for the extremal function f given by (8).

Proof. For the case 1, notice that for given $f \in \mathcal{T}_\xi$ and f is starlike of order δ, we have

$$\Big|\frac{(z-\xi)f'(z)}{f(z)} - 1\Big| < 1-\delta. \tag{10}$$

For the left hand side of (10) we obtain

$$\Big|\frac{(z-\xi)f'(z)}{f(z)} - 1\Big| \leq \frac{\sum_{n=2}^{\infty}(n-1)a_n|z-\xi|^{n-1}}{1-\sum_{n=2}^{\infty}a_n|z-\xi|^{n-1}}.$$

The last expression is less than $1-\delta$ if

$$\sum_{n=2}^{\infty}\frac{n-\delta}{1-\delta}a_n|z-\xi|^{n-1} < 1.$$

Using the fact, that $f \in \mathcal{TS}_\xi(\alpha,\beta,\gamma)$ if and only if

$$\sum_{n=2}^{\infty}\frac{\Theta_d(n,\alpha,\beta,\gamma)}{(1-\alpha)+|\gamma|(1-\beta)}\Upsilon^l_m(n)a_n < 1.$$

We can say (10) is true if

$$\frac{n-\delta}{1-\delta}|z-\xi|^{n-1} < \frac{\Theta_d(n,\alpha,\beta,\gamma)}{(1-\alpha)+|\gamma|(1-\beta)}\Upsilon^l_m(n).$$

Or equivalently,

$$R_3^{n-1} < \frac{(1-\delta)\Theta_d(n,\alpha,\beta,\gamma)}{(n-\delta)[(1-\alpha)+|\gamma|(1-\beta)]}\Upsilon^l_m(n)$$

which yields the starlikeness of the family.

Notice that we can prove case 2, on lines similar the proof of case 1, it is sufficient to use the fact that f is convex if and only if $(z-\xi)f'$ is starlike.

4. Modified Hadamard products

For functions of the form

$$f_j(z) = (z-\xi) - \sum_{n=2}^{\infty}a_{n,j}(z-\xi)^n, \qquad j=1,2$$

we define the modified Hadamard product as

$$(f_1 * f_2)(z) = (z-\xi) - \sum_{n=2}^{\infty}a_{n,1}a_{n,2}(z-\xi)^n.$$

Theorem 4.1
If $f_j \in \mathcal{TS}_\xi(\alpha,\beta,\gamma)$, $j=1,2$, *then* $(f_1 * f_2)(z) \in \mathcal{TS}_\xi(\alpha,\beta,\gamma)$, *where*

$$\xi = \frac{(2-\beta)\Theta_d(2,\alpha,\beta,\gamma)\Upsilon_m^l(2) - 2(1-\beta)[(1-\alpha)+|\gamma|(1-\beta)]}{(2-\beta)\Theta_d(2,\alpha,\beta,\gamma)\Upsilon_m^l(2) - (1-\beta)[(1-\alpha)+|\gamma|(1-\beta)]},$$

with $\Upsilon_m^l(2)$ *be defined as in* (7).

Proof. Since $f_j \in \mathcal{TS}_\xi(\alpha,\beta,\gamma)$, $j=1,2$, we have

$$\sum_{n=2}^{\infty} \Theta_d(n,\alpha,\beta,\gamma)\Upsilon_m^l(n) a_{n,j} \le (1-\alpha)+|\gamma|(1-\beta), \qquad j=1,2.$$

The Cauchy-Schwartz inequality leads to

$$\sum_{n=2}^{\infty} \frac{\Theta_d(n,\alpha,\beta,\gamma)\Upsilon_m^l(n)}{(1-\alpha)+|\gamma|(1-\beta)} \sqrt{a_{n,1}a_{n,2}} \le 1. \tag{11}$$

Note that we need to find the largest ρ such that

$$\sum_{n=2}^{\infty} \frac{\Theta_d(n,\alpha,\rho,\gamma)\Upsilon_m^l(n)}{(1-\alpha)+|\gamma|(1-\rho)} a_{n,1}a_{n,2} \le 1. \tag{12}$$

