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On the differential first-order invariants for the non-splitting subgroups of the generalized Poincare group $P(1,4)$

Dedicated to Professor Andrzej Zajtz on his seventieth birthday

Abstract. The functional bases of the differential first-order invariants for all non-splitting subgroups of the generalized Poincaré group $P(1,4)$ are constructed. Some applications of the results obtained are considered.

Introduction

The developement of the theoretical physics has required various extensions of the four–dimensional Minkowski space and, correspondingly, various extensions of the Poincaré group $P(1,3)$. The natural extension of this group is the generalized Poincaré group $P(1,4)$. The group $P(1,4)$ is the group of rotations and translations of the five-dimensional Minkowski space $M(1,4)$. This group has many applications in theoretical and mathematical physics [1-3]. The group $P(1,4)$ has many subgroups used in theoretical physics [4-8]. Among these subgroups there are the Poincaré group $P(1,3)$ and the extended Galilei group $\widetilde{G}(1,3)$ (see also [9]). Thus, the results obtained with the help of the subgroup structure of the group $P(1,4)$ will be useful in relativistic and non-relativistic physics.

The papers [10-12] are devoted to the construction of the first-order differential invariants for the splitting subgroups [4, 5, 7] of the generalized Poincaré group $P(1,4)$.

The present paper is devoted to the construction of functional bases of the differential first-order invariants for the non-splitting subgroups [4, 6-8] of the group $P(1,4)$.

Our paper is organized as follows. In the first section we introduce some notation and results concerning the Lie algebra of the group $P(1,4)$ which we use in the following. Sections 2 and 3 present our main results.

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1. The Lie algebra of the group $P(1,4)$ and its non-conjugate subalgebras

The Lie algebra of the group $P(1,4)$ is given by the 15 basis elements $M_{\mu\nu} = -M_{\nu\mu}$ $(\mu,\nu = 0,1,2,3,4)$ and P'_μ $(\mu = 0,1,2,3,4)$, satisfying the commutation relations

$$\begin{aligned} \left[P'_\mu, P'_\nu\right] &= 0, \\ \left[M'_{\mu\nu}, P'_\sigma\right] &= g_{\mu\sigma}P'_\nu - g_{\nu\sigma}P'_\mu, \\ \left[M'_{\mu\nu}, M'_{\rho\sigma}\right] &= g_{\mu\rho}M'_{\nu\sigma} + g_{\nu\sigma}M'_{\mu\rho} - g_{\nu\rho}M'_{\mu\sigma} - g_{\mu\sigma}M'_{\nu\rho}, \end{aligned}$$

where $g_{00} = -g_{11} = -g_{22} = -g_{33} = -g_{44} = 1$, $g_{\mu\nu} = 0$, if $\mu \neq \nu$. Here, and in what follows, $M'_{\mu\nu} = iM_{\mu\nu}$.

In order to study the subgroup structure of the group $P(1,4)$ we used the method proposed in [13]. Continuous subgroups of the group $P(1,4)$ have been described in [4–8].

Further we will use the following basis elements:

$$\begin{aligned} &G = M'_{40}, && L_1 = M'_{32}, \quad L_2 = -M'_{31}, \quad L_3 = M'_{21}, \\ &P_a = M'_{4a} - M'_{a0}, && C_a = M'_{4a} + M'_{a0}, \quad (a = 1,2,3), \\ &X_0 = \frac{1}{2}(P'_0 - P'_4), && X_k = P'_k \quad (k = 1,2,3), \\ &X_4 = \frac{1}{2}(P'_0 + P'_4). && \end{aligned}$$

2. The differential first-order invariants of the non-splitting subgroups of the group $P(1,4)$

The group $P(1,4)$ acts on $M(1,3) \times R(u)$ (i.e., on the Cartesian product of the four-dimensional Minkowski space (of the independent variables x_0, x_1, x_2, x_3) and the number axis of the dependent variable u). The group $P(1,4)$ usually acts on $M(1,3) \times R(u)$ as a group generated by translations and rotations of this space.

