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On generalized M-projectively recurrent manifolds

Abstract. The purpose of the present paper is to study generalized M-projectively recurrent manifolds. Some geometric properties of generalized M-projectively recurrent manifolds have been studied under certain curvature conditions. An application of such a manifold in the theory of relativity has also been shown. Finally, we give an example of a generalized M-projectively recurrent manifold.

1. Introduction

As is well known, symmetric spaces play an important role in differential geometry. The study of Riemannian symmetric spaces was initiated in the late twenties by Cartan [4], who, in particular, obtained a classification of those spaces. Let (M^n, g), $n = \dim M$, be a Riemannian manifold, i.e., a manifold M with the Riemannian metric g and let ∇ be the Levi-Civita connection on (M^n, g). A Riemannian manifold is called locally symmetric [4] if $\nabla R = 0$, where R is the Riemannian curvature tensor of (M^n, g). This condition of local symmetry is equivalent to the fact that at every point $P \in M$, the local geodesic symmetry $F(P)$ is an isometry [22]. The class of Riemannian locally symmetric manifolds is a very natural generalization of the class of manifolds of constant curvature. During the last six decades the notion of locally symmetric manifolds have been weakened by many authors in several ways and to various extent such as: conformally symmetric manifolds by Chaki and Gupta [6], recurrent manifolds introduced by Walker [36], conformally recurrent manifolds by Adati and Miyazawa [1], conformally symmetric Ricci-recurrent spaces by Roter [29], pseudo symmetric manifolds introduced by Chaki [7] etc.

The notion of recurrent manifolds have been generalized by various authors to: Ricci-recurrent manifolds by Patterson [26], 2-recurrent manifolds by Lichnerowicz [18], projective 2-recurrent manifolds by D. Ghosh [15] and others.

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A tensor field T of type $(0,q)$ is said to be recurrent [29] if the relation

$$(\nabla_X T)(Y_1,Y_2,\dots,Y_q)T(Z_1,Z_2,\dots,Z_q) \\ -T(Y_1,Y_2,\dots,Y_q)(\nabla_X T)(Z_1,Z_2,\dots,Z_q)=0$$

holds on (M^n,g). From the definition it follows that if at a point $x\in M$, $T(x)\neq 0$, then in some neighbourhood of x, there exists a unique 1-form A satisfying

$$(\nabla_X T)(Y_1,Y_2,\dots,Y_q)=A(X)T(Y_1,Y_2,\dots,Y_q).$$

In 1952, Patterson [26] introduced Ricci-recurrent manifolds. According to Patterson, a manifold (M^n,g) of dimension n, is called Ricci-recurrent if

$$(\nabla_X S)(Y,Z)=A(X)S(Y,Z)$$

for some 1-form A. He denoted such a manifold by R_n. Ricci-recurrent manifolds have been studied by several authors [5, 28, 29, 37] and many others. In a recent paper De, Guha and Kamilya [12] introduced the notion of generalized Ricci recurrent manifolds which is defined as follows:
a non-flat Riemannian manifold (M^n,g), $n>2$ is called generalized Ricci recurrent if the Ricci tensor S is non-zero and satisfies the condition

$$(\nabla_X S)(Y,Z)=A(X)S(Y,Z)+B(X)g(Y,Z),$$

where A and B are non-zero 1-forms. Such a manifold shall be denoted by GR_n. If the associated 1-form B becomes zero, then the manifold GR_n reduces to a Ricci-recurrent manifold R_n. This justifies the name generalized Ricci-recurrent manifold and the symbol GR_n for it. Also De and Guha in [11] introduced a non-flat Riemannian manifold (M^n,g), $n>2$ called a generalized recurrent manifold if its curvature tensor $\mathcal{R}$ of type $(1,3)$ satisfies the condition

$$(\nabla_X\mathcal{R})(Y,Z)U=A(X)\mathcal{R}(Y,Z)U+B(X)[g(Z,U)Y-g(Y,U)Z],$$

where A and B are non-zero 1-forms, and ∇_x has the meaning already mentioned. Such a manifold has been denoted by GK_n. If the associated 1-form B becomes zero, then the manifold GK_n reduces to a recurrent manifold introduced by Ruse [30] and Walker [36] which is denoted by K_n.

Generalized recurrent and generalized Ricci recurrent manifolds have been studied by several authors such as Özgür [23, 24, 25], Mallick, De and De [19], Arslan et al [2] and many others.

On the other hand, quasi Einstein manifolds arose during the study of exact solutions of the Einstein field equations as well as during considerations of quasi-umbilical hypersurfaces of semi-Euclidean spaces. A non-flat Riemannian manifold (M^n,g), $n>2$ is defined to be a quasi Einstein manifold if its Ricci tensor S of type $(0,2)$ is not identically zero and satisfies the following condition

$$S(X,Y)=ag(X,Y)+b\eta(X)\eta(Y),$$

where $a,b\in\mathbb{R}$ and η is a non-zero 1-form such that

$$g(X,\xi)=\eta(X)$$

for all vector fields X, η is the 1-form metrically equivalent to the vector field ξ.

Also Mantica and Suh [20] studied quasi-conformally recurrent Riemannian manifolds. In [10] De and Gazi proved that a generalized concircularly recurrent manifold with constant scalar curvature is a GR_n. Motivated by the above studies in the present paper we have studied a type of non-flat connected semi-Riemannian manifold which is called generalized M-projectively recurrent manifolds.

In 1971, Pokhariyal and Mishra [27] introduced a new curvature tensor of type $(1,3)$ in an n-dimensional Riemannian manifold (M^n, g), $n > 2$ denoted by $\mathcal{M}$ and defined by

$$\begin{aligned}\mathcal{M}(Y,Z)U = \mathcal{R}(Y,Z)U - \frac{1}{2(n-1)}[&S(Z,U)Y \\ &- S(Y,U)Z + g(Z,U)LY - g(Y,U)LZ],\end{aligned} \tag{1.1}$$

where $\mathcal{R}$ and L denote the Riemannian curvature tensor of type $(1,3)$ and the Ricci operator defined by $g(LX,Y) = S(X,Y)$, respectively. Such a tensor $\mathcal{M}$ is known as an M-projective curvature tensor. In the same paper the authors studied relativistic significance of such a tensor $\mathcal{M}$. The M-projective curvature tensor have been studied by J.P. Singh [33], S.K. Chaubey and R.H. Ojha [8], S.K. Chaubey [9], R.N. Singh and S.K. Pandey [34] and many others. Recently De and Mallick [13] studied M-projectively flat spacetime and also the divergence of the M-projective curvature tensor in a perfect fluid spacetime with the energy momentum tensor of Codazzi type.

From (1.1) we can define a $(0,4)$ type M-projective curvature tensor M as follows

$$\begin{aligned}M(Y,Z,U,V) = R(Y,Z,U,V) - \frac{1}{2(n-1)}[&S(Z,U)g(Y,V) \\ &- S(Y,U)g(Z,V) + S(Y,V)g(Z,U) - S(Z,V)g(Y,U)],\end{aligned} \tag{1.2}$$

where R denotes the Riemannian curvature tensor of type $(0,4)$ defined by

$$R(Y,Z,U,V) = g(\mathcal{R}(Y,Z)U,V),$$

and

$$M(Y,Z,U,V) = g(\mathcal{M}(Y,Z)U,V).$$

The M-projective curvature tensor satisfies the properties of the Riemannian curvature tensor. In this paper we consider a non-flat n-dimensional connected semi-Riemannian manifold (M^n, g), $n \geq 3$ in which the M-projective curvature tensor of type $(0,4)$ satisfies the condition

$$\begin{aligned}(\nabla_X M)(Y,Z,U,V) = {}& A(X)M(Y,Z,U,V) \\ &+ B(X)[g(Z,U)g(Y,V) - g(Y,U)g(Z,V)],\end{aligned} \tag{1.3}$$

where A and B are 1-forms. Such an n-dimensional connected semi-Riemannian manifold will be called a generalized M-projectively recurrent manifold and it is denoted by $G\{MP(K_n)\}$. If the 1-form B is zero, then the manifold reduces to an M-projectively recurrent manifold.

The paper is organized as follows: After preliminaries in Section 2, we obtain a necessary and sufficient condition for constant scalar curvature of a $G\{MP(K_n)\}$.