Therefore, in view of (11) and (12), whenever

$$\frac{n-\xi}{1-\xi}\sqrt{a_{n,1}a_{n,2}} \le \frac{n-\beta}{1-\beta}, \qquad n \ge 2$$

holds, then (12) is satisfied. We have, from (11),

$$\sqrt{a_{n,1}a_{n,2}} \le \frac{(1-\alpha)+|\gamma|(1-\beta)}{\Theta_d(n,\alpha,\beta,\gamma)\Upsilon_m^l(n)}, \qquad n \ge 2. \tag{13}$$

Thus, if

$$\Big(\frac{n-\xi}{1-\xi}\Big)\Big[\frac{(1-\alpha)+|\gamma|(1-\beta)}{\Theta_d(n,\alpha,\beta,\gamma)\Upsilon_m^l(n)}\Big] \le \frac{n-\beta}{1-\beta}, \qquad n \ge 2,$$

or, if

$$\xi = \frac{(n-\beta)\Theta_d(n,\alpha,\beta,\gamma)\Upsilon_m^l(n) - n(1-\beta)[(1-\alpha)+|\gamma|(1-\beta)]}{(n-\beta)\Theta_d(n,\alpha,\beta,\gamma)\Upsilon_m^l(n) - (1-\beta)[(1-\alpha)+|\gamma|(1-\beta)]}, \qquad n \ge 2,$$

then (11) is satisfied. Note that the right hand side of the above expression is an increasing function on n. Hence, setting $n=2$ in the above inequality gives the required result. Finally, by taking the function

$$f(z) = (z-\xi) - \frac{(1-\alpha)+|\gamma|(1-\beta)}{(2-\beta)[\Theta_d(2,\alpha,\beta,\gamma)]\Upsilon_m^l(n)}(z-\xi)^2,$$

we see that the result is sharp.

5. Integral means

In order to find the integral means inequality and to verify the Silverman Conjuncture [20] for $f \in \mathcal{TS}_\xi(\alpha,\beta,\gamma)$ we need the following subordination result due to Littlewood [12].

Lemma 5.1 ([12])
If the functions f and g are analytic in $\mathbb{U}$ with $g \prec f$, then

$$\int_0^{2\pi} |g(re^{i\theta})|^\eta d\theta \le \int_0^{2\pi} |f(re^{i\theta})|^\eta \, d\theta, \qquad \eta > 0,\ z = re^{i\theta} \text{ and } 0 < r < 1.$$

Applying Theorem 2.1 with extremal function given by (8) and Lemma 5.1, we prove the following theorem.

Theorem 5.2
Let $\eta > 0$. If $f \in \mathcal{TS}_\xi(\alpha,\beta,\gamma)$ and $\{\Phi(\alpha,\beta,\gamma,n)\}_{n=2}^\infty$ is non-decreasing sequence, then for $(z-\xi) = re^{i\theta}$ and $0 < r+d < 1$ we have

$$\int_0^{2\pi} |f(re^{i\theta})|^\eta \, d\theta \le \int_0^{2\pi} |f_2(re^{i\theta})|^\eta \, d\theta,$$

where

$$f_2(z) = (z-\xi) - \frac{(1-\alpha)+|\gamma|(1-\beta)}{\Theta_d(2,\alpha,\beta,\gamma)\Upsilon_m^l(2)}(z-\xi)^2.$$

Proof. Let $f(z)$ of the form (2) and

$$f_2(z) = (z-\xi) - \frac{(1-\alpha)+|\gamma|(1-\beta)}{\Theta_d(2,\alpha,\beta,\gamma)\Upsilon_m^l(2)}(z-\xi)^2,$$

then we must show that

$$\int_0^{2\pi} \left|1 - \sum_{n=2}^\infty a_n (z-\xi)^{n-1}\right|^\eta d\theta \le \int_0^{2\pi} \left|1 - \frac{(1-\alpha)+|\gamma|(1-\beta)}{\Theta_d(2,\alpha,\beta,\gamma)\Upsilon_m^l(2)}(z-\xi)\right|^\eta d\theta.$$

