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On a sum form functional equation containing three unknown mappings

Abstract. The general solutions of a sum form functional equation containing three unknown mappings have been obtained without imposing any regularity condition on any of three mappings.

1. Introduction

For $n = 1, 2, \ldots$; let

$$\Gamma_n = \left\{ (p_1, \ldots, p_n) : \; p_i \geq 0, \, i = 1, \ldots, n; \; \sum_{i=1}^{n} p_i = 1 \right\}$$

denote the set of all n-component complete discrete probability distributions with nonnegative elements. Let $\mathbb{R}$ denote the set of all real numbers; $I = \{x \in \mathbb{R} : 0 \leq x \leq 1\}$, the unit closed interval; $]0,1[= \{x \in \mathbb{R} : \; 0 < x < 1\}$, the unit open interval; $]0,1] = \{x \in \mathbb{R} : \; 0 < x \leq 1\}$ and $[0,1[= \{x \in \mathbb{R} : \; 0 \leq x < 1\}$.

Recently, P. Nath and D.K. Singh [8] (see also [3, 5, 6]) obtained the general solutions of

$$\sum_{i=1}^{n} \sum_{j=1}^{m} F(p_i q_j) = \sum_{i=1}^{n} G(p_i) + \sum_{j=1}^{m} H(q_j) + \sum_{i=1}^{n} K(p_i) \sum_{j=1}^{m} L(q_j) \tag{FE1}$$

by assuming F, G, H, K and L to be real-valued mappings each with domain I; without imposing any regularity condition on any of the mappings F, G, H, K and L; but assuming $(p_1, \ldots, p_n) \in \Gamma_n$, $(q_1, \ldots, q_m) \in \Gamma_m$; $n \geq 3$, $m \geq 3$ to be fixed integers. During the process of finding such general solutions, they came across three functional equations. The first one is

$$\sum_{i=1}^{n} \sum_{j=1}^{m} F_1(p_i q_j) = \sum_{i=1}^{n} F_1(p_i) + \sum_{j=1}^{m} F_1(q_j) + \sum_{i=1}^{n} K_1(p_i) \sum_{j=1}^{m} L_1(q_j) \tag{FE2}$$

with $(p_1, \ldots, p_n) \in \Gamma_n$, $(q_1, \ldots, q_m) \in \Gamma_m$; $n \geq 3$, $m \geq 3$ being fixed integers and

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$F_1\colon I \to \mathbb{R}$, $K_1\colon I \to \mathbb{R}$ and $L_1\colon I \to \mathbb{R}$ are mappings which satisfy the conditions

$$\begin{aligned} F_1(1) &= (n-1)(m-1)F_1(0),\\ K_1(1) &= -(n-1)K_1(0),\\ L_1(1) &= -(m-1)L_1(0). \end{aligned}$$

The second one is

$$\begin{aligned} \sum_{i=1}^{n}\sum_{j=1}^{m} F_1(p_i q_j) = \sum_{i=1}^{n} F_1(p_i) + \sum_{j=1}^{m} F_1(q_j) + c\sum_{i=1}^{n} K_1(p_i)\sum_{j=1}^{m} K_1(q_j)\\ + c(n-m)K_1(0)\sum_{i=1}^{n} K_1(p_i), \end{aligned} \tag{FE3}$$

where $F_1\colon I \to \mathbb{R}$, $K_1\colon I \to \mathbb{R}$ are the same mappings which appear in (FE2); $c \neq 0$ is a given real constant; $(p_1,\ldots,p_n) \in \Gamma_n$, $(q_1,\ldots,q_m) \in \Gamma_m$; $n \geq 3$, $m \geq 3$ being fixed integers. The third one is

$$\sum_{i=1}^{n}\sum_{j=1}^{m} F_2(p_i q_j) = \sum_{i=1}^{n} F_2(p_i) + \sum_{j=1}^{m} F_2(q_j) + c\sum_{i=1}^{n} K_2(p_i)\sum_{j=1}^{m} K_2(q_j), \tag{FE4}$$

where $F_2\colon I \to \mathbb{R}$, $K_2\colon I \to \mathbb{R}$ are mappings which satisfy the conditions

$$F_2(0) = 0, \qquad K_2(0) = 0, \tag{1.1}$$

$$F_2(1) = 0, \qquad K_2(1) = 0; \tag{1.2}$$

$c \neq 0$ is a given real constant (same as in (FE3)); $(p_1,\ldots,p_n) \in \Gamma_n$, $(q_1,\ldots,q_m) \in \Gamma_m$; $n \geq 3$, $m \geq 3$ being fixed integers.