We study a Ricci-symmetric $G\{MP(K_n)\}$ and an Einstein $G\{MP(K_n)\}$ in Section 4 and 5, respectively. In Section 6, we study conformally flat $G\{MP(K_n)\}$, $n > 3$. Next we obtain a sufficient condition for a $G\{MP(K_n)\}$ to be a quasi Einstein manifold. Section 8 deals with decomposable $G\{MP(K_n)\}$. Section 9 is devoted to study $G\{MP(K_n)\}$ warped product manifolds. Also a relativistic application is shown in Section 10. Finally, we give an example of a $G\{MP(K_n)\}$.

2. Preliminaries

Let S and r denote the Ricci tensor of type $(0,2)$ and the scalar curvature, respectively. L denotes the symmetric tensor of type $(1,1)$ corresponding to the Ricci tensor S, that is,

$$g(LX,Y) = S(X,Y). \tag{2.1}$$

In this section, some formulas useful while studying $G\{MP(K_n)\}$ are derived. Let $\{e_i\}$ be an orthonormal basis of the tangent space at each point of the semi-Riemannian manifold, where $1 \leq i \leq n$ such that $g(e_i,e_j) = 0$ for $i \neq j$ and $g(e_i,e_i) = \epsilon_i$, $\epsilon_i = \pm 1$.

From (1.1) we can easily verify that the tensor M satisfies the following properties:

$$\begin{aligned} &\text{i)}\ \ \mathcal{M}(Y,Z)U = -\mathcal{M}(Z,Y)U, \\ &\text{ii)}\ \ \mathcal{M}(Y,Z)U + \mathcal{M}(Z,U)Y + \mathcal{M}(U,Y)Z = 0. \end{aligned} \tag{2.2}$$

From (1.2) and (2.2) it follows that

$$\begin{aligned} &\text{(i)}\ \ M(Y,Z,U,V) = -M(Z,Y,U,V), \\ &\text{(ii)}\ \ M(Y,Z,U,V) = -M(Y,Z,V,U), \\ &\text{(iii)}\ \ M(Y,Z,U,V) = M(U,V,Y,Z), \\ &\text{(iv)}\ \ M(Y,Z,U,V) + M(Z,U,Y,V) + M(U,Y,Z,V) = 0. \end{aligned}$$

Also from (1.2) we have

$$\sum_{i=1}^{n} M(Y,Z,e_i,e_i) = 0 = \sum_{i=1}^{n} M(e_i,e_i,U,V)$$

and

$$\begin{aligned} \sum_{i=1}^{n} \epsilon_i M(e_i,Z,U,e_i) &= \sum_{i=1}^{n} \epsilon_i M(Z,e_i,e_i,U) \\ &= \frac{n}{2(n-1)}\Big[S(Z,U) - \frac{r}{n}g(Z,U)\Big], \end{aligned} \tag{2.3}$$

where $r = \sum_{i=1}^{n} \epsilon_i S(e_i,e_i)$ is the scalar curvature.

Let

$$\tilde{M}(Z,U) = S(Z,U) - \frac{r}{n}g(Z,U).$$

Therefore,

$$\sum_{i=1}^{n} \epsilon_i \tilde{M}(e_i,e_i) = 0. \tag{2.4}$$

3. Necessary and sufficient condition for constant scalar curvature of a generalized M-projectively recurrent manifold

In this section we would like to obtain a necessary and sufficient condition for constant scalar curvature of a generalized M-projectively recurrent manifold. From (1.2) and (1.3) we get

$$\begin{aligned}(\nabla_X R)&(Y,Z,U,V)\\ &= A(X)M(Y,Z,U,V) + B(X)[g(Z,U)g(Y,V) - g(Y,U)g(Z,V)]\\ &\quad + \frac{1}{2(n-1)}[(\nabla_X S)(Z,U)g(Y,V) - (\nabla_X S)(Y,U)g(Z,V)\\ &\quad + (\nabla_X S)(Y,V)g(Z,U) - (\nabla_X S)(Z,V)g(Y,U)].\end{aligned} \tag{3.1}$$

Using (3.1) and Bianchi's 2nd identity we get

$$\begin{aligned}&[A(X)M(Y,Z,U,V) + A(Y)M(Z,X,U,V) + A(Z)M(X,Y,U,V)]\\ &\quad + [B(X)\{g(Z,U)g(Y,V) - g(Y,U)g(Z,V)\}\\ &\quad + B(Y)\{g(X,U)g(Z,V) - g(Z,U)g(X,V)\}\\ &\quad + B(Z)\{g(Y,U)g(X,V) - g(X,U)g(Y,V)\}]\\ &\quad + \frac{1}{2(n-1)}[\{(\nabla_X S)(Z,U)g(Y,V) - (\nabla_X S)(Y,U)g(Z,V)\\ &\quad + (\nabla_X S)(Y,V)g(Z,U) - (\nabla_X S)(Z,V)g(Y,U)\}\\ &\quad + \{(\nabla_Y S)(X,U)g(Z,V) - (\nabla_Y S)(Z,U)g(X,V)\\ &\quad + (\nabla_Y S)(Z,V)g(X,U) - (\nabla_Y S)(X,V)g(Z,U)\}\\ &\quad + \{(\nabla_Z S)(Y,U)g(X,V) - (\nabla_Z S)(X,U)g(Y,V)\\ &\quad + (\nabla_Z S)(X,V)g(Y,U) - (\nabla_Z S)(Y,V)g(X,U)\}] = 0.\end{aligned} \tag{3.2}$$

Contracting (3.2) over Y and V, we get

$$\begin{aligned}A(X)&\Big[\frac{n}{2(n-1)}\{S(Z,U) - \frac{r}{n}g(Z,U)\}\Big] + A(\mathcal{M}(Z,X)U)\\ &- A(Z)\Big[\frac{n}{2(n-1)}\{S(X,U) - \frac{r}{n}g(X,U)\}\Big] + (n-1)B(X)g(Z,U)\\ &+ B(Z)g(X,U) - B(X)g(Z,U) + B(Z)g(X,U) - nB(Z)g(X,U)\\ &+ \frac{1}{2(n-1)}[\{n(\nabla_X S)(Z,U) - (\nabla_X S)(Z,U)\\ &+ dr(X)g(Z,U) - (\nabla_X S)(Z,U)\}\\ &+ \Big\{(\nabla_Z S)(X,U) - (\nabla_X S)(Z,U) + \frac{1}{2}dr(Z)g(X,U) - \frac{1}{2}dr(X)g(Z,U)\Big\}\\ &+ \{(\nabla_Z S)(X,U) - n(\nabla_Z S)(X,U) + (\nabla_Z S)(X,U) - dr(Z)g(X,U)\}] = 0.\end{aligned} \tag{3.3}$$

Again contracting (3.3) over Z and U, we get

$$\begin{aligned}-\frac{n}{n-1}&\Big\{A(LX) - \frac{r}{n}A(X)\Big\} + (n^2 - 3n + 2)B(X)\\ &+ \frac{1}{2(n-1)}\Big[(2n-2)dr(X) - \frac{n}{2}dr(X) - \frac{n}{2}dr(X)\Big] = 0,\end{aligned}$$

which implies that

$$rA(X) = nA(LX) - (n-1)(n^2 - 3n + 2)B(X) - \frac{(n-2)}{2} dr(X). \tag{3.4}$$

Thus we can state the following:

Theorem 3.1
The scalar curvature r of a generalized M-projectively recurrent manifold is constant if and only if $rA(X) = nA(LX) - (n-1)(n^2 - 3n + 2)B(X)$ holds for all vector fields X.

Now we suppose that the scalar curvature r is constant in a $G\{MP(K_n)\}$, that is, $dr = 0$. Then from (3.4) we get

$$rA(X) = nA(LX) - (n-1)^2(n-2)B(X). \tag{3.5}$$

Contracting (3.1) over Y and V we get

$$\begin{aligned}
&(\nabla_X S)(Z,U) \\
&\quad = \frac{n}{2(n-1)} A(X)\Big[S(Z,U) - \frac{r}{n} g(Z,U)\Big] + (n-1)B(X)g(Z,U) \\
&\qquad + \frac{1}{2(n-1)}[(n-2)(\nabla_X S)(Z,U) + dr(X)g(Z,U)].
\end{aligned} \tag{3.6}$$

Using (3.5) and $dr = 0$ in (3.6) we get

$$(\nabla_X S)(Z,U) = A(X)S(Z,U) + [-A(LX) + (n-1)^2 B(X)]g(Z,U).$$

This can be written as

$$(\nabla_X S)(Z,U) = A(X)S(Z,U) + D(X)g(Z,U),$$

where $D(X) = [-A(LX) + (n-1)^2 B(X)]$. Hence the manifold is a generalized Ricci-recurrent manifold or Ricci-recurrent manifold. Thus we have the following theorem:

Theorem 3.2
A generalized M-projectively recurrent manifold with constant scalar curvature is a generalized Ricci-recurrent manifold or Ricci-recurrent manifold.