By Lemma 5.1, it suffices to show that

$$1 - \sum_{n=2}^\infty a_n (z-\xi)^{n-1} \prec 1 - \frac{(1-\alpha)+|\gamma|(1-\beta)}{\Theta_d(2,\alpha,\beta,\gamma)\Upsilon_m^l(2)}(z-\xi).$$

Setting

$$1 - \sum_{n=2}^\infty a_n (z-\xi)^{n-1} = 1 - \frac{(1-\alpha)+|\gamma|(1-\beta)}{\Theta_d(2,\alpha,\beta,\gamma)\Upsilon_m^l(2)} w(z). \tag{14}$$

From (14) and (6) we obtain

$$\begin{aligned}|w(z)| &= \left|\sum_{n=2}^{\infty}\frac{\Theta_d(n,\alpha,\beta,\gamma)\Upsilon_m^l(n)}{(1-\alpha)+|\gamma|(1-\beta)}a_n(z-\xi)^{n-1}\right|\\ &\le |z-\xi|\sum_{n=2}^{\infty}\frac{\Theta_d(n,\alpha,\beta,\gamma)\Upsilon_m^l(n)}{(1-\alpha)+|\gamma|(1-\beta)}a_n\\ &\le |z-\xi|\\ &< 1.\end{aligned}$$

This completes the proof of the Theorem 5.2.

6. Inclusion relations involving $N_\delta(e)$

In this section following [14, 17], we define the n, δ neighborhood of function $f \in \mathcal{T}_\xi$ and discuss the inclusion relations involving $N_\delta(e)$.

$$N_\delta(f) = \left\{g \in \mathcal{T}_\xi:\ g(z) = (z-\xi) - \sum_{n=2}^{\infty} b_n(z-\xi)^n \ \text{ and } \ \sum_{n=2}^{\infty} n|a_n - b_n| \le \delta\right\}.$$

In particular, for the identity function $e(z) = z$ we have

$$N_\delta(e) = \left\{g \in \mathcal{T}_\xi:\ g(z) = (z-\xi) - \sum_{n=2}^{\infty} b_n z^n \ \text{ and } \ \sum_{n=2}^{\infty} n|b_n| \le \delta\right\}.$$

Theorem 6.1
Let

$$\delta = \frac{2[(1-\alpha)+|\gamma|(1-\beta)]}{\Theta_d(2,\alpha,\beta,\gamma)\Upsilon_m^l(2)},$$

where $-1 \le \alpha < 1$, $\beta \ge 0$ *and* $\gamma \in \mathbb{C}\setminus\{0\}$. *Then* $\mathcal{TS}_\xi(\alpha,\beta,\gamma) \subset N_\delta(e)$.

Proof. For $f \in \mathcal{TS}_\xi(\alpha,\beta,\gamma)$ Theorem 2.1 yields

$$\Theta_d(2,\alpha,\beta,\gamma)\Upsilon_m^l(2)\sum_{n=2}^{\infty} a_n \le (1-\alpha)+|\gamma|(1-\beta)$$

so that

$$\sum_{n=2}^{\infty} a_n \le \frac{(1-\alpha)+|\gamma|(1-\beta)}{[\Theta_d(2,\alpha,\beta,\gamma)\Upsilon_m^l(2)}. \tag{15}$$