The main object of this paper is to determine the general solutions of the functional equations (FE2) without imposing a regularity condition on any of the mappings $F_1\colon I \to \mathbb{R}$, $K_1\colon I \to \mathbb{R}$ and $L_1\colon I \to \mathbb{R}$, assuming it to be valid for all $(p_1,\ldots,p_n) \in \Gamma_n$, $(q_1,\ldots,q_m) \in \Gamma_m$; $n \geq 3$, $m \geq 3$ being fixed integers. To achieve this objective, we need the general solutions of the equations (FE3) and (FE4) (assuming only (1.1)).

2. Some known definitions and results

In this section, we mention some known definitions and results which are needed to develop the remaining sections 3 to 5 of this paper.

A mapping $a\colon I \to \mathbb{R}$ is said to be additive on I or on the unit triangle $\Delta = \{(x,y) : 0 \leq x \leq 1, 0 \leq y \leq 1, 0 \leq x+y \leq 1\}$ if it satisfies the equation $a(x+y) = a(x) + a(y)$ for all $(x,y) \in \Delta$. A mapping $A\colon \mathbb{R} \to \mathbb{R}$ is said to be additive on $\mathbb{R}$ if the equation $A(x+y) = A(x) + A(y)$ holds for all $x \in \mathbb{R}$, $y \in \mathbb{R}$. It is known (see Z. Daróczy and L. Losonczi [2]) that if a mapping $a\colon I \to \mathbb{R}$ is additive on I, then it has a unique additive extension $A\colon \mathbb{R} \to \mathbb{R}$ in the sense that $A\colon \mathbb{R} \to \mathbb{R}$ is additive on $\mathbb{R}$ and $A(x) = a(x)$ for all $x \in I$.

A mapping $M\colon I \to \mathbb{R}$ is said to be multiplicative if $M(pq) = M(p)M(q)$ for all $p \in I$, $q \in I$.

A mapping $\ell\colon I \to \mathbb{R}$ is said to be logarithmic if $\ell(0) = 0$ and $\ell(pq) = \ell(p)+\ell(q)$ for all $p \in]0,1]$, $q \in]0,1]$.

Result 2.1 ([4])
Let $h\colon I \to \mathbb{R}$ be a mapping which satisfies the equation $\sum_{i=1}^{n} h(p_i) = d$ for all $(p_1,\dots,p_n) \in \Gamma_n$, d a given real constant and $n \geq 3$ a fixed integer. Then, there exists an additive mapping $b\colon \mathbb{R} \to \mathbb{R}$ such that $h(p) = b(p) - \frac{1}{n}b(1) + \frac{d}{n}$ for all $p \in I$.

T.W. Chaundy and J.B. Mcleod [1] considered the functional equation

$$\sum_{i=1}^{n}\sum_{j=1}^{m} f(p_i q_j) = \sum_{i=1}^{n} f(p_i) + \sum_{j=1}^{m} f(q_j), \tag{2.1}$$

where $f\colon I \to \mathbb{R}$, $(p_1,\dots,p_n) \in \Gamma_n$, $(q_1,\dots,q_m) \in \Gamma_m$; n and m being positive integers.

Result 2.2 ([4])
If a mapping $f\colon I \to \mathbb{R}$ satisfies (2.1) for all $(p_1,\dots,p_n) \in \Gamma_n$, $(q_1,\dots,q_m) \in \Gamma_m$, $n \geq 3$, $m \geq 3$ being fixed integers, then f is of the form

$$f(p) = \begin{cases} f(0) + f(0)(nm-n-m)p + a(p) + D(p,p) & \text{if } 0 < p \leq 1, \\ f(0) & \text{if } p = 0, \end{cases}$$

where $f(0)$ is an arbitrary real constant; $a\colon \mathbb{R} \to \mathbb{R}$ is an additive mapping; the mapping $D\colon \mathbb{R}\times]0,1] \to \mathbb{R}$ is additive in the first variable; there exists a mapping $E\colon \mathbb{R}\times\mathbb{R} \to \mathbb{R}$ additive in both variables such that $a(1) = E(1,1)$ and $D(pq,pq) = D(pq,p) + D(pq,q) + E(p,q)$ for all $p \in]0,1]$, $q \in]0,1]$.