4. Ricci-symmetric generalized M-projectively recurrent manifold

In this section we assume that $G\{MP(K_n)\}$ is Ricci-symmetric, that is, $\nabla S = 0$, that is, $\nabla L = 0$. Then the scalar curvature r is constant and $dr = 0$. So we have from (3.6)

$$\frac{n}{2(n-1)} A(X)\Big[S(Z,U) - \frac{r}{n} g(Z,U)\Big] + (n-1)B(X)g(Z,U) = 0. \tag{4.1}$$

Again, since r is constant we can use (3.5). Putting the value of $B(X)$ from (3.5) in (4.1) we get

$$\frac{n}{2(n-1)}A(X)S(Z,U) = \frac{r}{2(n-1)}A(X)g(Z,U) + (n-1)\Big[\frac{rA(X)-nA(LX)}{(n-1)^2(n-2)}\Big]g(Z,U),$$

which implies

$$S(Z,U) = \Big\{\frac{r}{n-2} - \frac{2}{(n-2)}\frac{A(LX)}{A(X)}\Big\}g(Z,U),$$

where we take X so that (at least locally) $A(X) \neq 0$. In order to guarantee that $A \neq 0$ we have to assume that M is not locally symmetric. This can be written as

$$S(Z,U) = \lambda g(Z,U),$$

where $\lambda = \{\frac{r}{n-2} - \frac{2}{n-2}\frac{A(LX)}{A(X)}\}$ is a scalar.

Hence the manifold M is an Einstein manifold. This leads to the following theorem:

Theorem 4.1
A Ricci-symmetric generalized M-projectively recurrent manifold is an Einstein manifold, provided that it is not locally symmetric.

5. Einstein $G\{MP(K_n)\}$

This section deals with an Einstein $G\{MP(K_n)\}$. Then the Ricci tensor satisfies

$$S(Y,Z) = \frac{r}{n}g(Y,Z), \tag{5.1}$$

from which it follows that

$$dr(X) = 0$$

and

$$(\nabla_X S)(Y,Z) = 0 \qquad \text{for all } X, Y, Z. \tag{5.2}$$

Using (5.1) and (5.2) we get from (1.2)

$$(\nabla_X M)(Y,Z,U,V) = (\nabla_X R)(Y,Z,U,V). \tag{5.3}$$

Now using (5.3) in (1.3) we have

$$\begin{aligned}(\nabla_X R)&(Y,Z,U,V)\\ &= A(X)M(Y,Z,U,V) + B(X)[g(Z,U)g(Y,V) - g(Y,U)g(Z,V)].\end{aligned} \tag{5.4}$$

Again using (1.2) in (5.4) we get

$$\begin{aligned}(\nabla_X R)&(Y,Z,U,V)\\ &= A(X)[R(Y,Z,U,V) - \frac{1}{2(n-1)}\{S(Z,U)g(Y,V) - S(Y,U)g(Z,V)\\ &\quad + g(Z,U)S(Y,V) - g(Y,U)S(Z,V)]\\ &\quad + B(X)[g(Z,U)g(Y,V) - g(Y,U)g(Z,V)].\end{aligned} \tag{5.5}$$

Since the manifold is Einstein, so using (5.1) in (5.5) we obtain

$$(\nabla_X R)(Y,Z,U,V) = A(X)R(Y,Z,U,V) + \Big\{B(X) - \frac{r}{n(n-1)}A(X)\Big\} \\ \times [g(Z,U)g(Y,V) - g(Y,U)g(Z,V)],$$

which implies

$$(\nabla_X R)(Y,Z,U,V) = A(X)R(Y,Z,U,V) \\ + D(X)[g(Z,U)g(Y,V) - g(Y,U)g(Z,V)], \tag{5.6}$$

where $D(X) = B(X) - \frac{r}{n(n-1)}A(X)$. Let the 1-forms A and B be metrically equivalent to the vector fields P and Q, respectively.

From (5.6) we conclude that an Einstein $G\{MP(K_n)\}$ is a GK_n, provided $Q \neq \frac{r}{n(n-1)}P$.

Hence we have the following:

Theorem 5.1
An Einstein $G\{MP(K_n)\}$, $n > 2$ is a GK_n, provided $Q \neq \frac{r}{n(n-1)}P$.

6. Conformally flat $G\{MP(K_n)\}, n > 3$

Suppose (M^n, g) is a semi-Riemannian manifold of dimension n and X is any vector field on M. Then the divergence of the vector field X, denoted by $\operatorname{div} X$, is defined as $\operatorname{div} X = \sum_{i=1}^n \epsilon_i g(\nabla_{e_i} X, e_i)$, where $\{e_i\}$ is an orthonormal basis of the tangent space T_pM at any point $p \in M$. Again, if K is a tensor field of type $(1,3)$, then its divergence $\operatorname{div} K$ is a tensor field of type (0,3) defined as $(\operatorname{div} K)(X_1, \dots, X_3) = \sum_{i=1}^n \epsilon_i g((\nabla_{e_i} K)(X_1, \dots, X_3), e_i)$.

In this section we assume that the manifold $G\{MP(K_n)\}$, $n > 3$ is conformally flat. Then $\operatorname{div} C = 0$, where C denotes the Weyl's conformal curvature tensor and 'div' denotes divergence. Hence we have (see [14]),

$$(\nabla_X S)(Y,Z) - (\nabla_Z S)(X,Y) = \frac{1}{2(n-1)}[g(Y,Z)dr(X) - g(X,Y)dr(Z)]. \tag{6.1}$$

Now using (1.2) in (1.3) and then contracting over Y and V we get

$$\frac{n}{2(n-1)}(\nabla_X S)(Z,U) - \frac{1}{2(n-1)}dr(X)g(Z,U) \\ = \frac{n}{2(n-1)}A(X)\Big[S(Z,U) - \frac{r}{n}g(Z,U)\Big] + (n-1)B(X)g(Z,U),$$

from which it follows that

$$(\nabla_X S)(Y,Z) = A(X)\Big\{S(Y,Z) - \frac{r}{n}g(Y,Z)\Big\} \\ + \frac{2(n-1)^2}{n}B(X)g(Y,Z) + \frac{1}{n}dr(X)g(Y,Z). \tag{6.2}$$

Using (6.2) in (6.1) we obtain

$$\begin{aligned} A(X)\Big\{S(Y,Z) - \frac{r}{n}g(Y,Z)\Big\} + \frac{2(n-1)^2}{n}B(X)g(Y,Z) \\ + \frac{1}{n}dr(X)g(Y,Z) - A(Z)\Big\{S(X,Y) - \frac{r}{n}g(X,Y)\Big\} \\ - \frac{2(n-1)^2}{n}B(Z)g(X,Y) - \frac{1}{n}dr(Z)g(X,Y) \\ = \frac{1}{2(n-1)}[g(Y,Z)dr(X) - g(X,Y)dr(Z)]. \end{aligned} \tag{6.3}$$

Now contracting (6.2) over X and Z, we get

$$dr(Y) = \frac{2n}{n-2}\Big[A(LY) - \frac{r}{n}A(Y)\Big] + \frac{4(n-1)^2}{n-2}B(Y). \tag{6.4}$$

Replacing Y by X in (6.4) we obtain

$$dr(X) = \frac{2n}{n-2}\Big[A(LX) - \frac{r}{n}A(X)\Big] + \frac{4(n-1)^2}{n-2}B(X). \tag{6.5}$$

Contracting (6.3) over Y and Z, we get

$$dr(X) = \frac{2n}{n-2}\Big[A(LX) - \frac{r}{n}A(X)\Big] - \frac{4(n-1)^3}{n-2}B(X). \tag{6.6}$$

From (6.5) and (6.6) it follows that

$$B(X) = 0.$$

Then the $G\{MP(K_n)\}$, $n > 3$ is reduced to a $MP(K_n)$. Thus we have the following:

Theorem 6.1
A conformally flat $G\{MP(K_n)\}$, $n > 3$ is a $MP(K_n)$.

