DE GRUYTER
OPEN

Ann. Univ. Paedagog. Crac. Stud. Math. 13 (2014), 19-43
DOI: 10.2478/aupcsm-2014-0003

FOLIA 149

Annales Universitatis Paedagogicae Cracoviensis
Studia Mathematica XIII (2014)

Maria Robaszewska

On some flat connection associated with locally symmetric surface

Abstract. For every two-dimensional manifold M with locally symmetric linear connection ∇, endowed also with ∇-parallel volume element, we construct a flat connection on some principal fibre bundle $P(M,G)$. Associated with – satisfying some particular conditions – local basis of TM local connection form of such a connection is an $\mathcal{R}(G)$-valued 1-form Ω build from the dual basis ω^1, ω^2 and from the local connection form ω of ∇. The structural equations of (M,∇) are equivalent to the condition $d\Omega - \Omega\wedge\Omega = 0$.

This work was intended as an attempt to describe in a unified way the construction of similar 1-forms known for constant Gauss curvature surfaces, in particular of that given by R. Sasaki for pseudospherical surfaces.

1. Introduction

In the paper [7] R. Sasaki considered the soliton equations which can be solved by the 2×2 inverse scattering method − for example the sine-Gordon equation $u_{xt} = \sin u$, the Korteweg de Vries equation $u_t + 6uu_x + u_{xxx} = 0$ or the modified Korteweg de Vries equation $u_t + 6u^2u_x + u_{xxx} = 0$. To the known remarkable properties of those equations - such as possessing infinite number of conservation laws and possessing the Bäcklund transformation - he added the property that they describe pseudospherical surfaces.

One of the facts on which the inverse scattering method is based is that each of those nonlinear equations may be written as the integrability condition of some linear system $dv = \Omega v$, $v = \binom{v_1}{v_2}$. Sasaki has explained how to build an $\mathbf{sl}(2,\mathbb{R})$-valued 1-form Ω satisfying the condition $d\Omega - \Omega\wedge\Omega = 0$, using the 1-forms ω^1, ω^2, which are the basis dual to the g-orthonormal local basis of TM, and the local connection form ω:

$$\Omega = \begin{pmatrix} -\frac{1}{2}\omega^2 & \frac{1}{2}(\omega+\omega^1) \\ \frac{1}{2}(-\omega+\omega^1) & \frac{1}{2}\omega^2 \end{pmatrix}.$$

AMS (2010) Subject Classification: 53B05, 53B15, 58J60.

Conversely, if an $\mathbf{sl}(2,\mathbb{R})$-valued 1-form $\Omega = \begin{pmatrix} \Omega_{11} & \Omega_{12} \\ \Omega_{21} & -\Omega_{11} \end{pmatrix}$ satisfies the conditions $d\Omega - \Omega \wedge \Omega = 0$ and $(\Omega_{12} + \Omega_{21}) \wedge \Omega_{11} \neq 0$, then the metric $g = \omega^1 \otimes \omega^1 + \omega^2 \otimes \omega^2$ with $\omega^1 = \Omega_{12} + \Omega_{21}$, $\omega^2 = -2\Omega_{11}$ has constant negative Gaussian curvature, whereas $\omega := \Omega_{12} - \Omega_{21}$ is the local connection form of the Levi-Civita connection of g.

Sasaki also mentioned that in the case of surfaces of constant positive curvature it is also possible to construct from 1-forms ω^1, ω^2 and ω a 1-form Ω in such a way that the structural equations of the surface are written as $d\Omega - \Omega \wedge \Omega = 0$, $\operatorname{tr}\Omega = 0$. The corresponding Lie algebraic structure is that of $SO(3)$, being the isometry group of the sphere S^2.

A $\mathbf{g}$-valued 1-form Ω can be interpreted itself as a local connection form of some connection on a principal G-bundle, where G is a Lie group with Lie algebra $\mathbf{g}$. The condition $d\Omega - \Omega \wedge \Omega = 0$ means that the curvature form of this connection vanishes. Therefore such a 1-form Ω is called a zero-curvature representation of the given differential equation.

In order that $d\Omega - \Omega \wedge \Omega = 0$ is a differential equation, the entries of Ω or equivalently the forms ω^1, ω^2 and ω must depend on some function and its derivatives. Such dependence arises in a natural way when we consider for example surfaces immersed in $\mathbb{R}^3$ and the induced connection. Furthermore, if the differential equation describes a surface M immersed in $\mathbb{R}^3$, then it is possible to associate with the immersion some mapping from M into $GL(3,\mathbb{R})$ and then the pull-back of the Maurer-Cartan form is also a zero-curvature representation of this equation. In this case the flat connection concerned is the standard connection in $\mathbb{R}^3$.

Not every equation which possesses a zero-curvature representation is a soliton equation. An important thing in soliton theory is the dependence of Ω on some spectral parameter λ, so in fact we have a family of flat connection forms Ω_λ. Moreover, parameters introduced through the gauge transformation $\Omega_\lambda = S\Omega S^{-1} + dSS^{-1}$ play no role in soliton theory [3]. The issue of the spectral parameter will not be considered in the present paper.

Apart from constant Gauss curvature surfaces there are other kinds of submanifolds described by soliton equations, there exist also higher dimension generalisations (see [8] and the references given there). Affine spheres with indefinite Blaschke metric are examples of soliton surfaces in affine geometry [9].

It is possible that one differential equation has zero-curvature representations within different, non-isomorphic Lie algebras. For example, for describing pseudospherical surfaces sin-Gordon equation $u_{xy} = \sin u$ we have the following parametrized by λ $\mathbf{sl}(2,\mathbb{R})$-valued Sasaki form [7]

$$\Omega_\lambda = \begin{pmatrix} \lambda & -\frac{1}{2}u_x \\ \frac{1}{2}u_x & -\lambda \end{pmatrix} dx + \frac{1}{4\lambda} \begin{pmatrix} \cos u & \sin u \\ \sin u & -\cos u \end{pmatrix} dy,$$

whereas from the Maurer-Cartan form on $SO(3,\mathbb{R})$ one can obtain one-parameter family of $\mathbf{so}(3)$-valued 1-forms (cf [8])

$$\Omega_\lambda = \begin{pmatrix} 0 & u_x & 0 \\ -u_x & 0 & 2\lambda \\ 0 & -2\lambda & 0 \end{pmatrix} dx + \frac{1}{2\lambda} \begin{pmatrix} 0 & 0 & -\sin u \\ 0 & 0 & -\cos u \\ \sin u & \cos u & 0 \end{pmatrix} dy.$$

The aim of this paper is to construct similar 1-form Ω satisfying the condition $d\Omega-\Omega\wedge\Omega=0$ for surfaces with non-metrizable locally symmetric connection. We use an elementary method, which is applicable to all locally symmetric surfaces. We show that the Sasaki 1-form may be also obtained in this way.

In section 2 we recall some results concerning locally symmetric connections on surfaces. In section 3 we choose some special local bases of TM which will be used in the construction of Ω, for example the orthogonal bases in the metrizable case. Those bases are local sections of a subbundle $\widetilde{Q}(M,H)$ of $LM(M,GL(2,\mathbb{R}))$, where H is one-dimensional Lie subgroup of $GL(2,\mathbb{R})$. The considered locally symmetric connection is reducible to $\widetilde{Q}$.

In section 4 for any given homomorphism $\iota\colon H\to G$ of Lie groups we construct some principal fibre bundle $P(M,G)$ and a homomorphism of fibre bundles $Q(M,H)\to P(M,G)$. To every local section σ of Q we want to assign an $\mathcal{R}(G)$-valued 1-form Ω_σ. We explain how Ω_σ should vary with σ, if the family $\{\Omega_\sigma\}$ has to define a connection on P. In section 5 we add to this the condition – which is satisfied in particular by the 1-form of Sasaki – that the entries of Ω_σ are linear combinations with constant coefficients of the 1-forms ω^1, ω^2 and ω corresponding to the section σ. Those two conditions together with the condition of flatness allow us in each case to find all classes of possible 1-forms Ω_σ with respect to the equivalence relation $\Omega_\sigma\sim S^{-1}\Omega_\sigma S$, where $S\in GL(N,\mathbb{R})$. In the case of surfaces of constant negative curvature we also use the homomorphism $\iota\colon SO(2)\ni a\mapsto\sqrt{a}\in SL(2,\mathbb{R})/\{I,-I\}$ in order to look directly for an $\mathbf{sl}(2,\mathbb{R})$ valued Ω_σ.

2. Locally symmetric connections on two-dimensional manifolds

Let M be a connected, two-dimensional real manifold and let ∇ be a torsion-free, non-flat, locally symmetric linear connection on M. From the equality $\dim\operatorname{im}R_p+\dim\ker\operatorname{Ric}_p=2$ [5], where R is the curvature tensor of ∇, Ric its Ricci tensor, $\operatorname{im}R_p=\operatorname{span}\{R(X,Y)Z:\ X,Y,Z\in T_pM\}$ and $\ker\operatorname{Ric}_p=\{X\in T_pM:\ \forall Y\in T_pM,\ \operatorname{Ric}(X,Y)=0\}$, it follows that either $\dim\operatorname{im}R=1$ or Ric is non-degenerate. The number $\dim\operatorname{im}R$ is called the rank of the connection ∇.

In the case of $\dim\operatorname{im}R=1$ we shall use

PROPOSITION 2.1 ([5])
Let ∇ be a locally symmetric connection of rank 1 on a 2-dimensional manifold M. For every $p\in M$ there is a coordinate system (u,v) around p such that

$$\nabla_{\partial_u}\partial_u=\nabla_{\partial_u}\partial_v=0\qquad\text{and}\qquad\nabla_{\partial_v}\partial_v=\varepsilon u\partial_u,\tag{1}$$

where $\varepsilon=\operatorname{sign}\operatorname{Ric}$.

The Ricci tensor of such connection ∇ is symmetric [5].

In the case of $\dim\operatorname{im}R=2$ we use

PROPOSITION 2.2 ([6])
If M is a 2-dimensional manifold with a locally symmetric connection ∇ of rank 2, then the Ricci tensor of ∇ is symmetric.

In this case ∇ is the Levi-Civita connection of Ric [6]. If Ric is definite, then Ric or $-\operatorname{Ric}$ is a metric, if Ric is indefinite, then it is a pseudometric. The curvature κ of this metric or pseudometric is constant.

It follows that we only have to consider the following cases:

I^+**:** $\dim \operatorname{im} R = 1$ and $\varepsilon = 1$,

I^-**:** $\dim \operatorname{im} R = 1$ and $\varepsilon = -1$,

IId^+**:** ∇ is metrizable of constant positive curvature,

IId^-**:** ∇ is metrizable of constant negative curvature,

IIi**:** ∇ is pseudometrizable of constant curvature.

If ∇ is metrizable and M is orientable, then there exists globally defined ∇-parallel volume element vol. If M is not orientable, then we can define vol on some open subset V of M. The last is true also in cases I^+ and I^-, because an affine connection ∇ with zero torsion has a symmetric Ricci tensor if and only if there is a parallel volume element around each point [4]. In the canonical coordinates (u, v) from Proposition 2.1, $\operatorname{vol} = c\, du \wedge dv$ with any $c \in \mathbb{R} \setminus \{0\}$.

From now on we assume that M is connected and that there exists on M a ∇-parallel volume element vol.

3. Reduction of LM to one-dimensional subgroup H of $GL(2, \mathbb{R})$

In this section we will consider a reduction of $LM(M, GL(2, \mathbb{R}))$ to some one-dimensional subgroup H of $GL(2, \mathbb{R})$.

Cases I^+ and I^-.
Let

$$\widetilde{Q} := \{(v_1, v_2) \in LM : \ v_1 \in \ker \operatorname{Ric}, \ \operatorname{vol}(v_1, v_2) = 1 \text{ and } \operatorname{Ric}(v_2, v_2) = \varepsilon\}$$

and let

$$H := \left\{ \begin{pmatrix} 1 & t \\ 0 & 1 \end{pmatrix} : \ t \in \mathbb{R} \right\} \cup \left\{ \begin{pmatrix} -1 & t \\ 0 & -1 \end{pmatrix} : \ t \in \mathbb{R} \right\}.$$

The subset $\widetilde{Q}$ of $LM(M, GL(2, \mathbb{R}))$ satisfies the assumptions of the following lemma.