Modified Form of Result 2.2
If a mapping $f\colon I \to \mathbb{R}$ satisfies (2.1) for all $(p_1,\dots,p_n) \in \Gamma_n$, $(q_1,\dots,q_m) \in \Gamma_m$, $n \geq 3$, $m \geq 3$ being fixed integers, then f is of the form

$$f(p) = f(0) + f(0)(nm-n-m)p + a(p) + D(p,p) \tag{2.2}$$

for all $p \in I$; $f(0)$ is an arbitrary real constant; $a\colon \mathbb{R} \to \mathbb{R}$ is an additive mapping; the mapping $D\colon \mathbb{R}\times I \to \mathbb{R}$ is additive in the first variable; there exists a mapping $E\colon \mathbb{R}\times\mathbb{R} \to \mathbb{R}$ additive in both variables such that $a(1) = E(1,1)$ and

$$D(pq,pq) = D(pq,p) + D(pq,q) + E(p,q) \tag{2.3}$$

for all $p \in I$, $q \in I$.

Using the fact that $a(1) = E(1,1)$, it can be easily deduced from (2.3) that

$$a(1) + D(1,1) = 0. \tag{2.4}$$

Result 2.3 ([7])
Let $c \neq 0$ be a given constant and $F_2\colon I \to \mathbb{R}$, $K_2\colon I \to \mathbb{R}$ be mappings which satisfy (FE4) for all $(p_1,\dots,p_n) \in \Gamma_n$, $(q_1,\dots,q_m) \in \Gamma_m$; $n \geq 3$, $m \geq 3$ being

fixed integers. Suppose further that $F_2\colon I \to \mathbb{R}$, $K_2\colon I \to \mathbb{R}$ satisfy (1.1). *Then, the mapping $K_2\colon I \to \mathbb{R}$ satisfies the functional equation*

$$\left[\sum_{j=1}^{m} K_2(xq_j) - K_2(x)\right] \sum_{t=1}^{m} K_2(r_t) = \left[\sum_{t=1}^{m} K_2(xr_t) - K_2(x)\right] \sum_{j=1}^{m} K_2(q_j) \tag{2.5}$$

for all $x \in I$ and $(r_1, \ldots, r_m) \in \Gamma_m$, $(q_1, \ldots, q_m) \in \Gamma_m$, $m \geq 3$ being a fixed integer.

For the proof of Result 2.3, see pp. 90-91 in [7] (take F_2 as f and K_2 as g).

Result 2.4 ([8])
Let $F_1\colon I \to \mathbb{R}$, $K_1\colon I \to \mathbb{R}$ and $L_1\colon I \to \mathbb{R}$ be mappings which satisfy (FE2) *for all $(p_1, \ldots, p_n) \in \Gamma_n$, $(q_1, \ldots, q_m) \in \Gamma_m$; $n \geq 3$, $m \geq 3$ being fixed integers. Then, the mappings K_1 and L_1 satisfy the equation*

$$\begin{aligned}\left[\sum_{t=1}^{m} K_1(r_t) + (n-m)K_1(0)\right] \sum_{j=1}^{m} L_1(q_j) \\ = \left[\sum_{j=1}^{m} K_1(q_j) + (n-m)K_1(0)\right] \sum_{t=1}^{m} L_1(r_t)\end{aligned} \tag{2.6}$$

for all $(r_1, \ldots, r_m) \in \Gamma_m$, $(q_1, \ldots, q_m) \in \Gamma_m$; $m \geq 3$ being fixed integers. Moreover, for all $p \in I$, any general solution (K_1, L_1) of (2.6) *is of the form*