On the other hand, from (6) and (15) we have

$$
\begin{aligned}
&(1-\beta)(r+d)\Upsilon_m^l(2)\sum_{n=2}^{\infty} na_n \\
&\quad\le (1-\alpha)+|\gamma|(1-\beta)+[(\alpha-\beta)-|\gamma|(1-\beta)](r+d)\Upsilon_m^l(2)\sum_{n=2}^{\infty} a_n \\
&\quad\le (1-\alpha)+|\gamma|(1-\beta)+[(\alpha-\beta)-|\gamma|(1-\beta)](r+d)\Upsilon_m^l(2) \\
&\qquad\times \frac{(1-\alpha)+|\gamma|(1-\beta)}{[(2-\alpha+\beta)+|\gamma|(1-\beta)](r+d)\Upsilon_m^l(2)} \\
&\quad\le \frac{[(1-\alpha)+|\gamma|(1-\beta)]2(1-\beta)}{(2-\alpha+\beta)+|\gamma|(1-\beta)}.
\end{aligned}
$$

Hence

$$\sum_{n=2}^{\infty} na_n \le \frac{2[(1-\alpha)+|\gamma|(1-\beta)]}{[(2-\alpha+\beta)+|\gamma|(1-\beta)](r+d)\Upsilon_m^l(2)}$$

and

$$\sum_{n=2}^{\infty} na_n \le \frac{2[(1-\alpha)+|\gamma|(1-\beta)]}{\Theta_d(2,\alpha,\beta,\gamma)\Upsilon_m^l(2)} = \delta. \tag{16}$$

Now we determine the neighborhood for each of the function class $\mathcal{TS}_\xi(\alpha,\beta,\gamma)$ which we define as follows:

A function $f \in \mathcal{T}_\xi$ is said to be in the class $\mathcal{TS}_\xi(\alpha,\beta,\gamma,\eta)$ if there exists a function $g \in \mathcal{TS}_\xi(\alpha,\beta,\gamma)$ such that

$$\left|\frac{f(z)}{g(z)} - 1\right| < 1-\eta, \qquad (z-\xi)\in\mathbb{U},\ 0\le\eta<1. \tag{17}$$

Theorem 6.2
If $g \in \mathcal{TS}_\xi(\alpha,\beta,\gamma)$ and

$$\eta = 1 - \frac{\delta\Theta_d(2,\alpha,\beta,\gamma)\Upsilon_m^l(2)}{\Theta_d(2,\alpha,\beta,\gamma)\Upsilon_m^l(2) - 2[(1-\alpha)+|\gamma|(1-\beta)]}. \tag{18}$$

Then $N_\delta(g) \subset \mathcal{TS}_\xi(\alpha,\beta,\gamma,\eta)$.

Proof. Suppose that $f \in N_\delta(g)$, then we find from (16) that

$$\sum_{n=2}^{\infty} n|a_n - b_n| \le \delta,$$

which implies the coefficient inequality

$$\sum_{n=2}^{\infty} |a_n - b_n| \le \frac{\delta}{2}.$$

Next, since $g \in \mathcal{TS}_\xi(\alpha,\beta,\gamma)$, we have

$$\sum_{n=2}^{\infty} b_n \le \frac{2[(1-\alpha)+|\gamma|(1-\beta)]}{\Theta_d(2,\alpha,\beta,\gamma)\Upsilon_m^l(2)}.$$

So that

$$\begin{aligned}\Big|\frac{f(z)}{g(z)}-1\Big| &< \frac{\sum_{n=2}^{\infty}|a_n-b_n|}{1-\sum_{n=2}^{\infty}b_n}\\ &\le \frac{\delta}{2}\times\frac{\Theta_d(2,\alpha,\beta,\gamma)\Upsilon_m^l(2)}{\Theta_d(2,\alpha,\beta,\gamma)\Upsilon_m^l(2)-2[(1-\alpha)+|\gamma|(1-\beta)]}\\ &\le 1-\eta,\end{aligned}$$

provided that η is given precisely by (18). Thus by definition, $f\in\mathcal{TS}_\xi(\alpha,\beta,\gamma,\eta)$ for η given by (18), which completes the proof.

Concluding Remarks: By suitably specializing the various parameters involved in Theorem 6 to Theorem 6.2 we can state the corresponding results for the new subclasses defined in Example 1 and also for many relatively more familiar function classes.

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