Lemma 3.1 ([2])
Let $\widetilde{Q}$ be a subset of $P(M, G)$ and H a Lie subgroup of G. Assume:

(1) *the projection $\pi \colon P \to M$ maps $\widetilde{Q}$ onto M;*

(2) *$\widetilde{Q}$ is stable by H;*

(3) *if $p, q \in \widetilde{Q}$ and $\pi(p) = \pi(q)$, then there is an element $a \in H$ such that $q = pa$;*

(4) *every point of M has a neighbourhood U and a cross section $\sigma \colon U \to P$ such that $\sigma(U) \subset \widetilde{Q}$.*

Then $\widetilde{Q}(M,H)$ is a reduced subbundle of $P(M,G)$.

Indeed, if (u,v) are canonical coordinates on U, then $\mathrm{vol} = c\,du \wedge dv$ for some $c \in \mathbb{R} \setminus \{0\}$ and the cross section $\sigma := (\frac{1}{c}\partial_u, \partial_v)$ satisfies (4). The condition (1) follows from (4).

Assume that $\pi((v_1,v_2)) = \pi((w_1,w_2))$. Then $(w_1,w_2) = (v_1,v_2)\cdot a$ with some $a = \begin{pmatrix} {a^1}_1 & {a^1}_2 \\ {a^2}_1 & {a^2}_2 \end{pmatrix} \in GL(2,\mathbb{R})$. To check the condition (3) we have to show that if $(v_1,v_2) \in \widetilde{Q}$ and $(w_1,w_2) \in \widetilde{Q}$, then $a \in H$. We have

$$w_1 = {a^1}_1 v_1 + {a^2}_1 v_2, \qquad w_2 = {a^1}_2 v_1 + {a^2}_2 v_2.$$

Since $v_1, w_1 \in \ker \mathrm{Ric}$ and $\dim\ker\mathrm{Ric} = 1$, we have ${a^2}_1 = 0$. From $\mathrm{vol}(v_1,v_2) = \mathrm{vol}(w_1,w_2)$ it follows that $\det({a^i}_j) = 1$. Consequently ${a^1}_1{a^2}_2 = 1$. Comparing $\mathrm{Ric}(v_2,v_2)$ and $\mathrm{Ric}(w_2,w_2)$ we obtain $({a^2}_2)^2 = 1$.

It is easily seen that if $(v_1,v_2) \in \widetilde{Q}$ and $a \in H$, then $(v_1,v_2)\cdot a \in \widetilde{Q}$, hence $\widetilde{Q}$ satisfies (2).

Cases IId^+ and IId^-.

We take as $\widetilde{Q}$ the bundle of orthonormal frames satisfying the condition $\mathrm{vol}(v_1,v_2) > 0$. The structure group is $H = SO(2,\mathbb{R})$.

Case IIi.

Let g be a pseudometric such that ∇ is the Levi-Civita connection of g. Let

$$\widetilde{Q} := \{(v_1,v_2) \in LM : \ g(v_1,v_1) = -g(v_2,v_2) = 1, \ g(v_1,v_2) = 0, \ \mathrm{vol}(v_1,v_2) > 0\}.$$

The structure group of the reduced bundle $\widetilde{Q}$ is

$$SO(1,1) = \left\{ A \in GL(2,\mathbb{R}) : \ A^T \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} A = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \text{ and } \det A = 1 \right\}.$$

In the following we will use a principal fibre bundle $Q(M,H)$ with H acting on Q on the left. As a set Q is equal to $\widetilde{Q}$, and the left action of H is $L_a((v_1,v_2)) := (v_1,v_2)\cdot a^{-1}$ for $a \in H$.

4. Extension $P(M,G)$ of $Q(M,H)$ and a connection on P

Unless otherwise stated we will consider principal fibre bundles with the structural groups acting on the left. By $\mathcal{R}(G)$ we denote the Lie algebra of right-invariant vector fields on the Lie group G and ϑ_G stands for the $\mathcal{R}(G)$-valued Maurer-Cartan form on G.

PROPOSITION 4.1

Let $\iota\colon H \to G$ be a continuous homomorphism of Lie groups H and G. Let $Q(M,H)$ be a principal fibre bundle. Then there exist a principal fibre bundle $P(M,G)$ and a mapping $f\colon Q \to P$ such that $(f, \mathrm{id}_M, \iota)$ is a homomorphism of principal fibre bundles.

If ι is an imbedding, then the same holds for f.

Proof. The proposition is a slight modification of Theorem 26.12, page 224 in [1]. We only replace the inclusion $H \subset G$ by a homomorphism $\iota\colon H \to G$ and the right action of G by the left action. The main idea of the proof is similar. Some parts of it we describe here with more details.

We define the left action L of H and the left action $\widetilde{L}$ of G on $G \times Q$:

$$L_a(b,q) := (b\iota(a^{-1}), aq), \quad \widetilde{L}_c(b,q) := (cb, q) \qquad \text{for } a \in H,\ b, c \in G.$$

Then we define an equivalence relation on $G \times Q$:

$$(b_1, q_1) \sim (b_2, q_2) \iff \exists\, a \in H :\ (b_2, q_2) = L_a(b_1, q_1).$$

Let $P := (G \times Q)/\sim$.

1. *P with the quotient topology is a Hausdorff space.*

The canonical projection $\rho\colon G \times Q \ni (b,q) \mapsto [(b,q)] \in P$ is an open mapping, because

$$\rho^{-1}(\rho(U)) = \bigcup_{a \in H} L_a(U)$$

is open for an open subset $U \subset G \times Q$. Let

$$R_0 = \{(b,q,c,r) \in G \times Q \times G \times Q :\ (b,q) \sim (c,r)\}.$$

It suffices to check that R_0 is closed. Let $(b,q,c,r) \in (G \times Q \times G \times Q) \setminus R_0$. Let $\pi\colon Q \to M$ denote the projection in $Q(M,H)$.

If $\pi(q) \neq \pi(r)$, then there exist disjoint neighbourhoods U_1 and U_2 of $\pi(q)$ and $\pi(r)$, respectively. Then $G \times \pi^{-1}(U_1) \times G \times \pi^{-1}(U_2)$ is an open neighbourhood of (b,q,c,r) in $G \times Q \times G \times Q$ and

$$(G \times \pi^{-1}(U_1) \times G \times \pi^{-1}(U_2)) \cap R_0 = \emptyset,$$

because if $(b_1, q_1, c_1, r_1) \in R_0$, then $(c_1, r_1) = (b_1\iota(a^{-1}), aq_1)$ for some $a \in H$, hence $\pi(r_1) = \pi(q_1)$.

Assume now that $\pi(q) = \pi(r)$, so $r = aq$ with some $a \in H$. From $(b,q,c,aq) \notin R_0$ it follows that $b^{-1}c\iota(a) \neq e_G$. Let $U_1 \subset G$ be an open neighbourhood of $b^{-1}c\iota(a)$ such that $e_G \notin U_1$.

The continuity of the mapping $G \times G \times H \ni (\xi, \eta, \zeta) \mapsto \xi^{-1}\eta\iota(\zeta) \in G$ implies that there exist open neighbourhoods $U_2, U_3 \subset G$, $U_4 \subset H$ of b, c, a, respectively, such that $(\xi, \eta, \zeta) \in U_2 \times U_3 \times U_4$ implies $\xi^{-1}\eta\iota(\zeta) \in U_1$. Next we use the continuity of $H \times H \ni (\alpha, \beta) \mapsto \alpha a \beta^{-1} \in H$ and find the neighbourhoods U_5, U_6 of e_H such that $\alpha a \beta^{-1} \in U_4$ if $(\alpha, \beta) \in U_5 \times U_6$.

Let $\varphi = (\psi, \pi)\colon \pi^{-1}(U) \to H \times U$ with $U \ni \pi(q)$ be a local trivialisation of $Q(M,H)$. Then $\widetilde{U}_5 := \varphi^{-1}((U_5 a \psi(q)) \times U) \subset Q$ is an open neighbourhood of aq, and $\widetilde{U}_6 := \varphi^{-1}((U_6 \psi(q)) \times U) \subset Q$ is an open neighbourhood of q.

We check that $(U_2 \times \widetilde{U}_6 \times U_3 \times \widetilde{U}_5) \cap R_0 = \emptyset$.

Let $(b', q', c', r') \in U_2 \times \widetilde{U}_6 \times U_3 \times \widetilde{U}_5$. If $\pi(q') \neq \pi(r')$, then $(b', q', c', r') \notin R_0$. If $\pi(q') = \pi(r')$, then $r' = a'q'$ with some $a' \in H$.

From $q' \in \widetilde{U}_6 = \varphi^{-1}((U_6\psi(q)) \times U)$ it follows that $\psi(q') \in U_6\psi(q)$, hence $\psi(q') = \beta\psi(q)$ with some $\beta \in U_6$. Similarly, from $r' \in \widetilde{U}_5 = \varphi^{-1}((U_5 a\psi(q)) \times U)$ it follows that $\psi(r') = \alpha a\psi(q)$ with some $\alpha \in U_5$. But $r' = a'q'$, so $\psi(r') = a'\psi(q')$, hence $a'\beta\psi(q) = \alpha a\psi(q)$ and consequently $a' = \alpha a\beta^{-1}$, which implies $a' \in U_4$ and $b'^{-1}c'\iota(a') \in U_1$. Therefore $b'^{-1}c'\iota(a') \neq e_G$ and $(b', q', c', r') = (b', q', c', a'q') \notin R_0$.

2. *G acts freely on P on the left.*

From

$$(\widetilde{L}_c \circ L_a)(b, q) = \widetilde{L}_c(b\iota(a^{-1}), aq) = (cb\iota(a^{-1}), aq) = L_a(cb, q) = (L_a \circ \widetilde{L}_c)(b, q)$$

it follows that $\rho(b_1, q_1) = \rho(b_2, q_2)$ implies $\rho(\widetilde{L}_c(b_1, q_1)) = \rho(\widetilde{L}_c(b_2, q_2))$, and the left action of G on P

$$c[(b, q)] := [\widetilde{L}_c(b, q)] = [(cb, q)]$$

is well defined. If $c[(b, q)] = [(b, q)]$, then for some $a \in H$ we have $(cb, q) = (b\iota(a^{-1}), aq)$. From $aq = q$ it follows that $a = e_H$, because H acts freely on Q. Now from $cb = b$ we conclude that $c = e_G$.

3. *The projection $\widetilde{\pi}\colon P \to M$.*

The projection $\widetilde{\pi}\colon P \to M$, $\widetilde{\pi}([(b, q)]) := \pi(q)$, is defined in such a way that the diagram

$$\begin{array}{ccc} G \times Q & \xrightarrow{p_2} & Q \\ \rho \downarrow & & \downarrow \pi \\ (G \times Q)/\sim\, = P & \xrightarrow{\widetilde{\pi}} & M \end{array}$$

is commutative. The mapping $\widetilde{\pi}$ is continuous, because so is $\pi \circ p_2$.

Let $\widetilde{\pi}([(b_1, q_1)]) = \widetilde{\pi}([(b_2, q_2)])$. Then $\pi(q_1) = \pi(q_2)$ which means $q_2 = aq_1$ with some $a \in H$. It follows that

$$[(b_2, q_2)] = [(b_2, aq_1)] = [(b_2\iota(a)\iota(a^{-1}), aq_1)] = [(b_2\iota(a), q_1)] = b_2\iota(a)b_1^{-1}[(b_1, q_1)].$$

Conversely, for any $c \in G$, $\widetilde{\pi}(c[(b, q)]) = \widetilde{\pi}([(cb, q)]) = \pi(q) = \widetilde{\pi}([(b, q)])$.

4. *Local trivialisations.*

Let $\varphi\colon \pi^{-1}(U) \to H \times U$, $\varphi = (\psi, \pi)$, be a local trivialisation of $Q(M, H)$. We define a homeomorphism $\widetilde{\varphi}\colon \widetilde{\pi}^{-1}(U) \to G \times U$. Let $\widetilde{\varphi}([(b, q)]) := (b\iota(\psi(q)), \pi(q))$. The mapping $\widetilde{\varphi}$ is well defined, because if $(b_2, q_2) = (b_1\iota(a^{-1}), aq_1)$ with some $a \in H$, then

$$\begin{aligned} (b_2\iota(\psi(q_2)), \pi(q_2)) &= (b_1\iota(a^{-1})\iota(\psi(aq_1)), \pi(aq_1)) \\ &= (b_1\iota(a^{-1})\iota(a\psi(q_1)), \pi(q_1)) = (b_1\iota(a^{-1}a\psi(q_1)), \pi(q_1)) \\ &= (b_1\iota(\psi(q_1)), \pi(q_1)). \end{aligned}$$

The continuity of $\widetilde{\varphi}$ follows from that of $\widetilde{\varphi} \circ \rho$.