$$K_1(p) = \bar{A}_1(p) + K_1(0) \text{ with } \bar{A}_1(1) = -nK_1(0), \quad L_1 \text{ arbitrary} \tag{2.7}$$

or

$$K_1 \text{ arbitrary}, \quad L_1(p) = \bar{A}_2(p) + L_1(0) \text{ with } \bar{A}_2(1) = -mL_1(0) \tag{2.8}$$

or else K_1 and L_1 are related to each other as

$$L_1(p) = c[K_1(p) - K_1(0)] + \bar{A}_3(p) + L_1(0) \text{ with } \bar{A}_3(1) = -\,mL_1(0) + cnK_1(0), \tag{2.9}$$

where $\bar{A}_i\colon \mathbb{R} \to \mathbb{R}$ $(i = 1, 2, 3)$ are additive mappings and $c \neq 0$ an arbitrary real constant in (2.9).

Remark 2.5
Result 2.4 is a combination of Lemmas 3.3 and 3.2 in [8].

3. On the functional equation (FE4)

The main result of this section is the following:

Theorem 3.1
Let $c \neq 0$ be a given real constant and $F_2\colon I \to \mathbb{R}$, $K_2\colon I \to \mathbb{R}$ be mappings which satisfy (FE4) *for all $(p_1, \ldots, p_n) \in \Gamma_n, (q_1, \ldots, q_m) \in \Gamma_m$; $n \geq 3$, $m \geq 3$ being fixed integers. Suppose further that the mappings $F_2\colon \to \mathbb{R}$, $K_2\colon I \to \mathbb{R}$ satisfy* (1.1).

Then, for all $p \in I$, any general solution (F_2, K_2) of (FE4) *is one of the following forms:*

$$\begin{cases} \text{(i) } F_2(p) = -c\,[b_1(1)]^2 p + a(p) + D(p,p) \\ \text{(ii) } K_2(p) = b_1(p) \end{cases} \tag{α_1}$$

or

$$\begin{cases} \text{(i) } F_2(p) = \frac{1}{2} cp[\ell(p)]^2 + a(p) + D(p,p) \\ \text{(ii) } K_2(p) = p\ell(p) \end{cases} \tag{α_2}$$

or

$$\begin{cases} \text{(i) } F_2(p) = c\mu^2[M(p) - p] + a(p) + D(p,p) \\ \text{(ii) } K_2(p) = \mu[M(p) - p] \end{cases} \tag{α_3}$$

where $\mu \neq 0$ is an arbitrary real constant; $b_1 \colon \mathbb{R} \to \mathbb{R}$ is an additive mapping with $b_1(1)$ an arbitrary real constant; the mappings $a \colon \mathbb{R} \to \mathbb{R}$ and $D \colon \mathbb{R} \times I \to \mathbb{R}$ are as described in the Modified Form of Result 2.2; $M \colon I \to \mathbb{R}$ is a multiplicative mapping which is not additive and $M(0) = 0$, $M(1) = 1$; $\ell \colon I \to \mathbb{R}$ is a logarithmic mapping.

Note 3.2
Since $\ell \colon I \to \mathbb{R}$ is a logarithmic mapping, so $0\ell(0) = 0$ and $0[\ell(0)]^2 = 0$.

Proof of Theorem 3.1. Let us pay attention to equation (2.5) in Result 2.3. We divide our discussion into two cases:

Case 1. $\sum_{t=1}^m K_2(r_t) \equiv 0$ on Γ_m.

In this case, by using Result 2.1, it follows that there exists an additive mapping $b_1 : \mathbb{R} \to \mathbb{R}$ such that $K_2(p) = b_1(p)$ with $b_1(1) = 0$. This form of $K_2(p)$ is included in (α_1)(ii) when $b_1(1) = 0$.

Case 2. $\sum_{t=1}^m K_2(r_t)$ does not vanish identically on Γ_m.