7. Sufficient condition for a generalized M-projectively recurrent manifold to be a quasi Einstein manifold

From (6.2) we have

$$\begin{aligned} (\nabla_X S)(Y,Z) = A(X)\Big\{S(Y,Z) - \frac{r}{n}g(Y,Z)\Big\} \\ + \frac{2(n-1)^2}{n}B(X)g(Y,Z) + \frac{1}{n}dr(X)g(Y,Z). \end{aligned} \tag{7.1}$$

A vector field P on a manifold with a linear connection ∇ is said to be concircular if

$$\nabla_X P = \alpha X + \omega(X)P \tag{7.2}$$

for every vector field X, where α is a scalar function and ω is a closed 1-form ([32], pages 322, 10 and the table on page 323). If the manifold is a semi-Riemannian manifold and a concircular field P satisfies additional assumption that $g(P, P) \equiv 1$, then $g(\nabla_X P, P) = 0$ and consequently

$$\omega(X) = -\alpha A(X), \tag{7.3}$$

where A defined by

$$A(X) = g(X, P) \tag{7.4}$$

is the 1-form associated with the vector field P.

Using (7.2) and (7.3) we get

$$g(\alpha X, Y) - g(\alpha A(X)P, Y) = g(\nabla_X P, Y),$$

which implies

$$\alpha[g(X, Y) - A(X)A(Y)] = g(Y, \nabla_X P). \tag{7.5}$$

Now, we have

$$(\nabla_X A)(Y) = X(A(Y)) - A(\nabla_X Y),$$

which implies

$$(\nabla_X A)(Y) = X(g(Y, P)) - g(\nabla_X Y, P).$$

Since $(\nabla_X g)(Y, P) = 0$, so, we have

$$(\nabla_X A)(Y) = g(Y, \nabla_X P). \tag{7.6}$$

Now, since P is a unit one, using (7.5) in (7.6) we get

$$(\nabla_X A)(Y) = \alpha[g(X, Y) - A(X)A(Y)]. \tag{7.7}$$

Using (7.3) in (7.7) we get

$$(\nabla_X A)(Y) = \alpha g(X, Y) + \omega(X)A(Y).$$

Let (M^n, g) be a $G\{MP(K_n)\}$ with corresponding 1-forms A and B in (1.3), the vector field P defined by $g(X, P) = A(X)$ for any vector field X is a concircular vector field [32] with a constant function α and $g(P, P) = 1$, and ω is a closed 1-form and the scalar curvature r of this manifold is constant. We assume that $G\{MP(K_n)\}$ admits the associated vector field P defined by (7.4), with a non-zero constant α. Now we can state and prove the following theorem:

Theorem 7.1
If in a $G\{MP(K_n)\}$ with constant scalar curvature the associated unit vector field P is a unit concircular vector field whose associated scalar is a non-zero constant, then the manifold reduces to a quasi Einstein manifold.

Proof. The Ricci identity for 1-form A reads

$$\nabla_X(\nabla_Y A) - \nabla_Y(\nabla_X A) - \nabla_{[X,Y]}A = \mathcal{R}(X, Y).A,$$

and

$$(\mathcal{R}(X,Y).A)(Z) = -A(\mathcal{R}(X,Y)Z).$$

Applying Ricci identity to (7.7) we obtain

$$A(\mathcal{R}(X,Y)Z) = \alpha^2[g(X,Z)A(Y) - g(Y,Z)A(X)].$$

Putting $Y = Z = e_i$ in (7.4) and taking the summation over i, $1 \leq i \leq n$, where $\{e_i\}$ is an orthonormal basis of the tangent space at each point of the manifold, we get

$$A(LX) = -(n-1)\alpha^2 A(X), \tag{7.8}$$

where L is the Ricci operator defined by

$$g(LX,Y) = S(X,Y),$$

which implies

$$S(X,P) = -(n-1)\alpha^2 A(X). \tag{7.9}$$

Now,

$$(\nabla_X S)(Y,P) = X(S(Y,P)) - S(\nabla_X Y,P) - S(Y,\nabla_X P). \tag{7.10}$$

Applying (7.9) in (7.10) we get

$$(\nabla_X S)(Y,P) = -(n-1)\alpha^2 X(A(Y)) + (n-1)\alpha^2 A(\nabla_X Y) - S(Y,\nabla_X P).$$

This can be written as

$$(\nabla_X S)(Y,P) = -(n-1)\alpha^2(\nabla_X A)(Y) - S(Y,\nabla_X P). \tag{7.11}$$

Using (7.7) in (7.11), we have

$$(\nabla_X S)(Y,P) = -(n-1)\alpha^3[g(X,Y) - A(X)A(Y)] - S(Y,\nabla_X P). \tag{7.12}$$

Using (7.3) in (7.2) we get

$$\nabla_X P = \alpha X - \alpha A(X)P.$$

This yields

$$\nabla_X P = \alpha(X - A(X)P).$$

Therefore,

$$S(Y,\nabla_X P) = S(Y,\alpha X) - S(Y,\alpha A(X)P).$$

Hence

$$S(Y,\nabla_X P) = \alpha[S(X,Y) - A(X)S(Y,P)]. \tag{7.13}$$

Applying (7.13) in (7.12), we obtain

$$\begin{aligned}(\nabla_X S)(Y,P) = &-(n-1)\alpha^3[g(X,Y) - A(X)A(Y)] \\ &- \alpha[S(X,Y) - A(X)S(Y,P)].\end{aligned} \tag{7.14}$$

Using (7.9) in (7.14), we get

$$(\nabla_X S)(Y,P) = -(n-1)\alpha^3 g(X,Y) - \alpha S(X,Y).$$

Putting $Z = P$ in (7.1) we have

$$\begin{aligned}(\nabla_X S)(Y,P) = A(X)\Big\{S(Y,P) - \frac{r}{n}g(Y,P)\Big\}\\ + \frac{2(n-1)^2}{n}B(X)g(Y,P) + \frac{1}{n}dr(X)g(Y,P).\end{aligned} \tag{7.15}$$

Now using (7.14) and (7.9) in (7.15) yields

$$\begin{aligned}&-(n-1)\alpha^3 g(X,Y) - \alpha S(X,Y)\\ &\quad = -(n-1)\alpha^2 A(X)A(Y) - \frac{r}{n}A(X)A(Y)\\ &\qquad + \frac{2(n-1)^2}{n}B(X)A(Y) + \frac{1}{n}dr(X)A(Y).\end{aligned} \tag{7.16}$$

Also we assume that the scalar curvature of the $G\{MP(K_n)\}$, is constant. Hence

$$dr = 0. \tag{7.17}$$

Now using (7.8) and (7.17) in (3.4) we get

$$B(X) = -\frac{1}{(n-1)^2(n-2)}[r + n(n-1)\alpha^2]A(X). \tag{7.18}$$

Using (7.17) and (7.18) in (7.16) we get

$$S(X,Y) = -(n-1)\alpha^2 g(X,Y) + \Big[\frac{\alpha n(n-1)}{n-2} + \frac{r}{\alpha(n-2)}\Big]A(X)A(Y). \tag{7.19}$$

Since α is a non-zero constant, (7.19) can be written as

$$S(X,Y) = pg(X,Y) + qA(X)A(Y),$$

where $p = -(n-1)\alpha^2$ and $q = [\frac{\alpha n(n-1)}{n-2} + \frac{r}{\alpha(n-2)}]$ are two non-zero constants as α is a non-zero constant. Hence the manifold is a quasi Einstein manifold. Thus the theorem is proved.

8. Decomposable $G\{MP(K_n)\}$

A semi-Riemannian manifold (M^n, g) is said to be decomposable or a product manifold ([32]) if it can be expressed as $M_1^p \times M_2^{n-p}$ for some p in the range $2 \leq p \leq (n-2)$, that is, in some coordinate neighbourhood of the semi-Riemannian manifold (M^n, g), the metric can be expressed as

$$ds^2 = g_{ij}dx^i dx^j = \bar{g}_{ab}dx^a dx^b + g^*_{\alpha\beta}dx^\alpha dx^\beta, \tag{8.1}$$

where $\bar{g}_{ab}$ are functions of $x^1, x^2, \dots, x^p$ denoted by $\bar{x}$ and $g^*_{\alpha\beta}$ are functions of $x^{p+1}, x^{p+2}, \dots, x^n$ denoted by x^*. Here $a, b, c, \dots$ run from 1 to p and $\alpha, \beta, \gamma, \dots$ run from $p+1$ to n.

The two parts of (8.1) are the metrics of M_1^p, $p \geq 2$ and M_2^{n-p}, $n-p \geq 2$ which are called the components of the decomposable manifold $M^n = M_1^p \times M_2^{n-p}$.

Let (M^n, g) be a semi-Riemannian decomposable manifold such that M_1^p, $p \geq 2$ and M_2^{n-p}, $n-p \geq 2$ are components of this manifold.