To define the inverse mapping of $\widetilde{\varphi}$, we use the local section $\sigma\colon U \to Q$, $\sigma(x) := \varphi^{-1}(e_H, x)$. Let $\Phi(b, x) := [(b, \sigma(x))]$ for $b \in G$, $x \in U$. Then

$$\begin{aligned} (\widetilde{\varphi} \circ \Phi)(b, x) &= \widetilde{\varphi}([(b, \sigma(x))]) = (b\iota(\psi(\sigma(x))), \pi(\sigma(x))) = (b\iota(e_H), x) = (be_G, x) \\ &= (b, x) \end{aligned}$$

and

$$\begin{aligned}(\Phi\circ\widetilde{\varphi})([(b,q)]) &= \Phi((b\iota(\psi(q)),\pi(q))) = [(b\iota(\psi(q)),\sigma(\pi(q)))]\\ &= [(b\iota(\psi(q)),(\psi(q))^{-1}\psi(q)\sigma(\pi(q)))] = [(b,\psi(q)\sigma(\pi(q)))]\\ &= [(b,q)].\end{aligned}$$

The last equality follows from

$$\begin{aligned}\varphi(\psi(q)\sigma(\pi(q))) &= (\psi(\psi(q)\sigma(\pi(q))),\pi(\psi(q)\sigma(\pi(q))))\\ &= (\psi(q)\psi(\sigma(\pi(q))),\pi(\sigma(\pi(q))))\\ &= (\psi(q)e_H,\pi(q)) = (\psi(q),\pi(q))\\ &= \varphi(q).\end{aligned}$$

Since $\Phi = \rho\circ(\mathrm{id}_G,\sigma)$, it is continuous.

We see that $\widetilde{\varphi} = (\widetilde{\psi},\widetilde{\pi})$, where $\widetilde{\psi}([(b,q)]) := b\iota(\psi(q))$. The mapping $\widetilde{\psi}$ satisfies the condition $\widetilde{\psi}(c[(b,q)]) = c\widetilde{\psi}([(b,q)])$.

5. *Differentiable structure in P.*

Let $\varphi_\alpha\colon \pi^{-1}(U_\alpha)\to H\times U_\alpha$, $\varphi_\beta\colon \pi^{-1}(U_\beta)\to H\times U_\beta$ be two local trivialisations of Q with $U_\alpha\cap U_\beta\neq\emptyset$, σ_α, σ_β the corresponding local sections of Q and $\widetilde{\varphi}_\alpha$, $\widetilde{\varphi}_\beta$ the corresponding local trivialisations of P. Let $h_{\beta\alpha}\colon U_\alpha\cap U_\beta\to H$ be the transition function, $h_{\beta\alpha}(\pi(q)) = (\psi_\beta(q))^{-1}\psi_\alpha(q)$. Then $\sigma_\beta(x) = h_{\beta\alpha}(x)\sigma_\alpha(x)$ and

$$\begin{aligned}\widetilde{\varphi}_\beta\circ\widetilde{\varphi}_\alpha^{-1}(b,x) &= \widetilde{\varphi}_\beta\circ\Phi_\alpha(b,x) = \widetilde{\varphi}_\beta([(b,\sigma_\alpha(x))])\\ &= (b\iota(\psi_\beta(\sigma_\alpha(x))),\pi(\sigma_\alpha(x)))\\ &= (b\iota((h_{\beta\alpha}(x))^{-1}),x).\end{aligned}$$

It follows that we have an open covering $\{\widetilde{\pi}^{-1}(U_\alpha)\}_\alpha$ of P and a family of homeomorphisms $\{\widetilde{\varphi}_\alpha\}$ such that $\widetilde{\varphi}_\beta\circ\widetilde{\varphi}_\alpha^{-1}$ is smooth for any α and β. If this is so, then there exists exactly one differentiable structure in P such that all $\widetilde{\varphi}_\alpha$ are diffeomorphisms.

We see now that $\widetilde{\pi}\colon P\to M$ is differentiable, because $\widetilde{\pi}|_{\widetilde{\pi}^{-1}(U_\alpha)} = p_2\circ\widetilde{\varphi}_\alpha$ is differentiable and $\{\widetilde{\pi}^{-1}(U_\alpha)\}_\alpha$ is an open covering of P.

6. *Homomorphism $f\colon Q\to P$ of principal fibre bundles.*

Let $f(q) := [(e_G,q)]$. Let $\varphi\colon \pi^{-1}(U)\to H\times U$ be a local trivialisation of Q. Since we have the following commutative diagram

$$\begin{array}{ccc}\pi^{-1}(U) & \xrightarrow{f} & \widetilde{\pi}^{-1}(U)\\ {\scriptstyle\varphi}\downarrow & & \downarrow{\scriptstyle\tilde{\varphi}}\\ H\times U & \xrightarrow{\iota\times\mathrm{id}} & G\times U,\end{array}$$

f is differentiable. Moreover, from $\widetilde{\pi}(f(q)) = \pi(q)$ and

$$f(aq) = [(e_G,aq)] = [(e_G\iota(a)\iota(a^{-1}),aq)] = [(\iota(a),q)] = \iota(a)[(e_G,q)]$$

it follows that (f,id_M,ι) is a homomorphism of principal fibre bundles.

Assume now that ι is an imbedding. From $f|_{\pi^{-1}(U)} = \widetilde{\varphi}^{-1} \circ (\iota \times \mathrm{id}_M) \circ \varphi$ it follows that f is an immersion. Let $f(q_1) = f(q_2)$. Then $\pi(q_1) = \widetilde{\pi}(f(q_1)) = \widetilde{\pi}(f(q_2)) = \pi(q_2)$, hence $q_2 = aq_1$ for some $a \in H$, $[(e_G, aq_1)] = [(e_G, q_1)]$ and consequently $(e_G, aq_1) = (e_G\iota(b^{-1}), bq_1)$ with some $b \in H$, which implies $b = a$ and $\iota(a) = e_G$. Since ι is injective, we have $a = e_H$ and $q_1 = q_2$.

In the next proposition we state some condition on $\sigma \to \Omega_\sigma$ under which the family of 1-forms Ω_σ may define a connection on P.

Proposition 4.2
Let $(f, \mathrm{id}_M, \iota)$ be a homomorphism of principal fibre bundles $Q(M,H)$ and $P(M,G)$. Assume that with every local section σ of Q we associate some $\mathcal{R}(G)$-valued 1-form Ω_σ. Moreover, assume that if Ω_α and Ω_β are the 1-forms associated with $\sigma_\alpha\colon U_\alpha \to Q$, $\sigma_\beta\colon U_\beta \to Q$, respectively, and on $U_\alpha \cap U_\beta$ we have $\sigma_\beta = h_{\beta\alpha}\sigma_\alpha$ with $h_{\beta\alpha}\colon U_\alpha \cap U_\beta \to H$, then

$$\Omega_\beta = \mathrm{Ad}_{\iota\circ h_{\beta\alpha}} \cdot \Omega_\alpha + (\iota \circ h_{\beta\alpha})^* \vartheta_G. \tag{2}$$

Under the conditions stated above, there exists a unique connection Γ in P such that for every local section σ of Q the 1-form Ω_σ is the local connection form corresponding to the local section $f \circ \sigma$ of P.

Proof. We will define the connection form $\widetilde{\Omega}$ of Γ.

Let $\sigma\colon U \to Q$ be a local section of Q. Let $\widetilde{\varphi}\colon \pi_P^{-1}(U) \to G \times U$ be the local trivialisation associated with the local section $f \circ \sigma$ of P: $\widetilde{\varphi}(bf \circ \sigma(x)) = (b, x)$. Then $d_{(b,x)}(\widetilde{\varphi}^{-1})$ maps $T_bG \oplus T_xM$ isomorphically onto $T_{bf\circ\sigma(x)}P$. Consequently, for every $W \in T_{bf\circ\sigma(x)}P$ there exist $A \in \mathcal{R}(G)$ and $X_x \in T_xM$, such that $W = d_{(b,x)}(\widetilde{\varphi}^{-1})(A_b \oplus X_x)$. Let

$$\widetilde{\Omega}_{bf\circ\sigma(x)}(d_{(b,x)}(\widetilde{\varphi}^{-1})(A_b \oplus X_x)) := A + \mathrm{Ad}_b(\Omega_\sigma(X_x)). \tag{3}$$

We first check that in this way we may obtain a 1-form $\widetilde{\Omega}$ on the whole M. Let $\widehat{\sigma}\colon \widehat{U} \to Q$ be another local section of Q and we define $\widehat{\varphi}$ by $\widehat{\varphi}^{-1}(c, y) := cf \circ \widehat{\sigma}(y)$ for $c \in G$, $y \in \widehat{U}$. Assume that $U \cap \widehat{U} \neq \emptyset$, then $\widehat{\sigma} = h\sigma$ and $f \circ \widehat{\sigma} = (\iota \circ h)(f \circ \sigma)$ on $U \cap \widehat{U}$.

Let $p \in P$ and $x := \pi_P(p) \in U \cap \widehat{U}$. Let $\widetilde{\varphi}(p) = (b, x)$ and $\widehat{\varphi}(p) = (c, x)$. Then $p = bf \circ \sigma(x) = cf \circ \widehat{\sigma}(x)$ and consequently $b = c\iota \circ h(x)$.

Now we take $Z_p \in T_pP$. Let

$$Z_p = d_{(b,x)}(\widetilde{\varphi}^{-1})(A_b \oplus X_x) = d_{(c,x)}(\widehat{\varphi}^{-1})(B_c \oplus Y_x). \tag{4}$$

We have to check that $A + \mathrm{Ad}_b(\Omega_\sigma(X_x)) = B + \mathrm{Ad}_c(\Omega_{\hat{\sigma}}(Y_x))$.

Since $p_2 \circ \widetilde{\varphi} = \pi_P = p_2 \circ \widehat{\varphi}$, we have

$$\begin{aligned} X_x &= d_{(b,x)}p_2(A_b \oplus X_x) = d_{(b,x)}p_2 \circ d_p\widetilde{\varphi}(d_{(b,x)}(\widetilde{\varphi}^{-1})(A_b \oplus X_x)) \\ &= d_{(b,x)}p_2 \circ d_p\widetilde{\varphi}(Z_p) \\ &= d_p\pi_P(Z_p) \end{aligned}$$

and similarly $Y_x = d_p\pi_P(Z_p)$, which yields $X_x = Y_x$.

If $B \in \mathcal{R}(G)$ and $B_e = [t \mapsto b_t]$, then $B_g = [t \mapsto b_t g]$. Let $X_x = [t \mapsto \gamma(t)]$. We conclude from $\widetilde{\varphi} \circ \widehat{\varphi}^{-1}(g, y) = (g\iota \circ h(y), y)$ and from $\iota \circ h(x) = c^{-1}b$ that

$$d_{(c,x)}(\widetilde{\varphi} \circ \widehat{\varphi}^{-1})(B_c \oplus 0) = [t \mapsto (b_t c\iota \circ h(x), x)] = [t \mapsto (b_t cc^{-1}b, x)] = B_b \oplus 0$$

and

$$d_{(c,x)}(\widetilde{\varphi} \circ \widehat{\varphi}^{-1})(0 \oplus X_x) = [t \mapsto (c\iota \circ h(\gamma(t)), \gamma(t))] = [t \mapsto c\iota \circ h(\gamma(t))] \oplus X_x.$$

Let $((\iota \circ h)^* \vartheta_G)_x(X_x) = C \in \mathcal{R}(G)$, which means that $d_x(\iota \circ h)(X_x) = C_{\iota \circ h(x)} = C_{c^{-1}b}$, hence

$$[t \mapsto c\iota \circ h(\gamma(t))] = d_{c^{-1}b} l_c(C_{c^{-1}b}) = (\mathrm{Ad}_c(C))_b,$$

where l_c is the left translation on G. Consequently we have

$$d_{(c,x)}(\widetilde{\varphi} \circ \widehat{\varphi}^{-1})(B_c \oplus X_x) = (B_b + (\mathrm{Ad}_c(C))_b) \oplus X_x,$$

which implies

$$d_{(c,x)}(\widehat{\varphi}^{-1})(B_c \oplus X_x) = d_{(b,x)}(\widetilde{\varphi}^{-1})((B_b + (\mathrm{Ad}_c(C))_b) \oplus X_x).$$

But the left-hand side is equal to $d_{(b,x)}(\widetilde{\varphi}^{-1})(A_b \oplus X_x)$, therefore $A = B + \mathrm{Ad}_c(C)$.