Let

$$X = \sum_{i=0}^{3} \xi_i(x,u)\frac{\partial}{\partial x_i} + \eta(x,u)\frac{\partial}{\partial u}$$

be one of the basis infinitesimal operators. The first prolongation of X has the form

$$X^{(1)} = X + \sum_{i=0}^{3}\left(\frac{\partial\eta}{\partial x_i} + \frac{\partial\eta}{\partial u}u_i - \sum_{j=0}^{3} u_j\frac{\partial\xi_j}{\partial x_i} - \sum_{j=0}^{3} u_i u_j\frac{\partial\xi_j}{\partial u}\right)\frac{\partial}{\partial u_i}.$$

Now, a function $J(x,u^{(1)})$ is a first-order differential invariant if

$$X^{(1)} \cdot J(x,u^{(1)}) = 0.$$

Here $u^{(1)} = (u, u_0, u_1, u_2, u_3)$ is an element of the first prolongation $R(u)^{(1)}$.

Let us consider the following representation of the Lie algebra of the group $P(1,4)$:

$$P'_0 = \frac{\partial}{\partial x_0}, \qquad P'_1 = -\frac{\partial}{\partial x_1}, \qquad P'_2 = -\frac{\partial}{\partial x_2}, \qquad P'_3 = -\frac{\partial}{\partial x_3},$$
$$P'_4 = -\frac{\partial}{\partial u}, \qquad M'_{\mu\nu} = -\left(x_\mu P'_\nu - x_\nu P'_\mu\right), \qquad x_4 \equiv u.$$

More details about this representation can be found in [14-16].

In the construction of the differential invariants it has turned out that different non-splitting subalgebras of the Lie algebra of the group $P(1,4)$ can have the same functional basis of the first-order differential invariants. Consequently, there is no one-to-one correspondence between non-conjugate subalgebras of the Lie algebra of the group $P(1,4)$ and corresponding differential invariants.

Definition 1
We call two subalgebras L^1 and L^2 of the Lie algebra of the group $P(1,4)$ equivalent if they have the same functional basis of the first-order differential invariants.

It is possible to prove that the relation of equivalence of subalgebras L^1 and L^2 given by Definition 1 is the set-theoretical equivalence relation. With respect to this equivalence relation, all non-splitting subalgebras of the Lie algebra of the group $P(1,4)$ split into classes of equivalent subalgebras. Each two different classes have different functional bases of the first-order differential invariants.

Definition 2
We call two functional bases of the first-order differential invariants of the non-splitting subalgebras of the Lie algebra of the group $P(1,4)$ equivalent if they belong to the equivalent subalgebras.

One of the results in this section can be formulated as follows.

Proposition
The non-splitting subgroups of the group $P(1,4)$ *have* 264 *non-equivalent functional bases of the first-order differential invariants.*

Proof. Here, we only give a sketch of the proof. Following the sketch, for the purpose of proving the Proposition, we have to use:

— the list of the non-splitting subalgebras of the Lie algebra of the group $P(1,4)$ [17];

— the general ranks of the matrices which contain coordinates of the one time prolonged basis elements of the subalgebras of the considered Lie algebra;

— theorem on number of invariants of the Lie group of the point transformations (see, for example, [18, 19]);

— Definition 1 and Definition 2.

For all non-splitting subgroups of the group $P(1,4)$ the functional bases of the first-order differential invariants have been constructed.

Below, for some of the non-splitting subalgebras of the Lie algebra of the group $P(1,4)$ we give their respective basis elements and corresponding functional basis of differential invariants.

One-dimensional subalgebras

1. $\langle L_3 + eG + \kappa_3 X_3,\ e > 0,\ \kappa_3 < 0\rangle$,

$$J_1 = (x_0^2 - u^2)^{\frac{1}{2}}, \qquad J_2 = (x_1^2 + x_2^2)^{\frac{1}{2}},$$

$$J_3 = \kappa_3 \ln(x_0 + u) - e x_3, \qquad J_4 = x_3 + \kappa_3 \arctan \frac{x_1}{x_2},$$

$$J_5 = \frac{x_1 u_2 - x_2 u_1}{x_1 u_1 + x_2 u_2}, \qquad J_6 = u_3 \frac{x_0 + u}{u_0 + 1},$$