Throughout this section each object denoted by a 'bar' is assumed to come from M_1 and each object denoted by 'star' is assumed to come from M_2.

Let $\bar{X}, \bar{Y}, \bar{Z}, \bar{U}, \bar{V} \in \chi(M_1)$ and $X^*, Y^*, Z^*, U^*, V^* \in \chi(M_2)$. Then in a decomposable semi-Riemannian manifold $M^n = M_1^p \times M_2^{n-p}$, $2 \leq p \leq n-2$, the following relations hold [38]:

$$R(X^*, \bar{Y}, \bar{Z}, \bar{U}) = 0 = R(\bar{X}, Y^*, \bar{Z}, U^*) = R(\bar{X}, Y^*, Z^*, U^*),$$
$$(\nabla_{X^*} R)(\bar{Y}, \bar{Z}, \bar{U}, \bar{V}) = 0 = (\nabla_{\bar{X}} R)(\bar{Y}, Z^*, \bar{U}, V^*) = (\nabla_{X^*} R)(\bar{Y}, Z^*, \bar{U}, V^*),$$
$$R(\bar{X}, \bar{Y}, \bar{Z}, \bar{U}) = \bar{R}(\bar{X}, \bar{Y}, \bar{Z}, \bar{U})$$
$$R(X^*, Y^*, Z^*, U^*) = R^*(X^*, Y^*, Z^*, U^*),$$
$$S(\bar{X}, \bar{Y}) = \bar{S}(\bar{X}, \bar{Y}); \qquad S(X^*, Y^*) = S^*(X^*, Y^*),$$
$$(\nabla_{\bar{X}} S)(\bar{Y}, \bar{Z}) = (\bar{\nabla}_{\bar{X}} \bar{S})(\bar{Y}, \bar{Z}); \qquad (\nabla_{X^*} S)(Y^*, Z^*) = (\nabla^*_{X^*} S^*)(Y^*, Z^*),$$

where the meaning of $\bar{X}$, $\bar{Y}$ and $\bar{Z}$ is different on each side, that is, the left hand side of $S(\bar{X}, \bar{Y}) = \bar{S}(\bar{X}, \bar{Y})$ means the value of the Ricci tensor S on M for $\bar{X}, \bar{Y}, \bar{Z} \in \chi(M_1)$ and right hand side means the value of the Ricci tensor $\bar{S}$ on M_1 for $\bar{X}, \bar{Y}, \bar{Z} \in \chi(M_1)$. Similarly for X^*, Y^* and Z^*, and $r = \bar{r} + r^*$, where r, $\bar{r}$ and r^* are scalar curvatures of M, M_1 and M_2, respectively.

Let us consider a semi-Riemannian manifold (M^n, g), which is a decomposable $G\{MP(K_n)\}$. Then $M^n = M_1^p \times M_2^{n-p}$, $2 \leq p \leq n-2$.

Now from (1.2), we get

$$M(\bar{Y}, \bar{Z}, \bar{U}, \bar{V}) = \bar{M}(\bar{Y}, \bar{Z}, \bar{U}, \bar{V}), \tag{8.2}$$
$$M(Y^*, Z^*, U^*, V^*) = M^*(Y^*, Z^*, U^*, V^*),$$
$$\begin{aligned} M(Y^*, \bar{Z}, \bar{U}, \bar{V}) &= 0 = M(\bar{Y}, Z^*, U^*, V^*) = M(\bar{Y}, Z^*, \bar{U}, \bar{V}) \\ &= M(\bar{Y}, \bar{Z}, U^*, \bar{V}), \end{aligned}$$
$$M(\bar{Y}, Z^*, U^*, \bar{V}) = -\frac{1}{2(n-1)}[S(Z^*, U^*)g(\bar{Y}, \bar{V}) + S(\bar{Y}, \bar{V})g(Z^*, U^*)], \tag{8.3}$$
$$M(Y^*, \bar{Z}, \bar{U}, V^*) = -\frac{1}{2(n-1)}[S(\bar{Z}, \bar{U})g(Y^*, V^*) + S(Y^*, V^*)g(\bar{Z}, \bar{U})], \tag{8.4}$$
$$M(Y^*, \bar{Z}, U^*, \bar{V}) = \frac{1}{2(n-1)}[S(Y^*, U^*)g(\bar{Z}, \bar{V}) + S(\bar{Z}, \bar{V})g(Y^*, U^*)],$$
$$M(\bar{Y}, Z^*, \bar{U}, V^*) = \frac{1}{2(n-1)}[S(\bar{Y}, \bar{U})g(Z^*, V^*) + S(Z^*, V^*)g(\bar{Y}, \bar{U})],$$
$$(\nabla_{X^*} M)(\bar{Y}, \bar{Z}, \bar{U}, \bar{V}) = 0 = (\nabla_{\bar{X}} M)(Y^*, Z^*, U^*, V^*).$$

Again from (1.3), we get

$$(\nabla_{\bar{X}} M)(\bar{Y},\bar{Z},\bar{U},\bar{V}) = A(\bar{X})M(\bar{Y},\bar{Z},\bar{U},\bar{V}) + B(\bar{X})[g(\bar{Z},\bar{U})g(\bar{Y},\bar{V}) - g(\bar{Y},\bar{U})g(\bar{Z},\bar{V})],$$

$$A(X^*)M(\bar{Y},\bar{Z},\bar{U},\bar{V}) + B(X^*)[g(\bar{Z},\bar{U})g(\bar{Y},\bar{V}) - g(\bar{Y},\bar{U})g(\bar{Z},\bar{V})] = 0, \tag{8.5}$$

and

$$B_{(\bar{p},p^*)}(0 \oplus v) = 0$$

for every $\bar{p} \in M_1$, $p^* \in M_2$ and $v \in T_{p^*}M_2$. Also for every $(\bar{p},p^*) \in M$ from (1.3) we obtain

$$(\nabla_{X^*} M)_{(\bar{p},p^*)}(Y^*,Z^*,U^*,V^*) = (\nabla^*_{X^*} M^*)_{p^*}(Y^*,Z^*,U^*,V^*) \tag{8.6}$$

and the value of $(\nabla_{X^*} M)_{(\bar{p},p^*)}(Y^*,Z^*,U^*,V^*)$ does not depend on $\bar{p} \in M_1$ for every $\bar{p} \in M_1$ and $p^* \in M_2$.

If possible let $B(X^*) = 0$ for all $X^* \in \chi(M_2)$, then from (8.5) we get

$$A(X^*)M(\bar{Y},\bar{Z},\bar{U},\bar{V}) = 0. \tag{8.7}$$

Using (8.2) in (8.7) we get

$$A(X^*)\bar{M}(\bar{Y},\bar{Z},\bar{U},\bar{V}) = 0. \tag{8.8}$$

If M_1 is not M-projectively flat, that is, $\bar{M}_{\bar{p}_0} \neq 0$ for some $\bar{p}_0 \in M_1$, then from (8.7) and (8.8) it follows that

$$A_{(\bar{p},p^*)}(0 \oplus v) = 0 \tag{8.9}$$

for every $\bar{p} \in M_1$, $p^* \in M_2$ and for every $v \in T_{p^*}M_2$. Hence (1.3) yields

$$(\nabla_{X^*} M)_{(\bar{p},p^*)}(Y^*,Z^*,U^*,V^*) = 0$$

for every $\bar{p} \in M_1$ and $p^* \in M_2$. It follows that if M_1 is not M-projectively flat, then

$$A_{(\bar{p},p^*)}(X^*)M^*_{p^*}(Y^*,Z^*,U^*,V^*) = 0 \tag{8.10}$$

for all $\bar{p} \in M_1$ and $p^* \in M_2$.

Now we assume that

$$(\nabla_X M)(Y,Z,U,V) = \bar{A}(X)M(Y,Z,U,V) + \bar{B}(X)[g(Z,U)g(Y,V) - g(Y,U)g(Z,V)], \tag{8.11}$$

where $\bar{A}$ and $\bar{B}$ are 1-forms. Putting (8.11) in (1.3) we get

$$[A(X) - \bar{A}(X)]M(Y,Z,U,V) + [B(X) - \bar{B}(X)][g(Z,U)g(Y,V) - g(Y,U)g(Z,V)] = 0. \tag{8.12}$$

Contracting (8.12) over Y and V, and using (2.3) we obtain

$$\frac{n}{2(n-1)}[A(X) - \bar{A}(X)]\tilde{M}(Z,U) + (n-1)[B(X) - \bar{B}(X)]g(Z,U) = 0. \tag{8.13}$$

Again contracting (8.13) over Z and U and using (2.4) we get

$$B(X) = \bar{B}(X) \tag{8.14}$$

for all $X \in M$. From (8.14) in (8.12) it follows that

$$A(X) = \bar{A}(X)$$

for all $X \in M$, provided $M(Y,Z,U,V) \neq 0$, that is, if the manifold is not M-projectively flat manifold. Thus the 1-forms A and B in (1.3) are uniquely determined, provided that the manifold is not M-projectively flat manifold. Hence from (8.10) we obtain

$$A_{(\bar{p},p^*)}(X^*) = 0 \tag{8.15}$$

for all $\bar{p} \in M_1$ and $p^* \in M_2$.