From (2) it follows that

$$\Omega_{\hat{\sigma}}|_x = \mathrm{Ad}_{\iota \circ h(x)} \circ \Omega_\sigma|_x + ((\iota \circ h)^* \vartheta_G)|_x.$$

Now we obtain the desired equality

$$\begin{aligned} B + \mathrm{Ad}_c(\Omega_{\hat{\sigma}}(X_x)) &= B + \mathrm{Ad}_c(\mathrm{Ad}_{\iota \circ h(x)}(\Omega_\sigma(X_x)) + ((\iota \circ h)^* \vartheta_G)(X_x)) \\ &= B + (\mathrm{Ad}_c \circ \mathrm{Ad}_{c^{-1}b})(\Omega_\sigma(X_x)) + \mathrm{Ad}_c(C) \\ &= B + \mathrm{Ad}_b(\Omega_\sigma(X_x)) + A - B \\ &= A + \mathrm{Ad}_b(\Omega_\sigma(X_x)). \end{aligned}$$

We next prove that $\widetilde{\Omega}$ is a connection form. We have to check the following two conditions:

(i) $\widetilde{\Omega}(A^*) = A$ for every fundamental vertical vector field A^* on P,

(ii) $(L_c)^* \widetilde{\Omega} = \mathrm{Ad}_c \cdot \widetilde{\Omega}$ for every $c \in G$.

Let $p \in P$, $x := \pi_P(p) \in U$ and let $\sigma \colon U \to Q$ be a local section of Q. Similarly as before we define $\widetilde{\varphi}$ by $\widetilde{\varphi}(gf \circ \sigma(y)) = (g, y)$. Let $\widetilde{\varphi}(p) = (b, x)$.

Condition (i). Let $A \in \mathcal{R}(G)$. Since

$$\begin{aligned} A^*_p &= [t \mapsto a_t p] = [t \mapsto a_t bf \circ \sigma(x)] = [t \mapsto \widetilde{\varphi}^{-1}((a_t b, x))] \\ &= d_{(b,x)}(\widetilde{\varphi}^{-1})(A_b \oplus 0), \end{aligned}$$

we obtain from (3) that $\widetilde{\Omega}_p(A^*_p) = A$.

Condition (ii). Since $L_c \circ \widetilde{\varphi}^{-1} = \widetilde{\varphi}^{-1} \circ (l_c \times \mathrm{id}_U)$ we have

$$\begin{aligned}((L_c)^*\widetilde{\Omega})_p(d_{(b,x)}\widetilde{\varphi}^{-1}(A_b \oplus X_x)) &= \widetilde{\Omega}_{cp}((d_pL_c \circ d_{(b,x)}\widetilde{\varphi}^{-1})(A_b \oplus X_x))\\ &= \widetilde{\Omega}_{cp}(d_{(b,x)}(L_c \circ \widetilde{\varphi}^{-1})(A_b \oplus X_x))\\ &= \widetilde{\Omega}_{cp}(d_{(b,x)}(\widetilde{\varphi}^{-1} \circ (l_c \times \mathrm{id}_U))(A_b \oplus X_x))\\ &= \widetilde{\Omega}_{cp}(d_{(cb,x)}\widetilde{\varphi}^{-1}(d_{(b,x)}(l_c \times \mathrm{id}_U)(A_b \oplus X_x))).\end{aligned}$$

But

$$\begin{aligned}d_{(b,x)}(l_c \times \mathrm{id}_U)(A_b \oplus X_x) &= d_{(b,x)}(l_c \times \mathrm{id}_U)([t \mapsto (a_tb, \gamma(t))])\\ &= [t \mapsto (l_c \times \mathrm{id}_U)(a_tb, \gamma(t))] = [t \mapsto (ca_tb, \gamma(t))]\\ &= [t \mapsto (ca_tc^{-1}cb, \gamma(t))]\\ &= (\mathrm{Ad}_c(A))_{cb} \oplus X_x,\end{aligned}$$

which yields

$$\begin{aligned}((L_c)^*\widetilde{\Omega})_p(d_{(b,x)}\widetilde{\varphi}^{-1}(A_b \oplus X_x)) &= \widetilde{\Omega}_{cp}(d_{(cb,x)}\widetilde{\varphi}^{-1}((\mathrm{Ad}_c(A))_{cb} \oplus X_x))\\ &= \mathrm{Ad}_c(A) + \mathrm{Ad}_{cb}(\Omega_\sigma(X_x))\\ &= \mathrm{Ad}_c(A + \mathrm{Ad}_b(\Omega_\sigma(X_x)))\\ &= \mathrm{Ad}_c(\widetilde{\Omega}_p(d_{(b,x)}\widetilde{\varphi}^{-1}(A_b \oplus X_x))).\end{aligned}$$

Now we will look for the local connection form corresponding to the local section $f \circ \sigma$:

$$\begin{aligned}((f \circ \sigma)^*\widetilde{\Omega})_x(X_x) &= \widetilde{\Omega}_{f\circ\sigma(x)}(d_x(f \circ \sigma)(X_x))\\ &= \widetilde{\Omega}_{f\circ\sigma(x)}([t \mapsto f \circ \sigma \circ \gamma(t)]) = \widetilde{\Omega}_{f\circ\sigma(x)}([t \mapsto \widetilde{\varphi}^{-1}(e_G, \gamma(t))])\\ &= \widetilde{\Omega}_{f\circ\sigma(x)}(d_{(e_G,x)}\widetilde{\varphi}^{-1}(0 \oplus X_x)) = 0 + \mathrm{Ad}_{e_G}(\Omega_\sigma(X_x))\\ &= \Omega_\sigma(X_x).\end{aligned}$$

Uniqueness of $\widetilde{\Omega}$. Let $\widetilde{\widetilde{\Omega}}$ be a connection form on P such that $(f \circ \sigma)^*\widetilde{\widetilde{\Omega}} = \Omega_\sigma$ for any local section σ of Q. We will show that $\widetilde{\widetilde{\Omega}} = \widetilde{\Omega}$.

We have

$$\widetilde{\widetilde{\Omega}}_{bf\circ\sigma(x)}(d_{(b,x)}\widetilde{\varphi}^{-1}(A_b \oplus 0)) = \widetilde{\widetilde{\Omega}}_{bf\circ\sigma(x)}(A^*_{bf\circ\sigma(x)}) = A,$$

because $\widetilde{\widetilde{\Omega}}$ satisfies the condition (i), and

$$\begin{aligned}&\widetilde{\widetilde{\Omega}}_{bf\circ\sigma(x)}(d_{(b,x)}\widetilde{\varphi}^{-1}(0 \oplus X_x))\\ &\quad= \widetilde{\widetilde{\Omega}}_{bf\circ\sigma(x)}((d_{f\circ\sigma(x)}L_b \circ d_{bf\circ\sigma(x)}L_{b^{-1}} \circ d_{(b,x)}\widetilde{\varphi}^{-1})(0 \oplus X_x))\\ &\quad= (L_b^*\widetilde{\widetilde{\Omega}})_{f\circ\sigma(x)}((d_{(b,x)}(L_{b^{-1}} \circ \widetilde{\varphi}^{-1})(0 \oplus X_x))\\ &\quad= (L_b^*\widetilde{\widetilde{\Omega}})_{f\circ\sigma(x)}((d_{(b,x)}(\widetilde{\varphi}^{-1} \circ (l_{b^{-1}} \times \mathrm{id}_U))(0 \oplus X_x))\end{aligned}$$

$$= (L_b^*\widetilde{\widetilde{\Omega}})_{f\circ\sigma(x)}((d_{(e_G,x)}\widetilde{\varphi}^{-1} \circ d_{(b,x)}(l_{b^{-1}} \times \mathrm{id}_U))(0 \oplus X_x))$$
$$= (L_b^*\widetilde{\widetilde{\Omega}})_{f\circ\sigma(x)}(d_{(e_G,x)}\widetilde{\varphi}^{-1}(0 \oplus X_x))$$
$$= (L_b^*\widetilde{\widetilde{\Omega}})_{f\circ\sigma(x)}([t \mapsto \widetilde{\varphi}^{-1}(e_G, \gamma(t))])$$
$$= (L_b^*\widetilde{\widetilde{\Omega}})_{f\circ\sigma(x)}([t \mapsto f \circ \sigma \circ \gamma(t)]) = (L_b^*\widetilde{\widetilde{\Omega}})_{f\circ\sigma(x)}(d_x(f\circ\sigma)(X_x))$$
$$= \mathrm{Ad}_b(\widetilde{\widetilde{\Omega}}_{f\circ\sigma(x)}(d_x(f\circ\sigma)(X_x))) = \mathrm{Ad}_b(((f\circ\sigma)^*\widetilde{\widetilde{\Omega}})_x(X_x))$$
$$= \mathrm{Ad}_b(\Omega_\sigma(X_x))$$

because of the condition (ii).

It follows that

$$\widetilde{\widetilde{\Omega}}_{bf\circ\sigma(x)}(d_{(b,x)}\widetilde{\varphi}^{-1}(A_b \oplus X_x))$$
$$= \widetilde{\widetilde{\Omega}}_{bf\circ\sigma(x)}(d_{(b,x)}\widetilde{\varphi}^{-1}(A_b \oplus 0)) + \widetilde{\widetilde{\Omega}}_{bf\circ\sigma(x)}(d_{(b,x)}\widetilde{\varphi}^{-1}(0 \oplus X_x))$$
$$= A + \mathrm{Ad}_b(\Omega_\sigma(X_x))$$
$$= \widetilde{\Omega}_{bf\circ\sigma(x)}(d_{(b,x)}\widetilde{\varphi}^{-1}(A_b \oplus X_x)).$$

5. Construction of the 1-form Ω_σ

We apply Proposition 4.1 to the bundle $Q(M,H)$ constructed in section 3. We assume that G is some matrix Lie group and identify $\mathcal{R}(G)$ with the related subalgebra of $\mathbf{gl}(N,\mathbb{R})$.

Our goal is to find the formula for Ω_σ. It turns out, that the three conditions:

(i) entries of Ω_σ are linear combinations of the associated to the section σ one forms ω^1, ω^2 and ω with constant coefficients, the coefficients do not depend on σ;

(ii) condition (2) from Proposition 4.2;

(iii) flatness of the connection given by $\widetilde{\Omega}$

allow us to determine Ω_σ.

Cases I^+ and I^-.

From (1) we easily obtain the local connection form for the local section $X_1 = \frac{1}{c}\partial_u$, $X_2 = \partial_v$ of Q:

$$\omega^1{}_1 = \omega^2{}_1 = \omega^2{}_2 = 0, \qquad \omega^1{}_2 = \varepsilon c u \omega^2, \qquad \omega^1 = c\,du, \qquad \omega^2 = dv.$$

If we consider another local section

$$\widetilde{X}_1 = \delta X_1, \qquad \widetilde{X}_2 = tX_1 + \delta X_2 \tag{5}$$

of Q, $\delta \in \{1,-1\}$, then the dual basis is

$$\widetilde{\omega}^1 = \delta\omega^1 - t\omega^2, \qquad \widetilde{\omega}^2 = \delta\omega^2 \tag{6}$$

and the new local connection form is

$$\widetilde{\omega}^1{}_1 = \widetilde{\omega}^2{}_1 = \widetilde{\omega}^2{}_2 = 0, \qquad \widetilde{\omega}^1{}_2 = \omega^1{}_2 + \delta\, dt. \tag{7}$$

From now on, X_1, X_2 stands for an arbitrary local section of Q, its dual basis is ω^1, ω^2 and the transformation to another basis is described by (5), (6) and (7). For abbreviation, in cases I^+ and I^- we let ω stand for $\omega^1{}_2$.

We will use

$$\iota(A) := \begin{pmatrix} A & 0 \\ 0 & I_{N-2} \end{pmatrix}, \tag{8}$$

where I_{N-2} is the $(N-2) \times (N-2)$ identity matrix.

According to the condition (i) we have

$$\Omega_\sigma = A\omega^1 + B\omega^2 + C\omega \tag{9}$$

and

$$\Omega_{\tilde{\sigma}} = A\widetilde{\omega}^1 + B\widetilde{\omega}^2 + C\widetilde{\omega} = A(\delta\omega^1 - t\omega^2) + B\delta\omega^2 + C(\omega + \delta\, dt), \tag{10}$$

with $A, B, C \in \mathbf{gl}(N, \mathbb{R})$.