In this case, there exists a probability distribution $(r_1^*, \ldots, r_m^*) \in \Gamma_m$ such that $\sum_{t=1}^m K_2(r_t^*) \neq 0$. Putting $r_t = r_t^*$, $t = 1, \ldots, m$ in (2.5) and using $\sum_{t=1}^m K_2(r_t^*) \neq 0$, it follows that

$$\sum_{j=1}^m K_2(xq_j) = K_2(x) + M(x) \sum_{j=1}^m K_2(q_j), \tag{3.1}$$

where $M \colon I \to \mathbb{R}$ is defined as

$$M(x) = \left[\sum_{t=1}^m K_2(r_t^*)\right]^{-1} \left[\sum_{t=1}^m K_2(xr_t^*) - K_2(x)\right] \tag{3.2}$$

for all $x \in I$. Since $K_2(0) = 0$ by assumption, it follows from (3.2) that $M(0) = 0$. But since we are not assuming that $K_2(1) = 0$, it does not follow from (3.2) that $M(1) = 1$. So, **the technique adopted on pp. 88-89 in [7] does not work here**. Let us write (3.1) in the form

$$\sum_{j=1}^m \{K_2(xq_j) - M(x)K_2(q_j) - q_j K_2(x)\} = 0.$$

By Result 2.1, there exists a mapping $E\colon I \times \mathbb{R} \to \mathbb{R}$, additive in the second variable such that

$$K_2(xq) - M(x)K_2(q) - qK_2(x) = E(x;q) - \frac{1}{m}E(x;1). \tag{3.3}$$

Putting $q = 0$ in (3.3) and using $K_2(0) = 0$, $E(x;0) = 0$ for all $x \in I$, (3.3) gives $E(x;1) = 0$ for all $x \in I$. So, (3.3) reduces to the equation

$$K_2(xq) - M(x)K_2(q) - qK_2(x) = E(x;q) \tag{3.4}$$

valid for all $x \in I$, $q \in I$. Also from equation (3.4) it follows that $E(0;q) = 0$ for all $q \in I$.

Case 2.1. $E(x;q) \equiv 0$ on $I \times I$.

In this case, (3.4) reduces to the equation

$$K_2(xq) = M(x)K_2(q) + qK_2(x) \tag{3.5}$$

valid for all $x \in I$, $q \in I$. The left hand side of (3.5) is symmetric in x and q. Hence, so should be its right hand side. This fact gives rise to the equation

$$[M(x) - x]K_2(q) = [M(q) - q]K_2(x) \tag{3.6}$$

valid for all $x \in I$, $q \in I$.

Consider the case when the mapping $x \to M(x) - x$, $x \in I$, vanishes identically on I. This means that $M(x) = x$ for all $x \in I$. Making use of this form of $M\colon I \to \mathbb{R}$ in (3.5), we obtain the equation

$$K_2(xq) = xK_2(q) + qK_2(x) \tag{3.7}$$

valid for all $x \in I$, $q \in I$. The general solution of (3.7), for all $p \in I$, is $K_2(p) = p\ell(p)$, where $\ell\colon I \to \mathbb{R}$ is a logarithmic mapping. Thus, we have obtained (α_2)(ii).

Now consider the case when the mapping $x \mapsto M(x) - x$, $x \in I$, does not vanish identically on I. In this case, there exists an element $x_0 \in I$ such that $[M(x_0) - x_0] \neq 0$. Putting $x = x_0$ in (3.6) and using $[M(x_0) - x_0] \neq 0$, we get $K_2(q) = \mu[M(q) - q]$ for all $q \in I$, where $\mu = K_2(x_0)[M(x_0) - x_0]^{-1}$. If $\mu = 0$, then $K_2(q) = 0$ for all $q \in I$. Then $\sum_{t=1}^{m} K_2(r_t^*) = 0$ contradicting $\sum_{t=1}^{m} K_2(r_t^*) \neq 0$. Hence $\mu \neq 0$. So

$$K_2(q) = \mu[M(q) - q] \tag{3.8}$$

for all $q \in I$; $\mu \neq 0$ being an arbitrary real constant. Now, by assumption $K_2(0) = 0$ (see (1.1)). Hence, from (3.8), it follows that $M(0) = 0$. Also, from (3.5) and (3.8), it follows that M is multiplicative, that is,

$$M(xq) = M(x)M(q) \tag{3.9}$$

for all $x \in I$, $q \in I$. Thus we have to consider only those forms of M which are multiplicative and satisfy the condition $M(0) = 0$. Since $M(0) = 0$, therefore the possibility of $M(x) \equiv 1$ is ruled out. Also, $[M(x_0) - x_0] \neq 0$. It follows, from (3.8) that

$$K_2(x_0) \neq 0. \tag{3.10}$$

The possibility that $x_0 = 0$ is ruled out because, in this case, (3.10) gives $K_2(0) \neq 0$ contradicting the assumption $K_2(0) = 0$.