From (8.8) we conclude that either

(I) $A(X^*) = 0$ for all $X^* \in \chi(M_2)$, or

(II) M_1 is M-projectively flat.

Also from (1.3), we obtain

$$\begin{aligned}(\nabla_{X^*}M)&(Y^*,\bar{Z},\bar{U},V^*)\\ &= A(X^*)M(Y^*,\bar{Z},\bar{U},V^*)\\ &\quad + B(X^*)[g(\bar{Z},\bar{U})g(Y^*,V^*) - g(Y^*,\bar{U})g(\bar{Z},V^*)].\end{aligned} \tag{8.16}$$

Now we consider the case (I). From (8.16), it follows that

$$(\nabla_{X^*}M)(Y^*,\bar{Z},\bar{U},V^*) = 0,$$

which implies by the virtue of (8.4) that,

$$(\nabla_{X^*}S)(Y^*,V^*) = 0. \tag{8.17}$$

Hence the component M_2 is Ricci symmetric. Using (8.4), (8.6), (8.9), (8.10) and (8.15) and $A(X^*) = 0$, $B(X^*) = 0$ for all $X^* \in \chi(M_2)$, from (1.3), we have

$$(\nabla_{X^*}M)(Y^*,Z^*,U^*,V^*) = 0$$

and hence

$$\begin{aligned}(\nabla_{X^*}R)(Y^*,Z^*,U^*,V^*) &- \frac{1}{2(n-1)}[(\nabla_{X^*}S)(Z^*,U^*)g(Y^*,V^*)\\ &- (\nabla_{X^*}S)(Y^*,U^*)g(Z^*,V^*) + (\nabla_{X^*}S)(Y^*,V^*)g(Z^*,U^*)\\ &- (\nabla_{X^*}S)(Z^*,V^*)g(Y^*,U^*)] = 0,\end{aligned}$$

which yields by the virtue of (8.17) that

$$(\nabla_{X^*}R)(Y^*,Z^*,U^*,V^*) = 0,$$

that is, the component M_2 is locally symmetric. Similar result can be proved for M_1. Thus we can state the following:

Theorem 8.1
Let (M^n, g) be a semi-Riemannian manifold which is not M-projectively flat, such that $M = M_1^p \times M_2^{n-p}$ for some $2 \leq p \leq n-2$. If (M^n, g) is a $G\{MP(K_n)\}$ and $B(X^*) = 0$ for all $X^* \in \chi(M_2)$, (resp.$B(\bar{X}) = 0$ for all $\bar{X} \in \chi(M_1)$), then either (I) *or* (II) *holds.*

(I) *$A(X^*) = 0$ for all $X^* \in \chi(M_2)$ (resp. $A(\bar{X}) = 0$ for all $\bar{X} \in \chi(M_1)$), and hence M_2 (resp. M_1) is Ricci symmetric as well as locally symmetric.*

(II) *M_1 (resp. M_2) is M-projectively-flat.*

Also from (1.3) we get

$$\begin{aligned}(\nabla_{\bar{X}} M)(\bar{Y}, Z^*, U^*, \bar{V}) \\ &= A(\bar{X}) M(\bar{Y}, Z^*, U^*, \bar{V}) \\ &\quad + B(\bar{X})[g(Z^*, U^*) g(\bar{Y}, \bar{V}) - g(\bar{Y}, U^*) g(Z^*, \bar{V})].\end{aligned} \tag{8.18}$$

Using (8.3) in (8.18) we get

$$\begin{aligned}\frac{1}{2(n-1)}&[(\nabla_{\bar{X}} S)(\bar{Y}, \bar{V}) g(Z^*, U^*)] \\ &= A(\bar{X})\Big[\frac{1}{2(n-1)}\{S(Z^*, U^*) g(\bar{Y}, \bar{V}) + S(\bar{Y}, \bar{V}) g(Z^*, U^*)\}\Big] \\ &\quad - B(\bar{X}) g(Z^*, U^*) g(\bar{Y}, \bar{V}).\end{aligned} \tag{8.19}$$

Now assume that $S(Z^*, U^*) = 0$ and $g(Z^*, U^*) \neq 0$. Then from (8.19) we get

$$(\nabla_{\bar{X}} S)(\bar{Y}, \bar{V}) = A(\bar{X}) S(\bar{Y}, \bar{V}) - 2(n-1) B(\bar{X}) g(\bar{Y}, \bar{V}),$$

which implies

$$(\nabla_{\bar{X}} S)(\bar{Y}, \bar{V}) = C(\bar{X}) S(\bar{Y}, \bar{V}) + D(\bar{X}) g(\bar{Y}, \bar{V}),$$

where $C(\bar{X}) = A(\bar{X})$ and $D(\bar{X}) = -2(n-1)B(\bar{X})$ are two non-zero 1-forms. Therefore M_1 is a generalized Ricci-recurrent manifold if M_2 is Ricci-flat. Thus we have the following:

Theorem 8.2
Let (M^n, g) be a semi-Riemannian manifold such that $M = M_1^p \times M_2^{n-p}$, $2 \leq p \leq n-2$. If M^n is a $G\{MP(K_n)\}$ and M_2 is Ricci flat, then M_1 is a generalized Ricci-recurrent manifold.

9. $G\{MP(K_n)\}$ warped product manifolds

The study of warped product manifolds was initiated by Kručkovič [17] in 1957. In 1969 Bishop and O'Neill [3] also obtained the same notion of the warped product manifolds, while they were constructing a large class of manifolds of negative curvature. Warped product manifolds are generalizations of the Cartesian product of Riemannian manifolds. We extend this definition to semi-Riemannian manifolds. Let $(\bar{M}, \bar{g})$ and $(\tilde{M}, \tilde{g})$ be two semi-Riemannian manifolds. Let $\bar{M}$ and $\tilde{M}$ be covered by coordinate charts $(U; x^1, x^2, \dots, x^p)$ and $(V; y^1, y^2, \dots, y^{n-p})$, respectively. Then the warped product $M = \bar{M} \times_f \tilde{M}$ is the product manifold $\bar{M} \times \tilde{M}$ which is covered by the coordinate charts $(U \times V; x^1, x^2, \dots, x^p, x^{p+1} = y^{p+1}, x^{p+2} = y^{p+2}, \dots, x^n = y^n)$. Then the local components of the metric g with respect to such a coordinate chart are given by

$$g_{ij} = \begin{cases} \bar{g}_{ij} & \text{for } i = a \text{ and } j = b, \\ f\tilde{g}_{ij} & \text{for } i = \alpha \text{ and } j = \beta, \\ 0 & \text{otherwise,} \end{cases}$$

Here $a, b, c, \dots \in \{1, 2, \dots, p\}$ and $\alpha, \beta, \gamma, \dots \in \{p+1, p+2, \dots, n\}$, and $i, j, k, \dots \in \{1, 2, \dots, n\}$. Here $\bar{M}$ is called the base, $\tilde{M}$ is called the fiber and f is called the warping function of the warped product $M = \bar{M} \times_f \tilde{M}$. We denote by Γ^i_{jk}, R_{ijkl}, S_{ij} and κ the components of the Levi-Civita connection ∇, the Riemann-Christoffel curvature tensor R, Ricci tensor S and the scalar curvature of (M, g), respectively. Moreover, when Ω is a quantity formed with respect to g, we denote by $\bar{\Omega}$ and $\tilde{\Omega}$, the similar quantities formed with respect to $\bar{g}$ and $\tilde{g}$, respectively. Then the non-zero local components of Levi-Civita connection ∇ of (M, g) are given by

$$\Gamma^a_{bc} = \bar{\Gamma}^a_{bc}, \quad \Gamma^\alpha_{\beta\gamma} = \tilde{\Gamma}^\alpha_{\beta\gamma}, \quad \Gamma^a_{\beta\gamma} = -\frac{1}{2}\bar{g}^{ab} f_b \tilde{g}_{\beta\gamma}, \quad \Gamma^\alpha_{a\beta} = \frac{1}{2f} f_a \delta^\alpha_\beta,$$