Since $(\widetilde{X}_1, \widetilde{X}_2) = (X_1, X_2) \cdot a^{-1} = L_a((X_1, X_2))$ for $a = \begin{pmatrix} \delta & -t \\ 0 & \delta \end{pmatrix}$, we have $h(x) = \begin{pmatrix} \delta & -t(x) \\ 0 & \delta \end{pmatrix}$ and $\iota \circ h(x) = \begin{pmatrix} \delta & -t(x) & 0\dots0 \\ 0 & \delta & 0\dots0 \\ 0 & 0 & \\ \vdots & \vdots & I_{N-2} \\ 0 & 0 & \end{pmatrix}$. We will write it simply as $\iota \circ h(x) = \begin{pmatrix} \delta & -t(x) & 0 \\ 0 & \delta & 0 \\ 0 & 0 & I_{N-2} \end{pmatrix}$. For $G \subset GL(N, \mathbb{R})$ we have $(\vartheta_G)_b(Y_b) = Y_b b^{-1}$, hence

$$\begin{aligned} ((\iota \circ h)^*\vartheta_G)_x(X_x) &= (\vartheta_G)_{\iota\circ h(x)}(d_x(\iota \circ h)(X_x)) = (d_x(\iota \circ h)(X_x))(\iota \circ h(x))^{-1} \\ &= \begin{pmatrix} 0 & -d_x t(X_x) & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} \delta & t(x) & 0 \\ 0 & \delta & 0 \\ 0 & 0 & I_{N-2} \end{pmatrix} \\ &= \begin{pmatrix} 0 & -\delta\, d_x t(X_x) & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}. \end{aligned}$$

From (ii) we now obtain

$$\Omega_{\tilde{\sigma}} = \begin{pmatrix} \delta & -t & 0 \\ 0 & \delta & 0 \\ 0 & 0 & I_{N-2} \end{pmatrix} \Omega_\sigma \begin{pmatrix} \delta & t & 0 \\ 0 & \delta & 0 \\ 0 & 0 & I_{N-2} \end{pmatrix} + \begin{pmatrix} 0 & -\delta\, dt & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}.$$

Let $A = \begin{pmatrix} A_1 & A_2 \\ A_3 & A_4 \end{pmatrix}$, where $A_1 \in M(2,2;\mathbb{R})$, $A_2 \in M(2,N-2;\mathbb{R})$, $A_3 \in M(N-2,2;\mathbb{R})$, $A_4 \in M(N-2,N-2;\mathbb{R})$ and similarly $B = \begin{pmatrix} B_1 & B_2 \\ B_3 & B_4 \end{pmatrix}$, $C = \begin{pmatrix} C_1 & C_2 \\ C_3 & C_4 \end{pmatrix}$, $\Omega_\sigma = \begin{pmatrix} \Omega_1 & \Omega_2 \\ \Omega_3 & \Omega_4 \end{pmatrix}$, $\Omega_{\tilde{\sigma}} = \begin{pmatrix} \widehat{\Omega}_1 & \widehat{\Omega}_2 \\ \widehat{\Omega}_3 & \widehat{\Omega}_4 \end{pmatrix}$.

It is easy to check that

$$\widehat{\Omega}_1 = \begin{pmatrix} \delta & -t \\ 0 & \delta \end{pmatrix} \Omega_1 \begin{pmatrix} \delta & t \\ 0 & \delta \end{pmatrix} + \begin{pmatrix} 0 & -\delta\,dt \\ 0 & 0 \end{pmatrix},$$

$$\widehat{\Omega}_2 = \begin{pmatrix} \delta & -t \\ 0 & \delta \end{pmatrix} \Omega_2, \qquad \widehat{\Omega}_3 = \Omega_3 \begin{pmatrix} \delta & t \\ 0 & \delta \end{pmatrix}, \qquad \widehat{\Omega}_4 = \Omega_4.$$

We consider now the first block. Using (9) and (10) we obtain

$$\begin{aligned} &A_1(\delta\omega^1 - t\omega^2) + B_1\delta\omega^2 + C_1(\omega + \delta\,dt) \\ &\quad= \begin{pmatrix} \delta & -t \\ 0 & \delta \end{pmatrix} (A_1\omega^1 + B_1\omega^2 + C_1\omega) \begin{pmatrix} \delta & t \\ 0 & \delta \end{pmatrix} + \begin{pmatrix} 0 & -\delta\,dt \\ 0 & 0 \end{pmatrix} \end{aligned} \tag{11}$$

for every function t and for every $\delta \in \{1,-1\}$. For $t \equiv 0$, $\delta = -1$ we obtain $-A_1\omega^1 - B_1\omega^2 + C_1\omega = A_1\omega^1 + B_1\omega^2 + C_1\omega$ which implies $A_1\omega^1 + B_1\omega^2 = 0$. Computing the left-hand side on X_1 and X_2 succesively, we obtain $A_1 = 0$ and $B_1 = 0$.

Let $C_1 = \begin{pmatrix} c_{11} & c_{12} \\ c_{21} & c_{22} \end{pmatrix}$. From (11) we obtain

$$\begin{pmatrix} -c_{21}\delta t & -c_{21}t^2 + (c_{11} - c_{22})\delta t \\ 0 & c_{21}\delta t \end{pmatrix} \omega - \begin{pmatrix} c_{11} & c_{12} + 1 \\ c_{21} & c_{22} \end{pmatrix} \delta\,dt = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$$

for every function t and for every $\delta \in \{1,-1\}$. In particular, for every constant t we obtain $c_{21}\delta t = 0$ and $c_{21}t^2 + (c_{22} - c_{11})\delta t = 0$ because $\omega \neq 0$. It follows that $c_{21} = 0$ and $c_{22} = c_{11}$. Now we have $\begin{pmatrix} c_{11} & c_{12} + 1 \\ 0 & c_{11} \end{pmatrix} \delta\,dt = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$ for every t and δ, which implies $c_{11} = 0$, $c_{12} = -1$ and finally $C_1 = \begin{pmatrix} 0 & -1 \\ 0 & 0 \end{pmatrix}$.

A similar method applied to other blocks of $\Omega_{\tilde{\sigma}}$ gives

$$C_2 = 0, \qquad A_2 = \begin{pmatrix} \alpha_1 & \alpha_2 & \dots & \alpha_{N-2} \\ 0 & 0 & \dots & 0 \end{pmatrix}, \qquad B_2 = \begin{pmatrix} \beta_1 & \beta_2 & \dots & \beta_{N-2} \\ \alpha_1 & \alpha_2 & \dots & \alpha_{N-2} \end{pmatrix},$$

$$C_3 = 0, \qquad A_3 = \begin{pmatrix} 0 & \gamma_1 \\ 0 & \gamma_2 \\ \vdots & \vdots \\ 0 & \gamma_{N-2} \end{pmatrix}, \qquad B_3 = \begin{pmatrix} -\gamma_1 & \delta_1 \\ -\gamma_2 & \delta_2 \\ \vdots & \vdots \\ -\gamma_{N-2} & \delta_{N-2} \end{pmatrix}$$

and $A_4 = B_4 = C_4 = 0$.

We consider now condition (iii). A connection is flat if and only if the $\mathcal{R}(G)$-valued connection form $\widetilde{\Omega}$ satisfies the condition

$$d\widetilde{\Omega}(Z,W) + [\widetilde{\Omega}(Z), \widetilde{\Omega}(W)]_{\mathcal{R}(G)} = 0$$

for all vector fields Z, W on P, which is equivalent to

$$d\Omega_\sigma(X,Y) + [\Omega_\sigma(X), \Omega_\sigma(Y)]_{\mathcal{R}(G)} = 0$$

for all σ and for all vector fields X, Y on M. If G is a matrix group, then for $A, B \in \mathcal{R}(G)$, $[A,B]_{\mathcal{R}(G)} = -AB + BA = -[A,B]$. Using the matrix external product we may also write the zero curvature condition as

$$d\Omega_\sigma - \Omega_\sigma \wedge \Omega_\sigma = 0.$$

It is easy to obtain from (9)

$$\begin{aligned} &d\Omega_\sigma(X,Y) - [\Omega_\sigma(X), \Omega_\sigma(Y)] \\ &\quad = A\,d\omega^1(X,Y) + B\,d\omega^2(X,Y) + C\,d\omega(X,Y) \\ &\qquad - [A,B]\omega^1 \wedge \omega^2(X,Y) - [A,C]\omega^1 \wedge \omega(X,Y) - [B,C]\omega^2 \wedge \omega(X,Y). \end{aligned}$$

From the structural equations

$$d\omega^1 = -\omega \wedge \omega^2, \qquad d\omega^2 = 0, \qquad d\omega = \varepsilon\omega^1 \wedge \omega^2$$

it follows that

$$d\Omega_\sigma - [\Omega_\sigma, \Omega_\sigma] = (\varepsilon C - [A,B])\omega^1 \wedge \omega^2 - [A,C]\omega^1 \wedge \omega + (A - [B,C])\omega^2 \wedge \omega.$$

But $[A,C] = 0$ and $[B,C] = A$, therefore the connection is flat if and only if $[A,B] = \varepsilon C$. It follows that $\gamma_i\alpha_j = 0$ for all $i,j \in \{1,\dots,N-2\}$ and $\sum(\alpha_i\delta_i - \beta_i\gamma_i) = -\varepsilon$.

Let $E_{jk} \in M(N,N;\mathbb{R})$ denote the matrix, whose j-th row and k-th column entry is 1 and whose all other entries are 0.

Proposition 5.1
There exists $S \in GL(N;\mathbb{R})$ such that

$$S^{-1}AS = E_{13} \quad \textit{and} \quad S^{-1}BS = E_{23} - \varepsilon E_{32} \quad \textit{and} \quad S^{-1}CS = C = -E_{12} \tag{12}$$

or

$$S^{-1}AS = E_{13} \quad \textit{and} \quad S^{-1}BS = E_{14} + E_{23} - \varepsilon E_{32} \quad \textit{and} \quad S^{-1}CS = C = -E_{12} \tag{13}$$

or

$$S^{-1}AS = E_{32} \quad \textit{and} \quad S^{-1}BS = \varepsilon E_{13} - E_{31} \quad \textit{and} \quad S^{-1}CS = C = -E_{12} \tag{14}$$

or

$$S^{-1}AS = E_{32} \quad \textit{and} \quad S^{-1}BS = \varepsilon E_{13} - E_{31} + E_{42} \quad \textit{and} \quad S^{-1}CS = C = -E_{12}. \tag{15}$$

Proof. In fact, if $\alpha_{j_0} \neq 0$ for some j_0, then $\gamma_1 = \gamma_2 = \ldots = \gamma_{N-2} = 0$ and $\sum_{i=1}^{N-2} \alpha_i \delta_i = -\varepsilon$. Let $\alpha := (\alpha_1, \ldots, \alpha_{N-2}) \in \mathbb{R}^{N-2}$, $\beta := (\beta_1, \ldots, \beta_{N-2})$, $\gamma := (\gamma_1, \ldots, \gamma_{N-2})$ and $\Delta := (\delta_1, \ldots, \delta_{N-2})$. Let $(\cdot)^\perp$ denote the orthogonal complement with respect to the standard scalar product $\langle \xi, \eta \rangle = \sum_{i=1}^{N-2} \xi_i \eta_i$ in $\mathbb{R}^{N-2}$. If α and β are linearly dependent in $\mathbb{R}^{N-2}$, then $\alpha^\perp \cap \beta^\perp = \alpha^\perp$ is an $N-3$ dimensional subspace of $\mathbb{R}^{N-2}$. Let $v_1, \ldots, v_{N-3}$ be its basis. Let $v_k =: (\xi_{1k}, \xi_{2k}, \ldots, \xi_{N-2\ k})$, $k = 1, \ldots, N-3$. For

$$S = \begin{pmatrix} 1 & -\varepsilon \sum \beta_i \delta_i & 0 & 0 & \ldots & 0 \\ 0 & 1 & 0 & 0 & \ldots & 0 \\ 0 & 0 & -\varepsilon\delta_1 & \xi_{11} & \ldots & \xi_{1\ N-3} \\ \vdots & \vdots & \vdots & \vdots & & \vdots \\ 0 & 0 & -\varepsilon\delta_{N-2} & \xi_{N-2\ 1} & \ldots & \xi_{N-2\ N-3} \end{pmatrix}$$

we easily obtain $AS = SE_{13}$, $BS = S(E_{23} - \varepsilon E_{32})$ and $CS = SC$. Since $\Delta \notin \alpha^\perp$, S is invertible and conditions (12) are satisfied.