Now we discuss the case when $x_0 = 1$. In this case, (3.10) gives $K_2(1) \neq 0$. Now, (3.8) gives $M(1) \neq 1$. But from (3.9), $M(x)[M(1)-1] = 0$ holds for all $x \in I$. Hence $M(x) \equiv 0$. Consequently, (3.8) gives $K_2(q) = -\mu q$ which is contained in (α_1)(ii) (choose $b_1(q) = -\mu q$ with $b_1(1) = -\mu$).

Now, we have to discuss the case when $x_0 \in]0, 1[$, keeping in mind that $K_2(0) = 0$ (by assumption) and $K_2(1) = 0$ because we have already discussed above the case when $K_2(1) \neq 0$. Now, from (3.8), $0 = K_2(1) = \mu[M(1) - 1]$, $\mu \neq 0$. So, $M(1) = 1$. Hence, we get (α_3)(ii) with $M(0) = 0$ and $M(1) = 1$.

Now we prove that $M\colon I \to \mathbb{R}$ is not additive. To the contrary, suppose that $M\colon I \to \mathbb{R}$ is additive. Then, for all $(r_1, \ldots, r_m) \in \Gamma_m$, using (α_3)(ii) and $M(1) = 1$, we have

$$\sum_{t=1}^{m} K_2(r_t) = \mu\left(\sum_{t=1}^{m} M(r_t) - 1\right) = \mu(M(1) - 1) = 0$$

contradicting $\sum_{t=1}^{m} K_2(r_t^*) \neq 0$. So, $M\colon I \to \mathbb{R}$ is not additive. In particular, $M(q) \equiv q$ is ruled out because if $M(q) \equiv q$, then $K_2(q) = 0$ contradicting (3.10).

Case 2.2. $E(x; q)$ does not vanish identically on $I \times I$.

In this case, there exists an element $(x^*; q^*) \in I \times I$ such that $E(x^*; q^*) \neq 0$. Since $E(x; 1) = 0$ and $E(x; 0) = 0$ for all $x \in I$; $E(0; q) = 0$ for all $q \in I$, it follows that $E(x^*; 1) = 0$, $E(x^*; 0) = 0$ and $E(0; q^*) = 0$. Hence, we must have $x^* \in]0, 1]$ and $q^* \in]0, 1[$. So, $(x^*; q^*) \in]0, 1] \times]0, 1[$. Now we prove that

$$\begin{aligned} r = [E(x^*; q^*)]^{-1}\{&M(x^*)M(q^*)K_2(r) + M(x^*)E(q^*; r) + E(x^*; q^*r) \\ &- M(x^*q^*)K_2(r) - E(x^*q^*; r)\} \end{aligned} \tag{3.11}$$

holds for all $r \in I$. Using (3.4), we have

$$\begin{aligned} K_2((x^*q^*)r) = {}&M(x^*q^*)K_2(r) + rM(x^*)K_2(q^*) + rq^*K_2(x^*) \\ &+ rE(x^*; q^*) + E(x^*q^*; r) \end{aligned} \tag{3.12}$$

and

$$\begin{aligned} K_2(x^*(q^*r)) = {}&M(x^*)M(q^*)K_2(r) + rM(x^*)K_2(q^*) + M(x^*)E(q^*; r) \\ &+ q^*rK_2(x^*) + E(x^*; q^*r). \end{aligned} \tag{3.13}$$

Since $K_2((x^*q^*)r) = K_2(x^*(q^*r))$ and $E(x^*; q^*) \neq 0$, equations (3.12) and (3.13) give (3.11) for $r \in I$.