where $f_a = \partial_a f = \frac{\partial f}{\partial x^a}$. The local components $R_{hijk} = g_{hl} R^l_{ijk} = g_{hl}(\partial_k \Gamma^l_{ij} - \partial_j \Gamma^l_{ik} + \Gamma^m_{ij}\Gamma^l_{mk} - \Gamma^m_{ik}\Gamma^l_{mj})$, $\partial_k = \frac{\partial}{\partial x^k}$, of the Riemann-Christoffel curvature tensor R of (M, g) which may not vanish identically are the following:

$$R_{abcd} = \bar{R}_{abcd}, \quad R_{a\alpha b\beta} = -fT_{ab}\tilde{g}_{\alpha\beta}, \quad R_{\alpha\beta\gamma\delta} = f\tilde{R}_{\alpha\beta\gamma\delta} - f^2 P \tilde{G}_{\alpha\beta\gamma\delta},$$

where $G_{ijkl} = g_{il}g_{jk} - g_{ik}g_{jl}$ and

$$T_{ab} = -\frac{1}{2f}\Big(\nabla_b f_a - \frac{1}{2f} f_a f_b\Big), \qquad \mathrm{tr}(T) = g^{ab}T_{ab},$$
$$Q = f((n-p-1)P - tr(T)), \qquad P = \frac{1}{4f^2} g^{ab} f_a f_b.$$

The non-zero local components of the Ricci tensor $S_{jk} = g^{il}R_{ijkl}$ of (M, g) are given by

$$S_{ab} = \bar{S}_{ab} + (n-p)T_{ab}, \quad S_{\alpha\beta} = \tilde{S}_{\alpha\beta} - Q\tilde{g}_{\alpha\beta}. \tag{9.1}$$

The scalar curvature κ of (M, g) is given by

$$\kappa = \bar{\kappa} + \frac{\tilde{\kappa}}{f} - (n-p)[(n-p-1)P - 2\mathrm{tr}(T)].$$

Again the non-zero local components of ∇R and ∇S are given by [16]

(i) $\nabla_e R_{abcd} = \bar{\nabla}_e \bar{R}_{abcd}$,

(ii) $\nabla_e R_{a\alpha b\beta} = -f\bar{\nabla}_e T_{ab}\tilde{g}_{\alpha\beta}$,

$$\begin{aligned}
&\text{(iii)}\ \nabla_e R_{\alpha\beta\gamma\delta} = -f_e \tilde{R}_{\alpha\beta\gamma\delta} - f^2 P_e \tilde{G}_{\alpha\beta\gamma\delta},\\
&\text{(iv)}\ \nabla_\epsilon R_{\alpha\beta\gamma\delta} = f \tilde{\nabla}_\epsilon \tilde{R}_{\alpha\beta\gamma\delta},\\
&\text{(v)}\ \nabla_\epsilon R_{\alpha\beta\gamma d} = -\frac{f_d}{2}\tilde{R}_{\alpha\beta\gamma\epsilon} - \frac{f^2}{2} P_d \tilde{G}_{\alpha\beta\gamma\epsilon},\\
&\text{(vi)}\ \nabla_\epsilon R_{abc\delta} = -\frac{1}{2}\tilde{g}_{\epsilon\delta}(f_a T_{bc} - f_b T_{ac}) + \frac{1}{2} f^d \bar{R}_{abcd}\tilde{g}_{\epsilon\delta},\quad f^b = \bar{g}^{ab} f_a,
\end{aligned} \tag{9.2}$$

and

$$\begin{aligned}
&\text{(i)}\ \nabla_e S_{ab} = \bar{\nabla}_e \bar{S}_{ab} + (n-p)\bar{\nabla}_e T_{ab},\\
&\text{(ii)}\ \nabla_e S_{\alpha\beta} = Q_e \tilde{g}_{\alpha\beta} - \frac{f_e}{f}(\tilde{S}_{\alpha\beta} - Q\tilde{g}_{\alpha\beta}),\\
&\text{(iii)}\ \nabla_\epsilon S_{\alpha\beta} = \tilde{\nabla}_\epsilon \tilde{S}_{\alpha\beta},\\
&\text{(iv)}\ \nabla_\epsilon S_{\alpha a} = -\frac{1}{2f}\tilde{S}_{\alpha\epsilon} f_a + \frac{1}{2}\tilde{g}_{\alpha\epsilon}\Big[f^c(\bar{S}_{ca} + (n-p)T_{ca}) + \frac{Q}{f} f_a\Big].
\end{aligned} \tag{9.3}$$

Let $M = \bar{M} \times_f \tilde{M}$ be a non-flat warped product manifold and also let M be a $G\{MP(K_n)\}$. That is,

$$\nabla_e M_{abcd} = A_e M_{abcd} + B_e G_{abcd}, \tag{9.4}$$

where $G_{abcd} = \{g_{bc}g_{ad} - g_{ac}g_{bd}\}$.

Using (1.2) in (9.4) we have

$$\begin{aligned}
&\nabla_e R_{abcd} - \frac{1}{2(n-1)}[\nabla_e S_{bc} g_{ad} - \nabla_e S_{ac} g_{bd} + \nabla_e S_{ad} g_{bc} - \nabla_e S_{bd} g_{ac}]\\
&\quad = A_e\Big[R_{abcd} - \frac{1}{2(n-1)}\{S_{bc}g_{ad} - S_{ac}g_{bd} + S_{ad}g_{bc} - S_{bd}g_{ac}\}\Big]\\
&\qquad + B_e\{g_{bc}g_{ad} - g_{ac}g_{bd}\}.
\end{aligned}$$

Now putting (9.1), (9.2) and (9.3) to the above equation we get

$$\begin{aligned}
&\bar{\nabla}_e \bar{R}_{abcd} - \frac{1}{2(n-1)}[\{\bar{\nabla}_e \bar{S}_{bc} + (n-p)\bar{\nabla}_e T_{bc}\}\bar{g}_{ad}\\
&\qquad - \{\bar{\nabla}_e \bar{S}_{ac} + (n-p)\bar{\nabla}_e T_{ac}\}\bar{g}_{bd} + \{\bar{\nabla}_e \bar{S}_{ad} + (n-p)\bar{\nabla}_e T_{ad}\}\bar{g}_{bc}\\
&\qquad - \{\bar{\nabla}_e \bar{S}_{bd} + (n-p)\bar{\nabla}_e T_{bd}\}\bar{g}_{ac}]\\
&\quad = A_e\Big[\bar{R}_{abcd} - \frac{1}{2(n-1)}\big[\{\bar{S}_{bc} + (n-p)T_{bc}\}\bar{g}_{ad}\\
&\qquad - \{\bar{S}_{ac} + (n-p)T_{ac}\}\bar{g}_{bd} + \{\bar{S}_{ad} + (n-p)T_{ad}\}\bar{g}_{bc}\\
&\qquad - \{\bar{S}_{bd} + (n-p)T_{bd}\}\bar{g}_{ac}\big]\Big] + B_e\{\bar{g}_{bc}\bar{g}_{ad} - \bar{g}_{ac}\bar{g}_{bd}\},
\end{aligned}$$

which implies

$$\begin{aligned}
\bar{\nabla}_e \bar{M}_{abcd} = {} & \bar{A}_e \bar{M}_{abcd} + \bar{B}_e G_{abcd}\\
& + \frac{(n-p)}{2(n-1)}[\{\bar{\nabla}_e T_{bc}\bar{g}_{ad} - \bar{\nabla}_e T_{ac}\bar{g}_{bd} + \bar{\nabla}_e T_{ad}\bar{g}_{bc} - \bar{\nabla}_e T_{bd}\bar{g}_{ac}\}\\
& + A_e\{-T_{bc}\bar{g}_{ad} + T_{ac}\bar{g}_{bd} - T_{ad}\bar{g}_{bc} + T_{bd}\bar{g}_{ac}\}]
\end{aligned}$$

This can be written as

$$\bar{\nabla}_e \bar{M}_{abcd} = \bar{A}_e \bar{M}_{abcd} + \bar{B}_e G_{abcd},$$

provided that

$$\bar{\nabla}_e T_{bc} = A_e T_{bc} \tag{9.5}$$

holds. Hence we can conclude that, if $M = \bar{M} \times_f \tilde{M}$ is a generalized M-projectively warped product manifold, then $\bar{M}$ is also a $G\{MP(K_n)\}$, provided (9.5) holds. Hence we have the following:

Theorem 9.1
If $M = \bar{M} \times_f \tilde{M}$ is a $G\{MP(K_n)\}$ warped product manifold, then, provided (9.5) holds, $\bar{M}$ is a $G\{MP(K_n)\}$.