If α and β are linearly independent, then $\dim(\alpha^\perp \cap \beta^\perp) = N - 4$. Let $v_1 =: (\xi_{11}, \ldots, \xi_{N-2\ 1}), \ldots, v_{N-4} = (\xi_{1\,N-4}, \ldots, \xi_{N-2\,N-4})$ be a basis of $\alpha^\perp \cap \beta^\perp$. The vector $w := \langle \alpha, \beta \rangle \alpha - \|\alpha\|^2 \beta$ belongs to $\alpha^\perp$ and does not belong to $\beta^\perp$, because $w \in \beta^\perp$ would imply $\langle w, w \rangle = 0$ and $w = 0$, which contradicts the linear independence of α and β. Let $\eta = \frac{w}{\langle w, \beta \rangle}$ and

$$S = \begin{pmatrix} 1 & -\varepsilon \sum \beta_i \delta_i & 0 & 0 & 0 & \ldots & 0 \\ 0 & 1 & 0 & 0 & 0 & \ldots & 0 \\ 0 & 0 & -\varepsilon\delta_1 & \eta_1 & \xi_{11} & \ldots & \xi_{1\ N-4} \\ \vdots & \vdots & \vdots & \vdots & \vdots & & \vdots \\ 0 & 0 & -\varepsilon\delta_{N-2} & \eta_{N-2} & \xi_{N-2\ 1} & \ldots & \xi_{N-2\ N-4} \end{pmatrix},$$

then S is invertible and the conditions (13) hold.

Assume now that $\alpha_1 = \alpha_2 = \ldots = \alpha_{N-2} = 0$. Then $\beta \neq 0$ and $\gamma \neq 0$, because $\sum \beta_i \gamma_i = \varepsilon$. If γ and Δ are linearly dependent, then we take an arbitrary basis $v_1 = (\xi_{11}, \ldots, \xi_{N-2\ 1}), \ldots, v_{N-3} = (\xi_{1\ N-3}, \ldots, \xi_{N-2\ N-3})$ of $\beta^\perp$ and for

$$S = \begin{pmatrix} 1 & \varepsilon \sum \beta_i \delta_i & 0 & 0 & \ldots & 0 \\ 0 & 1 & 0 & 0 & \ldots & 0 \\ 0 & 0 & \gamma_1 & \xi_{11} & \ldots & \xi_{1\ N-3} \\ \vdots & \vdots & \vdots & \vdots & & \vdots \\ 0 & 0 & \gamma_{N-2} & \xi_{N-2\ 1} & \ldots & \xi_{N-2\ N-3} \end{pmatrix}$$

we have (14). Note that $\Delta = c\gamma$ and $\sum \beta_i \gamma_i = \varepsilon$ imply $\delta_k = \varepsilon \gamma_k \sum \beta_i \delta_i$ for all $k \in \{1, \ldots, N-2\}$.

If γ and Δ are linearly independent, then let $\eta = \Delta - \varepsilon \langle \beta, \Delta \rangle \gamma$. Then $\eta \in \beta^\perp$ and $\eta \neq 0$. Therefore we can find vectors $v_1, \ldots, v_{N-4}$ such that $\eta, v_1, \ldots, v_{N-4}$ is a basis of $\beta^\perp$. We denote the coordinates of v_k in the same manner as before. For

$$S = \begin{pmatrix} 1 & \varepsilon\sum\beta_i\delta_i & 0 & 0 & 0 & \dots & 0 \\ 0 & 1 & 0 & 0 & 0 & \dots & 0 \\ 0 & 0 & \gamma_1 & \eta_1 & \xi_{11} & \dots & \xi_{1\ N-4} \\ \vdots & \vdots & \vdots & \vdots & \vdots & & \vdots \\ 0 & 0 & \gamma_{N-2} & \eta_{N-2} & \xi_{N-2\ 1} & \dots & \xi_{N-2\ N-4} \end{pmatrix}$$

the conditions (15) hold.

From Proposition 5.1 it follows that in cases I^+ and I^- there are four 1-forms associated to a locally symmetric connection:

1° $N = 3$, $\Omega_\sigma = \begin{pmatrix} 0 & -\omega & \omega^1 \\ 0 & 0 & \omega^2 \\ 0 & -\varepsilon\omega^2 & 0 \end{pmatrix}$;

2° $N = 4$, $\Omega_\sigma = \begin{pmatrix} 0 & -\omega & \omega^1 & \omega^2 \\ 0 & 0 & \omega^2 & 0 \\ 0 & -\varepsilon\omega^2 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$;

3° $N = 3$, $\Omega_\sigma = \begin{pmatrix} 0 & -\omega & \varepsilon\omega^2 \\ 0 & 0 & 0 \\ -\omega^2 & \omega^1 & 0 \end{pmatrix}$;

4° $N = 4$, $\Omega_\sigma = \begin{pmatrix} 0 & -\omega & \varepsilon\omega^2 & 0 \\ 0 & 0 & 0 & 0 \\ -\omega^2 & \omega^1 & 0 & 0 \\ 0 & \omega^2 & 0 & 0 \end{pmatrix}$.

Cases IId^+ and IId^-.

We consider two local sections $\sigma = (X_1, X_2)$ and

$$\begin{aligned} \widetilde{\sigma} &= (\widetilde{X}_1, \widetilde{X}_2) = (\cos\varphi\, X_1 + \sin\varphi\, X_2, -\sin\varphi\, X_1 + \cos\varphi\, X_2) \\ &= (X_1, X_2) \cdot \begin{pmatrix} \cos\varphi & -\sin\varphi \\ \sin\varphi & \cos\varphi \end{pmatrix} \end{aligned}$$

of the bundle of g-orthonormal frames. Since the left action of $SO(2,\mathbb{R})$ on Q is given by $L_b(q) = qb^{-1}$, we have $\widetilde{\sigma} = h\sigma$ with $h = \begin{pmatrix} \cos\varphi & \sin\varphi \\ -\sin\varphi & \cos\varphi \end{pmatrix}$.

The new dual basis is

$$\begin{aligned} \widetilde{\omega}^1 &= \cos\varphi\, \omega^1 + \sin\varphi\, \omega^2, \\ \widetilde{\omega}^2 &= -\sin\varphi\, \omega^1 + \cos\varphi\, \omega^2 \end{aligned}$$

and the new local connection form is

$$\widetilde{\omega}^2{}_1 = \omega^2{}_1 + d\varphi.$$

From now on we will write ω and $\widetilde{\omega}$ instead of $\omega^2{}_1$, $\widetilde{\omega}^2{}_1$, respectively.

According to the condition (i) we have $\Omega_\sigma = A\omega^1 + B\omega^2 + C\omega$ and

$$\begin{aligned}\Omega_{\tilde\sigma} &= A\widetilde{\omega}^1 + B\widetilde{\omega}^2 + C\widetilde{\omega}\\ &= A(\cos\varphi\,\omega^1 + \sin\varphi\,\omega^2) + B(-\sin\varphi\,\omega^1 + \cos\varphi\,\omega^2) + C(\omega + d\varphi),\end{aligned}$$

with $A, B, C \in \mathbf{sl}(2, \mathbb{R})$.

We will firstly use the homomorphism $\iota\colon SO(2,\mathbb{R}) \to SL(2,\mathbb{R})/\{I, -I\}$, where

$$\iota\left(\begin{pmatrix} \cos\varphi & \sin\varphi \\ -\sin\varphi & \cos\varphi \end{pmatrix}\right) = \left[\begin{pmatrix} \cos\left(\frac{\varphi}{2}\right) & \sin\left(\frac{\varphi}{2}\right) \\ -\sin\left(\frac{\varphi}{2}\right) & \cos\left(\frac{\varphi}{2}\right) \end{pmatrix}\right]$$

and look directly for an $\mathbf{sl}(2,\mathbb{R})$-valued 1-form Ω_σ.

Let G denote the quotient group $SL(2,\mathbb{R})/\{I, -I\}$. The canonical projection $\pi_G\colon SL(2,\mathbb{R}) \to G$ is a covering of multiplicity 2. Each point of G has a neighbourhood U such that each of two components V_1, V_2 of $\pi_G^{-1}(U)$ is homeomorphic to U under π_G. The differentiable structure in G is introduced by requiring all such $\pi_G|_{V_i}\colon V_i \to U$ to be diffeomorphisms. For every $a \in SL(2,\mathbb{R})$ the differential $d_a\pi_G\colon T_aSL(2,\mathbb{R}) \to T_{[a]}G$ is an isomorphism.

If $a \in SL(2,\mathbb{R})$ and $V_{[a]} \in T_{[a]}G$, then $\vartheta_G(V_{[a]}) = \widehat{A}_{[I]}$, where $\widehat{A} \in \mathcal{R}(G)$ satisfies the condition $V_{[a]} = \widehat{A}_{[a]} = d_I R_{[a]}(\widehat{A}_{[I]})$. Assume that we have $V_{[a]} = d_a\pi_G(W_a)$ for $W_a \in T_aSL(2,\mathbb{R})$. Let $A \in \mathcal{R}(SL(2,\mathbb{R}))$ be such that $W_a = A_a$, then

$$\begin{aligned}V_{[a]} &= d_a\pi_G(d_I R_a(A_I)) = d_I(\pi_G \circ R_a)(A_I) = d_I(R_{[a]} \circ \pi_G)(A_I)\\ &= d_{[I]}R_{[a]}(d_I\pi_G(A_I)).\end{aligned}$$

It follows that $\widehat{A}_{[I]} = d_I\pi_G(A_I)$, where $A_I = \vartheta_{SL(2,\mathbb{R})}(W_a) = W_a a^{-1}$.

For $x \in M$ and $X_x \in T_xM$ we have locally

$$d_x(\iota \circ h)(X_x) = d_{\alpha(x)}\pi_G(d_x\alpha(X_x)),$$

where

$$\alpha = \begin{pmatrix} \cos\left(\frac{\varphi}{2}\right) & \sin\left(\frac{\varphi}{2}\right) \\ -\sin\left(\frac{\varphi}{2}\right) & \cos\left(\frac{\varphi}{2}\right) \end{pmatrix}.$$

It follows that

$$\begin{aligned}((\iota \circ h)^*\vartheta_G)_x(X_x) &= (\vartheta_G)_{(\iota\circ h)(x)}(d_x(\iota \circ h)(X_x))\\ &= (\vartheta_G)_{[\alpha(x)]}(d_{\alpha(x)}\pi_G(d_x\alpha(X_x)))\\ &= d_I\pi_G(\vartheta_{SL(2,\mathbb{R})}(d_x\alpha(X_x)))\\ &= d_I\pi_G(d_x\alpha(X_x)\,(\alpha(x))^{-1}).\end{aligned}$$

Let $U \subset G$ be a neighbourhood of $[I]$ such that $\pi_G^{-1}(U) = V_1 \cup V_2$, with V_1 and V_2 diffeomorphic to U under π_G. Let $I \in V_1$. Since $V_1 \subset SL(2,\mathbb{R}) \subset GL(2,\mathbb{R}) \subset \mathbb{R}^4$, we may replace every tangent vector $[\gamma]_\sim \in T_{[I]}G$ by $\frac{d}{dt}((\pi_G|_{V_1})^{-1} \circ \gamma)|_{t=0}$ and $[\delta]_\sim \in T_I SL(2,\mathbb{R})$ by $\frac{d}{dt}\delta|_{t=0}$. In this way we identify $T_{[I]}G$ and $T_I SL(2,\mathbb{R})$ with the subalgebra $\mathbf{sl}(2,\mathbb{R})$ of $\mathbf{gl}(2,\mathbb{R})$. After such identification $d_I\pi_G = \mathrm{id}_{\mathbf{sl}(2,\mathbb{R})}$ and we have simply

$$\begin{aligned}
&((\iota \circ h)^*\vartheta_G)_x(X_x) \\
&\quad = d_x\alpha(X_x)(\alpha(x))^{-1} \\
&\quad = \frac{1}{2}X_x(\varphi)\begin{pmatrix} -\sin(\frac{\varphi(x)}{2}) & \cos(\frac{\varphi(x)}{2}) \\ -\cos(\frac{\varphi(x)}{2}) & -\sin(\frac{\varphi(x)}{2}) \end{pmatrix}\begin{pmatrix} \cos(\frac{\varphi(x)}{2}) & -\sin(\frac{\varphi(x)}{2}) \\ \sin(\frac{\varphi(x)}{2}) & \cos(\frac{\varphi(x)}{2}) \end{pmatrix} \\
&\quad = \frac{1}{2}X_x(\varphi)\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}.
\end{aligned}$$