Equation (3.11) can be rewritten as

$$\begin{aligned} r - [E(x^*; q^*)]^{-1}&[M(x^*)E(q^*; r) + E(x^*; q^*r) - E(x^*q^*; r)] \\ &= [E(x^*; q^*)]^{-1}[M(x^*)M(q^*) - M(x^*q^*)]K_2(r). \end{aligned} \tag{3.14}$$

Putting $r = 1$ in equation (3.14) and using $E(x; 1) = 0$, we obtain

$$[M(x^*)M(q^*) - M(x^*q^*)]K_2(1) = 0 \tag{3.15}$$

for some $x^* \in]0,1]$, $q^* \in]0,1[$.

Case 2.2.1. $M(x^*)M(q^*) - M(x^*q^*) \neq 0$ for some $x^* \in]0,1]$, $q^* \in]0,1[$.

Then, from equation (3.15), we have $K_2(1) = 0$. Since the left hand side of equation (3.14) is additive in r, so the right hand side must also be additive in r, $r \in I$. But, the right hand side of equation (3.14) can not be additive because if it is so then $0 \neq \sum_{t=1}^{m} K_2(r_t^*) = K_2(1) = 0$, a contradiction. So, this case is not possible.

Case 2.2.2.

$$M(x^*)M(q^*) - M(x^*q^*) = 0 \tag{3.15a}$$

for some $x^* \in]0,1]$ and for some $q^* \in]0,1[$. Then, from (3.15), it follows that $K_2(1)$ is an arbitrary real number. Let us put $x = 1$ in equation (3.4). We obtain

$$K_2(q)[1 - M(1)] = E(1;q) + qK_2(1) \tag{3.16}$$

for all $q \in I$.

Case 2.2.2.1. $1 - M(1) \neq 0$.

From equation (3.16), we obtain $K_2(q) = b_1(q)$, where $b_1\colon \mathbb{R} \to \mathbb{R}$ is defined as $b_1(q) = [1 - M(1)]^{-1}[E(1;q) + qK_2(1)]$. Since $q \to E(1;q)$ and $q \to qK_2(1)$ are additive mappings, so $q \to b_1(q)$ is also additive. Now $0 \neq \sum_{t=1}^{m} K_2(r_t^*) = \sum_{t=1}^{m} b_1(r_t^*) = b_1(\sum_{t=1}^{m} r_t^*) = b_1(1)$. Thus $b_1(1) \neq 0$ and also $b_1(1) = [1 - M(1)]^{-1}K_2(1)$. Hence $K_2(1) \neq 0$. So, solution $(\alpha_1)(ii)$ arises when $b_1(1) \neq 0$.

Case 2.2.2.2. $1 - M(1) = 0$.

Putting $1 - M(1) = 0$ in (3.16), we obtain $0 = E(1;q) + qK_2(1)$ for all $q \in I$. Substituting $q = 1$ in this equation and using the fact that $E(1;1) = 0$ (because $E(x;1) = 0$ for all $x \in I$), we obtain $K_2(1) = 0$.

Substituting $p_1 = 1, p_2 = \ldots = p_n = 0$; $q_1 = 1, q_2 = \ldots = q_m = 0$ in (FE4); using (1.1) and the fact that $K_2(1) = 0$, we get $F_2(1) = 0$. So, we have $F_2(0) = 0$, $K_2(0) = 0$, $F_2(1) = 0$, $K_2(1) = 0$. These are precisely the assumptions which we made in [7]. Now we make use of the Theorem (p-86) in [7] and take only those solutions which satisfy (1.1), (1.2) and (3.15a). There is only one such solution, namely, (α_3)(ii) in which $M\colon I \to \mathbb{R}$ is multiplicative. Making use of (α_3)(ii) in (3.4) and using the multiplicativity of $M\colon I \to \mathbb{R}$, it follows that $E(x;q) = 0$ for all $x \in I$, $q \in I$; contradicting the fact that $E(x^*;q^*) \neq 0$ (Case 2.2). So, in this case, we do not get any solution.

So far, we have obtained all possible forms of $K_2\colon I \to \mathbb{R}$ mentioned in (α_1) to (α_3). Now our task is to find the corresponding forms of $F_2\colon I \to \mathbb{R}$ needed in (α_1) to (α_3).