$$J_7 = \frac{u_3^2}{u_0^2 - 1}, \qquad J_8 = \frac{u_1^2 + u_2^2}{u_3^2},$$

$$u_\mu \equiv \frac{\partial u}{\partial x_\mu}, \qquad \mu = 0, 1, 2, 3;$$

2. $\langle P_3 + X_0\rangle$,

$$J_1 = x_1, \qquad J_2 = x_2,$$

$$J_3 = (x_0 + u)^2 - 2x_3, \qquad J_4 = x_0 - u + \frac{2}{3}(x_0 + u)^3 - 2x_3(x_0 + u),$$

$$J_5 = x_0 + u + \frac{u_3}{u_0 + 1}, \qquad J_6 = \frac{u_1}{u_2},$$

$$J_7 = \frac{u_1}{u_0 + 1}, \qquad J_8 = \frac{u_3^2}{(u_0 + 1)^2} + \frac{2}{u_0 + 1}.$$

Two-dimensional subalgebras

1. $\langle G,\ L_3 + dX_3,\ d < 0\rangle$,

$$J_1 = x_3 + d\arctan\frac{x_1}{x_2}, \qquad J_2 = (x_1^2 + x_2^2)^{\frac{1}{2}},$$
$$J_3 = (x_0^2 - u^2)^{\frac{1}{2}}, \qquad J_4 = (x_0 + u)^2\frac{1 - u_0}{u_0 + 1},$$
$$J_5 = \frac{x_1u_2 - x_2u_1}{x_1u_1 + x_2u_2}, \qquad J_6 = \frac{u_3^2}{u_0^2 - 1},$$
$$J_7 = \frac{u_1^2 + u_2^2}{u_3^2};$$

2. $\langle L_3 - X_4,\ P_3\rangle$,

$$J_1 = x_0 + u, \qquad J_2 = (x_1^2 + x_2^2)^{\frac{1}{2}},$$
$$J_3 = x_0^2 - x_3^2 - u^2 + (x_0 + u)\arctan\frac{x_1}{x_2}, \qquad J_4 = \frac{x_3}{x_0 + u} + \frac{u_3}{u_0 + 1},$$
$$J_5 = \frac{x_1u_2 - x_2u_1}{x_1u_1 + x_2u_2}, \qquad J_6 = \frac{u_1^2 + u_2^2}{(u_0 + 1)^2},$$
$$J_7 = \frac{u_3^2}{(u_0 + 1)^2} + \frac{2}{u_0 + 1}.$$

Three-dimensional subalgebras

1. $\langle G + a_1X_1 + a_3X_3,\ P_3,\ X_4,\ a_1 < 0,\ a_3 < 0\rangle$,

$$J_1 = x_2, \qquad J_2 = x_1 - a_1\ln(x_0 + u),$$
$$J_3 = x_3 - a_3\ln(x_0 + u) + u_3\frac{x_0 + u}{u_0 + 1}, \qquad J_4 = (x_0 + u)\frac{u_1}{u_0 + 1},$$
$$J_5 = \frac{u_1}{u_2}, \qquad J_6 = \frac{u_0^2 - u_3^2 - 1}{u_1^2};$$

2. $\langle L_3 - P_3 + \alpha_0X_0,\ X_3,\ X_4,\ \alpha_0 < 0\rangle$,

$$J_1 = (x_1^2 + x_2^2)^{\frac{1}{2}}, \qquad J_2 = \alpha_0\arctan\frac{x_1}{x_2} - x_0 - u,$$
$$J_3 = \arctan\frac{u_1}{u_2} - \frac{u_3}{u_0 + 1}, \qquad J_4 = x_0 + u - \alpha_0\frac{u_3}{u_0 + 1},$$
$$J_5 = \frac{u_1^2 + u_2^2}{(u_0 + 1)^2}, \qquad J_6 = \frac{u_3^2 + 2(u_0 + 1)}{(u_0 + 1)^2}.$$