10. Perfect fluid Ricci symmetric $G\{MP(K_n)\}$ spacetime

This section is concerned with certain investigations in general relativity by the coordinate free method of differential geometry. In this method of study the spacetime of general relativity is regarded as a connected four-dimensional semi-Riemannian manifold $(\mathbb{R}^4, g)$ with Lorentz metric g with signature $(+,+,+,-)$. The geometry of the Lorentz manifold begins with the study of the casual character of vectors of the manifold. It is due to this causality that the Lorentz manifold becomes a convenient choice for the study of general relativity. Here we consider a special type of spacetime. Perfect fluid model is a spacetime (M, g) together with a triple (P, σ, p), where P is a timelike future-pointing unit vector field, and σ and p are functions called the energy density function and the pressure function respectively ([22, 31]). The energy momentum tensor of type $(0,2)$ for a perfect fluid is given by

$$T(X,Y) = (\sigma + p)A(X)A(Y) + pg(X,Y), \tag{10.1}$$

where A is the 1-form metrically equivalent to P such that $A(X) = g(X,P)$ and σ and p are the energy density and isotropic pressure of the fluid, respectively. As every spacetime model in general relativity, the perfect fluid model is required to obey the Einstein field equations. The Einstein field equations without cosmological constant is as follows

$$S(X,Y) - \frac{r}{2} g(X,Y) = kT(X,Y), \tag{10.2}$$

where k is the gravitational constant. A semi-Riemannian four-dimensional Ricci symmetric generalized M-projectively recurrent manifold may similarly be defined by taking a Lorentz metric g with signature $(+,+,+,-)$. In this case we consider a Ricci symmetric generalized M-projectively recurrent spacetime with the timelike velocity vector field P such that $g(P,P) = -1$. So, Theorem 4.1 will also hold in such a spacetime.

Using (10.1) we can express (10.2) as

$$S(X,Y) - \frac{r}{2}g(X,Y) = k[(\sigma + p)A(X)A(Y) + pg(X,Y)]. \tag{10.3}$$

Taking a frame field and contracting (10.3) over X and Y we get

$$r = k(\sigma - 3p). \tag{10.4}$$

Since the spacetime is Ricci symmetric, it is Einstein as well. Putting $S(X,Y) = \frac{r}{4}g(X,Y)$ in (10.3) and using (10.4) we obtain

$$-\frac{k}{4}(\sigma + p)g(X,Y) = k(\sigma + p)A(X)A(Y). \tag{10.5}$$

Since g is non-degenerate, it must be $\sigma + p = 0$. This implies that the fluid behaves as a cosmological constant [35]. This is also termed as a phantom barrier [21]. Now in the cosmology we know that the choice $\sigma = -p$ leads to rapid expansion of the spacetime which is now termed as inflation. Thus we can state the following:

Theorem 10.1
A perfect fluid Ricci symmetric $G\{MP(K_n)\}$ spacetime represents inflation and the fluid behaves as a cosmological constant. This is also termed as a phantom barrier.

11. Example of a $G\{MP(K_n)\}$

This section deals with an example of a $G\{MP(K_n)\}$. First, on the real number space R^n (with coordinates $x^1, x^2, \dots, x^n$) we define suitable semi-Riemannian metric g such that R^n becomes a semi-Riemannian manifold (M^n, g). We calculate the components of the curvature tensor, the Ricci tensor, the M-projective curvature tensor and its covariant derivative and then we verify the defining relation (1.3).

Example 11.1
We define a semi-Riemannian metric on the 4-dimensional real number space $\mathbb{R}^4$ by the formula

$$ds^2 = g_{ij}dx^i dx^j = f(dx^1)^2 + 2dx^1dx^2 + (dx^3)^2 + k(x^1)^2 v(x^4)(dx^4)^2, \tag{11.1}$$

where $i, j = 1, 2, \dots, 4$. Here $f = p_0 + p_1x^3 + p_2(x^3)^2$, p_0, p_1, p_2 are non-constant functions of x^1 only, $f > 0$, v is a function of x^4 and $k < 0$ is an arbitrary constant. Moreover, we assume $x^1 > 0$, $x^3 \neq 0$ and $x^4 > 0$.

Since we want M to be connected, $M \subset \{x^1 > 0\}$ and $M \subset \{x^4 > 0\}$. Then the only non-vanishing components of the Christoffel symbols, the curvature tensor and the Ricci tensor are respectively:

$$\Gamma^2_{11} = \frac{1}{2}f_{.1}, \quad \Gamma^2_{13} = -\Gamma^3_{11} = \frac{1}{2}f_{.3}, \quad \Gamma^4_{14} = \frac{1}{x^1}, \quad \Gamma^2_{44} = -kx^1v, \quad \Gamma^4_{44} = \frac{v_{.4}}{2v},$$

$$R_{1331} = \frac{1}{2} f_{.33}, \quad S_{11} = \frac{1}{2} f_{.33},$$

and the components which can be obtained from these by the symmetric properties. Here '.' denotes the partial differentiation with respect to the coordinates. Using the above relations, it can be easily shown that the scalar curvature of the manifold is zero. Therefore $\mathbb{R}^4$ with the considered metric is a semi-Riemannian manifold M^4 whose scalar curvature is zero. In the view of the above relations, (1.2) yields that the non-zero components of the M-projective curvature tensor are

$$M_{1331} = \frac{5}{12} f_{.33} = \frac{5}{6} p_2 \neq 0, \tag{11.2}$$

$$M_{1441} = -\frac{1}{12} f_{.33} k(x^1)^2 v(x^4) = -\frac{k(x^1)^2 v(x^4)}{6} p_2 \neq 0, \tag{11.3}$$

and the components which can be obtained from (11.2) and (11.3) by the symmetric properties. If, in particular, we take $x^1 = x^4$ and $v(x^4) = \frac{1}{(x^4)^2} = \frac{1}{(x^1)^2}$, then (11.3) can be written as follows

$$M_{1441} = -\frac{1}{12} f_{.33} k(x^1)^2 v(x^4) = -\frac{k(x^1)^2 v(x^4)}{6} p_2 = -\frac{k}{6} p_2 \neq 0. \tag{11.4}$$

The non-zero covariant derivatives of the M-projective curvature tensor M are

$$M_{1331,1} = M_{1331,4} = \frac{5}{12} f_{.331} = \frac{5}{6} (p_2)_{.1} \neq 0, \tag{11.5}$$

$$M_{1441,1} = M_{1441,4} = -\frac{k}{6} (p_2)_{.1} \neq 0 \tag{11.6}$$

and the components which can be obtained from (11.5) and (11.6) by the symmetric properties, where ',' denotes the covariant derivative with respect to the metric tensor. Hence the Riemannian manifold (M^4, g) is neither M-projectively flat nor M-projectively symmetric.

We shall now show that this M^4 is a $MP(K_4)$, that is a $G\{MP(K_4)\}$ with the 1-forms $A \neq 0$ and $B = 0$ in (1.3) satisfying (1.3). Let us now consider the following 1-forms A_i and B_i:

$$A_i(x) = \begin{cases} \frac{(p_2)_{.1}}{p_2}, & i = 1, 4 \\ 0, & \text{otherwise}, \end{cases} \tag{11.7}$$

$$B_i(x) = 0 \qquad \text{for } i = 1, 2, 3, 4, \tag{11.8}$$

at any point $x \in \mathbb{R}^4$. Now the equation (1.3) reduces to the equation

$$M_{1331,1} = M_{1331,4} = A_1 M_{1331} = A_4 M_{1331}, \tag{11.9}$$

$$M_{1441,1} = M_{1441,4} = A_1 M_{1441} = A_4 M_{1441}, \tag{11.10}$$

since, for the other cases (1.3) holds trivially. Using (11.7) and (11.8) in (11.9) we get

$$\begin{aligned}\text{R.H.S. of (11.9)} &= A_1 M_{1331} + B_1[g_{33}g_{11} - g_{13}g_{31}] \\ &= \frac{(p_2)_{.1}}{p_2}\frac{5p_2}{6} + 0 = \frac{5(p_2)_{.1}}{6} \\ &= \text{L.H.S. of (11.9).}\end{aligned}$$

In all other cases the proof is trivial. Therefore, $(\mathbb{R}^4, g)$ is a $MP(K_4)$.

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