Consequently

$$(\iota \circ h)^*\vartheta_G = \frac{1}{2}\begin{pmatrix} 0 & d\varphi \\ -d\varphi & 0 \end{pmatrix}.$$

Similar considerations lead to

$$\mathrm{Ad}_{[a]}(d_I\pi_G(B_I)) = d_I\pi_G(\mathrm{Ad}_a(B_I))$$

and, after the identification of $T_{[I]}G$ and $T_I SL(2,\mathbb{R})$ with $\mathbf{sl}(2,\mathbb{R})$, to

$$\begin{aligned}
&\mathrm{Ad}_{(\iota\circ h)(x)}(\Omega_\sigma(X_x)) \\
&\quad = \alpha(x)\Omega_\sigma(X_x)(\alpha(x))^{-1} \\
&\quad = \begin{pmatrix} \cos(\frac{\varphi(x)}{2}) & \sin(\frac{\varphi(x)}{2}) \\ -\sin(\frac{\varphi(x)}{2} & \cos(\frac{\varphi(x)}{2}) \end{pmatrix}\Omega_\sigma(X_x)\begin{pmatrix} \cos(\frac{\varphi(x)}{2}) & -\sin(\frac{\varphi(x)}{2}) \\ \sin(\frac{\varphi(x)}{2}) & \cos(\frac{\varphi(x)}{2}) \end{pmatrix}.
\end{aligned}$$

According to the condition (ii), for any function φ we have

$$\begin{aligned}
&A(\cos\varphi\,\omega^1 + \sin\varphi\,\omega^2) + B(-\sin\varphi\,\omega^1 + \cos\varphi\,\omega^2) + C(\omega + d\varphi) \\
&\quad = \begin{pmatrix} \cos(\frac{\varphi}{2}) & \sin(\frac{\varphi}{2}) \\ -\sin(\frac{\varphi}{2}) & \cos(\frac{\varphi}{2}) \end{pmatrix}(A\omega^1 + B\omega^2 + C\omega)\begin{pmatrix} \cos(\frac{\varphi}{2}) & -\sin(\frac{\varphi}{2}) \\ \sin(\frac{\varphi}{2}) & \cos(\frac{\varphi}{2}) \end{pmatrix} \qquad (16) \\
&\qquad + \frac{1}{2}\begin{pmatrix} 0 & d\varphi \\ -d\varphi & 0 \end{pmatrix}.
\end{aligned}$$

Taking $\varphi \equiv \pi = 3,14\ldots$ we obtain

$$-A\omega^1 - B\omega^2 + C\omega = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}(A\omega^1 + B\omega^2 + C\omega)\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}.$$

Let $A = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & -a_{11} \end{pmatrix}$, $B = \begin{pmatrix} b_{11} & b_{12} \\ b_{21} & -b_{11} \end{pmatrix}$ and $C = \begin{pmatrix} c_{11} & c_{12} \\ c_{21} & -c_{11} \end{pmatrix}$. Then

$$\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} A \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} = \begin{pmatrix} -a_{11} & -a_{21} \\ -a_{12} & a_{11} \end{pmatrix},$$

similarly for B and C. We have

$$\begin{pmatrix} -a_{11} & -a_{12} \\ -a_{21} & a_{11} \end{pmatrix} \omega^1 + \begin{pmatrix} -b_{11} & -b_{12} \\ -b_{21} & b_{11} \end{pmatrix} \omega^2 + \begin{pmatrix} c_{11} & c_{12} \\ c_{21} & -c_{11} \end{pmatrix} \omega$$
$$= \begin{pmatrix} -a_{11} & -a_{21} \\ -a_{12} & a_{11} \end{pmatrix} \omega^1 + \begin{pmatrix} -b_{11} & -b_{21} \\ -b_{12} & b_{11} \end{pmatrix} \omega^2 + \begin{pmatrix} -c_{11} & -c_{21} \\ -c_{12} & c_{11} \end{pmatrix} \omega$$

which implies $2c_{11}\omega = 0$,

$$(a_{21} - a_{12})\omega^1 + (b_{21} - b_{12})\omega^2 + (c_{12} + c_{21})\omega = 0,$$
$$(a_{12} - a_{21})\omega^1 + (b_{12} - b_{21})\omega^2 + (c_{12} + c_{21})\omega = 0.$$

Adding and subtracting the last two equations we obtain $(c_{12} + c_{21})\omega = 0$ and $(a_{12} - a_{21})\omega^1 + (b_{12} - b_{21})\omega^2 = 0$. It follows that

$$A = \begin{pmatrix} a_{11} & a_{12} \\ a_{12} & -a_{11} \end{pmatrix}, \qquad B = \begin{pmatrix} b_{11} & b_{12} \\ b_{12} & -b_{11} \end{pmatrix}, \qquad C = \begin{pmatrix} 0 & c_{12} \\ -c_{12} & 0 \end{pmatrix}.$$

If we insert such A, B and C into (17), then we obtain for an arbitrary function φ

$$\sin\varphi(-(b_{11} + a_{12})\omega^1 + (a_{11} - b_{12})\omega^2) = 0$$

and

$$\sin\varphi((-b_{12} + a_{11})\omega^1 + (a_{12} + b_{11})\omega^2) + \left(c_{12} - \frac{1}{2}\right) d\varphi = 0.$$

From the first equation we obtain $b_{11} = -a_{12}$ and $b_{12} = a_{11}$, then from the second equation it follows that $c_{12} = \frac{1}{2}$. We have now

$$\Omega_\sigma = \begin{pmatrix} \alpha & \beta \\ \beta & -\alpha \end{pmatrix} \omega^1 + \begin{pmatrix} -\beta & \alpha \\ \alpha & \beta \end{pmatrix} \omega^2 + \begin{pmatrix} 0 & \frac{1}{2} \\ -\frac{1}{2} & 0 \end{pmatrix} \omega.$$

The zero-curvature condition $d\Omega_\sigma - \Omega_\sigma \wedge \Omega_\sigma = 0$ and the structural equations

$$d\omega^1 = \omega \wedge \omega^2,$$
$$d\omega^2 = -\omega \wedge \omega^1,$$
$$d\omega = -\frac{\varepsilon}{\rho^2}\, \omega^1 \wedge \omega^2$$

yield

$$\alpha^2 + \beta^2 = -\frac{\varepsilon}{4\rho^2}.$$

Recall that $\kappa = \frac{\varepsilon}{\rho^2}$.

It follows that this method of finding an $\mathbf{sl}(2, \mathbb{R})$-valued Ω_σ is effective only in the case of constant negative curvature, i.e. $\varepsilon = -1$.

Let $\varepsilon = -1$. We have $\alpha = \frac{\sin\xi}{2\rho}$ and $\beta = \frac{\cos\xi}{2\rho}$ for some $\xi \in \mathbb{R}$. Let $S = \begin{pmatrix} \cos\frac{\xi}{2} & \sin\frac{\xi}{2} \\ -\sin\frac{\xi}{2} & \cos\frac{\xi}{2} \end{pmatrix}$. Then $S^{-1}AS = \frac{1}{2\rho}\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$, $S^{-1}BS = \frac{1}{2\rho}\begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}$, $S^{-1}CS = \frac{1}{2}\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$ and

$$\Omega_\sigma = \begin{pmatrix} -\frac{1}{2\rho}\omega^2 & \frac{1}{2\rho}\omega^1 + \frac{1}{2}\omega^2{}_1 \\ \frac{1}{2\rho}\omega^1 - \frac{1}{2}\omega^2{}_1 & \frac{1}{2\rho}\omega^2 \end{pmatrix}.$$

In case $\kappa = -1$ we have $\rho = 1$ and Ω_σ is the well known form of Sasaki.

Now we consider the cases IId^+ and IId^- again, using the homomorphism (8). We have now

$$\iota \circ h = \begin{pmatrix} \cos\varphi & \sin\varphi & 0 \\ -\sin\varphi & \cos\varphi & 0 \\ 0 & 0 & I_{N-2} \end{pmatrix}$$

and

$$\begin{aligned} ((\iota \circ h)^* \vartheta_G)_x(X_x) &= (d_x(\iota \circ h)(X_x))(\iota \circ h(x))^{-1} \\ &= \begin{pmatrix} -\sin\varphi & \cos\varphi & 0 \\ -\cos\varphi & -\sin\varphi & 0 \\ 0 & 0 & 0 \end{pmatrix} d_x\varphi(X_x) \begin{pmatrix} \cos\varphi & -\sin\varphi & 0 \\ \sin\varphi & \cos\varphi & 0 \\ 0 & 0 & I_{N-2} \end{pmatrix} \\ &= \begin{pmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} d_x\varphi(X_x). \end{aligned}$$

It follows from (ii) that

$$\begin{aligned} &A(\cos\varphi\,\omega^1 + \sin\varphi\,\omega^2) + B(-\sin\varphi\,\omega^1 + \cos\varphi\,\omega^2) + C(\omega + d\varphi) \\ &\quad = \begin{pmatrix} \cos\varphi & \sin\varphi & 0 \\ -\sin\varphi & \cos\varphi & 0 \\ 0 & 0 & I_{N-2} \end{pmatrix} (A\omega^1 + B\omega^2 + C\omega) \begin{pmatrix} \cos\varphi & -\sin\varphi & 0 \\ \sin\varphi & \cos\varphi & 0 \\ 0 & 0 & I_{N-2} \end{pmatrix} \\ &\qquad + \begin{pmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} d\varphi \end{aligned} \tag{17}$$

for an arbitrary function φ. Similarly as in case I we divide A, B, C into four blocks. If we write (18) with $\varphi \equiv \pi$, then we obtain easily $A_1 = 0$, $A_4 = 0$, $B_1 = 0$, $B_4 = 0$, $C_2 = 0$ and $C_3 = 0$. Writing (18) for an arbitrary φ again and comparing $(\cdot)_{11} + (\cdot)_{22}$ of both sides gives $c_{11} + c_{22} = 0$, comparing $(\cdot)_{12} - (\cdot)_{21}$ gives $c_{12} - c_{21} = 2$. Next we consider $(\cdot)_{12} + (\cdot)_{21}$ with $\varphi \equiv \frac{\pi}{4}$ and with $\varphi = \frac{\pi}{2}$, which gives $c_{12} + c_{21} - 2c_{11} = 0$ and $c_{12} + c_{21} = 0$. If we compute $(\cdot)_{ij}$ with $i, j > 2$ on both sides of (18), then we obtain $c_{ij}\,d\varphi = 0$ for an arbitrary φ, which implies $c_{ij} = 0$. In a similar way we consider the upper right block and the lower left block. We obtain $b_{1j} = -a_{2j}$, $b_{2j} = a_{1j}$, $b_{j1} = -a_{j2}$ and $b_{j2} = a_{j1}$ for $j > 2$. Now it is easy to check that $[B, C] = -A$ and $[A, C] = B$. The only possibly non-zero term in $d\Omega_\sigma - \Omega_\sigma \wedge \Omega_\sigma$, after we have used the structural equations, is equal to $(-\frac{\varepsilon}{\rho^2}C - [A, B])\omega^1 \wedge \omega^2$. Consequently, the connection associated with Ω_σ is flat if and only if $[A, B] = -\frac{\varepsilon}{\rho^2}C$. If we write $(\cdot)_{12}$ of this equality, then we obtain

$$\sum_{k=3}^{N} a_{1k}a_{k1} + \sum_{k=3}^{N} a_{2k}a_{k2} = -\frac{\varepsilon}{\rho^2}, \tag{18}$$

whereas $(\cdot)_{kl}$ with $k,l>2$ gives

$$a_{k2}a_{1l}-a_{k1}a_{2l}=0 \qquad \text{for all } k,l>2. \tag{19}$$

We will show that either

$$a_{1i}=a_{i1}=0 \quad \text{for all } i>2 \qquad \text{and} \qquad \sum_{k=3}^{N} a_{2k}a_{k2}=-\frac{\varepsilon}{\rho^2} \tag{20}$$

or there exists $\lambda\in\mathbb{R}$ such that

$$a_{2i}=\lambda a_{1i},\ a_{i2}=\lambda a_{i1} \quad \text{for all } i>2 \qquad \text{and} \qquad \sum_{k=3}^{N} a_{1k}a_{k1}=-\frac{\varepsilon}{\rho^2}\frac{1}{1+\lambda^2}. \tag{21}$$

Indeed, if $a_{1i}=0$ for all $i>2$, then (18) implies that $a_{2l_0}\neq 0$ for some l_0 and we obtain from (19) $a_{k1}=\frac{a_{k2}a_{1l_0}}{a_{2l_0}}=0$ for all $k>2$. Similarly, if $a_{i1}=0$ for all $i>2$, then $a_{1i}=0$ for all $i>2$. Assume now that $a_{1i_0}\neq 0$, then $a_{k_01}\neq 0$ for some k_0. From (19) we obtain $a_{k2}=\lambda a_{k1}$ for all $k>2$ with $\lambda=\frac{a_{2i_0}}{a_{1i_0}}$. Using (19) again gives $\lambda a_{k1}a_{1l}=a_{k1}a_{2l}$ for all $k,l>2$, in particular for $k=k_0$.