Four-dimensional subalgebras

1. $\langle G + a_3X_3,\ L_3,\ P_3,\ X_4,\ a_3 < 0\rangle$,

$$J_1 = (x_1^2 + x_2^2)^{\frac{1}{2}}, \qquad J_2 = \frac{x_1u_2 - x_2u_1}{x_1u_1 + x_2u_2},$$

$$J_3 = x_3 - a_3\ln(x_0 + u) + \frac{x_0 + u}{u_0 + 1}u_3, \qquad J_4 = (u_1^2 + u_2^2)\frac{(x_0 + u)^2}{(u_0 + 1)^2},$$

$$J_5 = \frac{u_0^2 - u_3^2 - 1}{u_1^2 + u_2^2};$$

2. $\langle L_3,\ P_1,\ P_2,\ P_3 + X_3\rangle$,

$$J_1 = x_0 + u,$$

$$J_2 = x_0^2 - x_1^2 - x_2^2 - u^2 - \frac{x_0 + u}{x_0 + u - 1}x_3^2,$$

$$J_3 = \frac{x_3}{x_0 + u - 1} + \frac{u_3}{u_0 + 1},$$

$$J_4 = \left(\frac{x_1}{x_0 + u} + \frac{u_1}{u_0 + 1}\right)^2 + \left(\frac{x_2}{x_0 + u} + \frac{u_2}{u_0 + 1}\right)^2,$$

$$J_5 = \frac{u_1^2 + u_2^2 + u_3^2 + 2(u_0 + 1)}{(u_0 + 1)^2}.$$

Five-dimensional subalgebras

1. $\langle G + a_2X_1,\ P_1,\ P_2,\ P_3,\ X_4,\ a_2 < 0\rangle$,

$$J_1 = x_1 + \frac{x_0 + u}{u_0 + 1}u_1 - a_2\ln(x_0 + u),$$

$$J_2 = x_2 + \frac{x_0 + u}{u_0 + 1}u_2,$$

$$J_3 = x_3 + (x_0 + u)\frac{u_3}{u_0 + 1},$$

$$J_4 = (u_0^2 - u_1^2 - u_2^2 - u_3^2 - 1)\frac{(x_0 + u)^2}{(u_0 + 1)^2};$$

2. $\langle L_3,\ P_1 + X_2,\ P_2 - X_1,\ X_3,\ X_4\rangle$,

$$J_1 = x_0 + u,$$

$$J_2 = \left(\frac{x_1}{x_0+u} + \frac{u_2}{(x_0+u)(u_0+1)} + \frac{u_1}{u_0+1}\right)^2 + \left(\frac{x_2}{x_0+u} - \frac{u_1}{(x_0+u)(u_0+1)} + \frac{u_2}{u_0+1}\right)^2,$$

$$J_3 = \frac{u_3}{u_0+1},$$

$$J_4 = \frac{u_1^2+u_2^2}{(u_0+1)^2} + \frac{2}{u_0+1}.$$

Six-dimensional subalgebras

1. $\langle G + aX_3,\ L_3 + dX_3,\ P_1,\ P_2,\ P_3,\ X_4,\ a<0,\ d<0\rangle$,

$$J_1 = (x_0+u)^2\left(\frac{u_1^2+u_2^2+u_3^2+2u_0+2}{(u_0+1)^2} - 1\right),$$

$$J_2 = \left(x_1 + \frac{x_0+u}{u_0+1}u_1\right)^2 + \left(x_2 + \frac{x_0+u}{u_0+1}u_2\right)^2,$$

$$J_3 = x_3 - a\ln(x_0+u) + (x_0+u)\frac{u_3}{u_0+1} + d\arctan\left(\frac{x_1(u_0+1)+u_1(x_0+u)}{x_2(u_0+1)+u_2(x_0+u)}\right);$$

2. $\langle P_1 + X_3,\ P_2,\ X_0,\ X_1,\ X_2,\ X_4\rangle$,

$$J_1 = \frac{u_1}{u_0+1} - x_3, \qquad J_2 = \frac{u_3}{u_0+1},$$

$$J_3 = \frac{u_1^2+u_2^2}{(u_0+1)^2} + \frac{2}{u_0+1}.$$

Seven-dimensional subalgebras

1. $\langle G + a_3X_3,\ L_3,\ P_1,\ P_2,\ X_1,\ X_2,\ X_4,\ a_3<0\rangle$,

$$J_1 = x_3 - a_3\ln(x_0+u), \qquad J_2 = (x_0+u)\frac{u_3}{u_0+1},$$

$$J_3 = \frac{u_0^2-u_1^2-u_2^2-1}{u_3^2};$$

2. $\langle L_3 - P_3 + \alpha_0X_0,\ P_1,\ P_2,\ X_1,\ X_2,\ X_3,\ X_4,\ \alpha_0<0\rangle$,

$$J_1 = x_0 + u - \alpha_0 \frac{u_3}{u_0+1}, \qquad J_2 = \frac{u_1^2 + u_2^2 + u_3^2 + 2(u_0+1)}{(u_0+1)^2}.$$

Eight-dimensional subalgebras

1. $\langle G + a_3 X_3,\ L_3,\ P_1,\ P_2,\ X_0,\ X_1,\ X_2,\ X_4,\ a_3 < 0\rangle$,

$$J_1 = x_3 + a_3 \ln\left(\frac{u_3}{u_0+1}\right), \qquad J_2 = \frac{u_0^2 - u_1^2 - u_2^2 - 1}{u_3^2};$$

2. $\langle L_3 - X_0,\ P_1,\ P_2,\ P_3,\ X_1,\ X_2,\ X_3,\ X_4\rangle$,

$$J_1 = \frac{u_1^2 + u_2^2 + u_3^2 + 2(u_0+1)}{(u_0+1)^2}.$$

3. On some applications of the results obtained

The differential invariants of the local Lie groups of the point transformations play an important role in the group-analysis of differential equations (see, for example [18-28]). In particule, with the help of these invariants we can construct differential equations with non-trivial symmetry groups.

In our case the considered equations can be written in the following form (see, for example [18-20]):

$$F(J_1, J_2, \ldots, J_t) = 0,$$

where F is an arbitrary smooth function of its arguments, $\{J_1, J_2, \ldots, J_t\}$ is a functional basis of the first-order differential invariants of the non-splitting subgroups of the group $P(1,4)$.

Since the Lie algebra of the group $P(1,4)$ contains, as subalgebras, the Lie algebra of the Poincaré group $P(1,3)$ and the Lie algebra of the extended Galilei group $\widetilde{G}(1,3)$ (see also [9]), the results obtained can be used in relativistic and non-relativistic physics.

References

[1] W.I. Fushchych, *Representations of full inhomogeneous de Sitter group and equations in five-dimensional approach. I*, Teoret. i mat. fizika V. 4, N 3 (1970), 360-367.

[2] W.I. Fushchych, A.G. Nikitin, *Symmetries of Equations of Quantum Mechanics*, Allerton Press Inc., New York, 1994.

[3] V.G. Kadyshevsky, *New approach to theory electromagnetic interactions*, Fizika elementar. chastitz. i atomn. yadra V. 11, N 1 (1980), 5-39.

[4] V.M. Fedorchuk, *Continuous subgroups of the inhomogeneous de Sitter group* $P(1,4)$, Akad. Nauk Ukrain. SSR Inst. Mat. Preprint **18** 1978.

[5] V.M. Fedorchuk, *Splitting subalgebras of the Lie algebra of the generalized Poincaré group* $P(1,4)$*,* Ukr. Mat. Zh. **31** (1979), no. 6. 717-722.

[6] V.M. Fedorchuk, *Nonsplitting subalgebras of the Lie algebra of the generalized Poincaré group* $P(1,4)$, Ukr. Mat. Zh. **33** (1981), no. 5, 696-700.

[7] V.M. Fedorchuk, W.I. Fushchych, *On subgroups of the generalized Poincaré group*, Proceedings of the International Seminar on Group Theoretical Methods in Physics, Moscow, Nauka, **1** (1980), 61-66.

[8] W.I. Fushchich, A.F. Barannik, L.F. Barannik, V.M. Fedorchuk, *Continuous subgroups of the Poincaré group* $P(1,4)$, J. Phys. A **18** (1985), no. 15, 2893-2899.

[9] W.I. Fushchich, A.G. Nikitin, *Reduction of the representations of the generalized Poincaré algebra by the Galilei algebra*, J. Phys. A **13** (1980), no. 7, 2319-2330.