If $a_{1i}=a_{i1}=0$ for all $i>2$, then we take the basis $v_4,\ldots,v_N$ of the subspace in $\mathbb{R}^{N-2}$ orthogonal to the non-zero vector $(a_{23},\ldots,a_{2N})$. Let $v_k=:(s_{3k},s_{4k},\ldots,s_{Nk})$ for $k=4,\ldots,N$. Let $s_{k3}=-\frac{\rho}{\varepsilon}a_{k2}$. From (18) it follows that $v_3,v_4,\ldots,v_N$ with $v_3:=(s_{33},s_{43},\ldots,s_{N3})$ are also linearly independent. We take $s_{11}=s_{22}=0$, $s_{12}=-s_{21}=1$ and $s_{1k}=s_{2k}=s_{k1}=s_{k2}=0$ for $k>2$.

If $w:=(a_{13},a_{14},\ldots,a_{1N})\neq 0\in\mathbb{R}^{N-2}$, then we take the basis $v_4,\ldots,v_N$ of $w^\perp$ in $\mathbb{R}^{N-2}$ and define $(s_{3k},s_{4k},\ldots,s_{Nk}):=v_k$ for $k\geq 4$. Let $s_{k3}:=\frac{\rho}{\varepsilon}(1+\lambda^2)a_{k1}$ for $k>2$. Then $v_3:=(s_{33},s_{43},\ldots,s_{N3})$ is not in $w^\perp$ and $v_3,v_4,\ldots,v_N$ are linearly independent. Let $s_{11}=s_{22}=1$, $s_{21}=-s_{12}=\lambda$ and $s_{1k}=s_{2k}=s_{k1}=s_{k2}=0$ for $k>2$.

In both cases S is invertible and it is easy to check that $AS=SA_0$, $BS=SB_0$ and $CS=SC$ with

$$A_0=-\frac{1}{\rho}E_{13}+\frac{\varepsilon}{\rho}E_{31}, \qquad B_0=-\frac{1}{\rho}E_{23}+\frac{\varepsilon}{\rho}E_{32}.$$

It follows that it suffices to consider the case $N=3$ and the 1-form

$$\Omega_\sigma=\begin{pmatrix} 0 & {\omega^2}_1 & -\frac{1}{\rho}\omega^1 \\ -{\omega^2}_1 & 0 & -\frac{1}{\rho}\omega^2 \\ \frac{\varepsilon}{\rho}\omega^1 & \frac{\varepsilon}{\rho}\omega^2 & 0 \end{pmatrix}.$$

Case IIi.

We consider two local sections of Q, (X_1,X_2) and

$$(\widetilde{X}_1,\widetilde{X}_2)=(\delta\cosh\varphi\, X_1+\delta\sinh\varphi\, X_2, \delta\sinh\varphi\, X_1+\delta\cosh\varphi\, X_2),$$

with $\delta \in \{1,-1\}$. For a local basis satisfying the conditions $g(X_1,X_1) = 1$, $g(X_1,X_2) = 0$ and $g(X_2,X_2) = -1$ the local connection form is $(\omega^i{}_j) = \begin{pmatrix} 0 & \omega^2{}_1 \\ \omega^2{}_1 & 0 \end{pmatrix}$. We denote $\omega^2{}_1$ by ω. The structural equations are

$$d\omega^1 = -\omega \wedge \omega^2, \qquad d\omega^2 = -\omega \wedge \omega^1, \qquad d\omega = -\kappa\, \omega^1 \wedge \omega^2.$$

The new dual basis and the new local connection form are

$$\begin{aligned} \widetilde{\omega}^1 &= \delta \cosh\varphi\, \omega^1 - \delta \sinh\varphi\, \omega^2, \\ \widetilde{\omega}^2 &= -\delta \sinh\varphi\, \omega^1 + \delta \cosh\varphi\, \omega^2, \\ \widetilde{\omega} &= \omega + d\varphi. \end{aligned}$$

The transition function is

$$h = \begin{pmatrix} \delta\cosh\varphi & -\delta\sinh\varphi \\ -\delta\sinh\varphi & \delta\cosh\varphi \end{pmatrix}$$

and its composition with $\iota\colon SO(1,1) \to GL(N,\mathbb{R})$ is

$$\iota \circ h = \begin{pmatrix} \delta\cosh\varphi & -\delta\sinh\varphi & 0 \\ -\delta\sinh\varphi & \delta\cosh\varphi & 0 \\ 0 & 0 & I_{N-2} \end{pmatrix}.$$

It follows that for $x \in M$, $X_x \in T_xM$

$$\begin{aligned} &((\iota\circ h)^*\vartheta_G)_x(X_x) \\ &\quad = \vartheta_{G\,\iota\circ h(x)}(d_x(\iota\circ h)(X_x)) = d_x(\iota\circ h)(X_x)(\iota\circ h(x))^{-1} \\ &\quad = \begin{pmatrix} \delta\sinh\varphi & -\delta\cosh\varphi & 0 \\ -\delta\cosh\varphi & \delta\sinh\varphi & 0 \\ 0 & 0 & 0 \end{pmatrix} d\varphi(X_x) \begin{pmatrix} \delta\cosh\varphi & \delta\sinh\varphi & 0 \\ \delta\sinh\varphi & \delta\cosh\varphi & 0 \\ 0 & 0 & I_{N-2} \end{pmatrix} \\ &\quad = \begin{pmatrix} 0 & -1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} d\varphi(X_x). \end{aligned}$$

We now look for A, B, C such that for all $\delta \in \{1,-1\}$ and for every function φ

$$\begin{aligned} &A(\delta\cosh\varphi\,\omega^1 - \delta\sinh\varphi\,\omega^2) + B(-\delta\sinh\varphi\,\omega^1 + \delta\cosh\varphi\,\omega^2) + C(\omega + d\varphi) \\ &\quad = \begin{pmatrix} \delta\cosh\varphi & -\delta\sinh\varphi & 0 \\ -\delta\sinh\varphi & \delta\cosh\varphi & 0 \\ 0 & 0 & I_{N-2} \end{pmatrix} (A\omega^1 + B\omega^2 + C\omega) \\ &\qquad \times \begin{pmatrix} \delta\cosh\varphi & \delta\sinh\varphi & 0 \\ \delta\sinh\varphi & \delta\cosh\varphi & 0 \\ 0 & 0 & I_{N-2} \end{pmatrix} + \begin{pmatrix} 0 & -1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} d\varphi. \end{aligned}$$

Analysis similar to that in the cases I and IId shows that $a_{ij} = 0$ and $b_{ij} = 0$ for $(i, j) \in (\{1, 2\} \times \{1, 2\}) \cup (\{3, \ldots, N\} \times \{3, \ldots, N\})$, $b_{1k} = a_{2k}$, $b_{2k} = a_{1k}$, $b_{k1} = -a_{k2}$, $b_{k2} = -a_{k1}$ for $k > 2$, and $C = -E_{12} - E_{21}$. Since $[A, C] = B$ and $[B, C] = A$, we have

$$d\Omega_\sigma - \Omega_\sigma \wedge \Omega_\sigma = (-\kappa C - [A, B])\omega^1 \wedge \omega^2.$$

The connection is flat if and only if $[A, B] = -\kappa C$. In particular $([A, B])_{12} = -\kappa c_{12}$ and $([A, B])_{kl} = -\kappa c_{kl}$ for all $k, l > 2$, which implies

$$-\sum_{j=3}^{N} a_{1j}a_{j1} - \sum_{j=3}^{N} a_{2j}a_{j2} = \kappa$$

and

$$a_{k1}a_{2l} = -a_{k2}a_{1l}$$

for all $k, l > 2$. It follows that either $a_{1i} = a_{i1} = 0$ for all $i > 2$, or for all $i > 2$, $a_{i2} = \lambda a_{i1}$ and $a_{2i} = -\lambda a_{1i}$ with some $\lambda \notin \{1, -1\}$. In both cases it is easy to find an automorphism S of $\mathbb{R}^N$ such that $S^{-1}AS = -\frac{1}{\rho}E_{13} + \frac{\varepsilon}{\rho}E_{31}$, $S^{-1}BS = -\frac{1}{\rho}E_{23} - \frac{\varepsilon}{\rho}E_{32}$ and $S^{-1}CS = C = -E_{12} - E_{21}$, where $\varepsilon \in \{1, -1\}$ and $\rho > 0$ are such that $\kappa = \frac{\varepsilon}{\rho^2}$. The corresponding $\mathbf{sl}(3, \mathbb{R})$ valued 1-form Ω_σ is

$$\Omega_\sigma = \begin{pmatrix} 0 & -\omega & -\frac{1}{\rho}\omega^1 \\ -\omega & 0 & -\frac{1}{\rho}\omega^2 \\ \frac{\varepsilon}{\rho}\omega^1 & -\frac{\varepsilon}{\rho}\omega^2 & 0 \end{pmatrix}.$$

6. Summary

For any two-dimensional manifold M with locally symmetric linear connection ∇ and with ∇-parallel volume element vol one can construct a flat connection. Its local connection forms Ω_σ are build of the dual basis forms ω^1, ω^2 and a local connection form of ∇. The structural equations of the surface are equivalent to the zero-curvature condition $d\Omega_\sigma - \Omega_\sigma \wedge \Omega_\sigma = 0$. The corresponding Lie algebras $\mathbf{g}$ may differ from case to case depending on algebraic properties of the curvature tensor.

If a locally symmetric surface is associated to every solution of some differential equation, then such 1-form Ω_σ constitutes a $\mathbf{g}$-valued zero-curvature representation of this equation.

References

[1] J. Gancarzewicz, *Zarys współczesnej geometrii różniczkowej*, Script, Warszawa 2010. Cited on 24.

[2] S. Kobayashi, K. Nomizu, *Foundations of differential geometry*, vol. I, Interscience Publishers, a division of John Wiley & Sons, New York-London 1963. Cited on 22.

[3] M. Marvan, *On the spectral parameter problem*, Acta Appl. Math. **109** (2010), no. 1, 239–255. Cited on 20.

[4] K. Nomizu, T. Sasaki, *Affine differential geometry. Geometry of affine immersions.* Cambridge Tracts in Mathematics **111**, Cambridge University Press, Cambridge, 1994. Cited on 22.

[5] B. Opozda, *Locally symmetric connections on surfaces*, Results Math. **20** (1991), no. 3-4, 725–743. Cited on 21.

[6] B. Opozda, *Some relations between Riemannian and affine geometry*, Geom. Dedicata **47** (1993), no. 2, 225–236. Cited on 21 and 22.

[7] R. Sasaki, *Soliton equations and pseudospherical surfaces*, Nuclear Phys. B **154** (1979), no. 2, 343–357. Cited on 19 and 20.

[8] C.L. Terng, *Geometric transformations and soliton equations*, Handbook of geometric analysis, No. 2, 301–358, Adv. Lect. Math. (ALM), 13, Int. Press, Somerville, MA, 2010. Cited on 20.

[9] E. Wang, *Tzitzéica transformation is a dressing action*, J. Math. Phys. **47** (2006), no. 5, 053502, 13 pp. Cited on 20.

Institute of Mathematics
Pedagogical University
Podchorążych 2
30-084 Kraków
Poland
E-mail: robaszew@up.krakow.pl

Received: February 2, 2014; final version: May 4, 2014;
available online: June 30, 2014.