[10] V.M. Fedorchuk, V.I. Fedorchuk, *The first-order differential invariants of the splitting subgroups of the generalized Poincaré group* $P(1,4)$, Mat. Metodi Fiz.-Mekh. Polya **44** (2001), no. 1, 16-21.

[11] V.M. Fedorchuk, V.I. Fedorchuk, *On differential first-order invariants of the splitting subgroups of the generalized Poincaré group* $P(1,4)$, Dopov. Nats. Akad. Nauk Ukr. Mat. Prirodozn. Tekh. Nauki 2002, no. 5, 36-42.

[12] V.M. Fedorchuk, V.I. Fedorchuk, *On differential invariants of first- and second-order of the splitting subgroups of the generalized Poincaré group* $P(1,4)$. Symmetry in nonlinear mathematical physics, Part 1, 2 (Kyiv, 2001), 140-144, Pr. Inst. Mat. Nats. Akad. Nauk Ukr. Mat. Zastos., 43, Part 1, 2, Natsīonal. Akad. Nauk Ukraini, Īnst. Mat., Kiev, 2002.

[13] J. Patera, P. Winternitz, H. Zassenhaus, *Continuous subgroups of the fundamental groups of physics. I. General method and the Poincaré group*, J. Math. Phys. **16** (1975), no 8, 1597-1614.

[14] W.I. Fushchich, W.M. Shtelen, *The symmetry and some exact solutions of the relativistic eikonal equation*, Lett. Nuovo Cimento (2) 34 (1982), no. 16, 498-502.

[15] W.I. Fushchich, N.I. Serov, *The symmetry and some exact solutions of the multidimensional Monge-Ampere equation*, Dokl. Akad. Nauk SSSR **273** (1983), no. 3, 543-546.

[16] W.I. Fushchich, N.I. Serov, *On some exact solutions of the multidimensional nonlinear Euler-Lagrange equation*, Dokl. Akad. Nauk SSSR **278** (1984), no. 4, 847-851.

[17] W. Fushchich, L. Barannik,A. Barannik, *Subgroup Analysis of the Galilei and Poincaré Groups and Reductions of Nonlinear Equations*, "Naukova Dumka", Kiev, 1991.

[18] L.V. Ovsiannikov, *Group Analysis of Differential Equations*, Academic Press, New York – London, 1982.

[19] P.J. Olver, *Applications of Lie Groups to Differential Equations*, Springer-Verlag, New York, 1986.

[20] S. Lie, *Über Differentialinvarianten*, Math. Ann. **24** (1884), no. 1, 52-89.

[21] S. Lie, *Vorlesungen über continuierliche Gruppen*, Teubner, Leipzig, 1893.

[22] A. Tresse, *Sur les invariants differentiels des groupes continus de transformations*, Acta math. **18** (1894), 1-88.

[23] E. Vessiot, *Sur l'integration des sistem differentiels qui admittent des groupes continues de transformations*, Acta math. **28** (1904), 307-349.

[24] W.I. Fushchych, I.A. Egorchenko, *Differential invariants for Galilei algebra*, Dokl. Akad. Nauk Ukrain. SSR Ser. A 1989, no. 4, 19-34.

[25] W.I. Fushchych, I.A. Egorchenko, *Differential invariants for Poincaré algebra and conformal algebra*, Dokl. Akad. Nauk Ukrain. SSR Ser. A 1989, no. 5, 46-53.

[26] H. Lahno, *Representations of subalgebras of a subdirect sum of the sextended Euclid algebras and invariant equations*, Symmetry in nonlinear mathematical physics, Vol. 1, 2 (Kyiv, 1997), 206-210, Natl. Acad. Sci. Ukraine, Inst. Math., Kiev, 1997.

[27] V.I. Lahno, *Realizations of the Poincare algebra and Poincare-invariant equations in three-dimensional space-time*, XXXI Symposium on Mathematical Physics (Toruń, 1999). Rep. Math. Phys. **46** (2000), no. 1-2, 137-142.

[28] R.O. Popovych, V.M. Boyko, *Differential invariants of one-parameter group of local transformations and integration Riccati equation*, Vestnik Samarskogo Gosuniversiteta **18** (2000), no. 4, 49-56.

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