Making use of (α_1)(ii) in (FE4), we obtain the equation

$$\sum_{i=1}^{n}\sum_{j=1}^{m} F_2(p_iq_j) = \sum_{i=1}^{n} F_2(p_i) + \sum_{j=1}^{m} F_2(q_j) + c[b_1(1)]^2$$

which can be written in the form

$$\sum_{i=1}^{n}\sum_{j=1}^{m}\{F_2(p_iq_j)+c[b_1(1)]^2p_iq_j\} = \sum_{i=1}^{n}\{F_2(p_i)+c[b_1(1)]^2p_i\}+\sum_{j=1}^{m}\{F_2(q_j)+c[b_1(1)]^2q_j\}. \tag{3.17}$$

Thus if we define $f\colon I\to\mathbb{R}$ as

$$f(x)=F_2(x)+c[b_1(1)]^2x \tag{3.18}$$

for all $x\in I$, then (3.17) reduces to the functional equation (2.1) with $f(0)=0$. Now, from $f(0)=0$, (3.18) and (2.2), (α_1)(i) follows. Similarly, making use of (α_2)(ii) and (α_3)(ii) in (FE4) and proceeding as above, we can obtain respectively (α_2)(i) and (α_3)(i).

Note 3.3
In the solution (α_1), if $b_1(1)\neq 0$, then $F_2(1)=-c[b_1(1)]^2\neq 0$ (because of (2.4)) and $K_2(1)=b_1(1)\neq 0$.

The above observations reveal that under the conditions stated in the statement of Theorem 3.1, there are three solutions, namely (α_1) to (α_3) which satisfy (1.1) and out of these three solutions, there is one solution namely (α_1) (with $b_1(1)\neq 0$), in which both $F_2(1)\neq 0$, $K_2(1)\neq 0$.

Note 3.4 ([7])
Let $c\neq 0$ be a given real constant and $F_2\colon I\to\mathbb{R}$, $K_2\colon I\to\mathbb{R}$ be mappings which satisfy (FE4) for all $(p_1,\dots,p_n)\in\Gamma_n$, $(q_1,\dots,q_m)\in\Gamma_m$; $n\geq 3$, $m\geq 3$ be fixed integers. Suppose further that the mappings $F_2\colon I\to\mathbb{R}$ and $K_2\colon I\to\mathbb{R}$ satisfy (1.1) and (1.2). Then, any general solution (F_2,K_2) of (FE4) is of the form (α_1)(with $b_1(1)=0$), (α_2) and (α_3); $\mu\neq 0$ is an arbitrary real constant; $b_1\colon\mathbb{R}\to\mathbb{R}$ is an additive mapping; the mappings $a\colon\mathbb{R}\to\mathbb{R}$, $D\colon\mathbb{R}\times I\to\mathbb{R}$ are as described in Modified Form of Result 2.2; and the mappings $M\colon I\to\mathbb{R}$ and $\ell\colon I\to\mathbb{R}$ are as described in the statement of Theorem 3.1.

4. On the functional equation (FE3)

In this section, we make use of Theorem 3.1 to prove:

Theorem 4.1
Suppose the mappings $F_1\colon I\to\mathbb{R}$, $K_1\colon I\to\mathbb{R}$ satisfy the functional equation (FE3) *for all $(p_1,\dots,p_n)\in\Gamma_n$, $(q_1,\dots,q_m)\in\Gamma_m$; $n\geq 3$, $m\geq 3$ being fixed integers. Then, any general solution (F_1,K_1) of* (FE3)*, for all $p\in I$, is of the form*

$$\begin{cases}\text{(i) } F_1(p)=F_1(0)-c[b_1(1)]^2p+(nm-n-m)F_1(0)p+a(p)+D(p,p)\\ \text{(ii) } K_1(p)=b_1(p)-nK_1(0)p+K_1(0)\end{cases} \tag{β_1}$$

or

$$\begin{cases} \text{(i) } F_1(p) = F_1(0) + (nm-n-m)F_1(0)p + \frac{1}{2}cp[\ell(p)]^2 + a(p) + D(p,p) \\ \text{(ii) } K_1(p) = p\ell(p) - nK_1(0)p + K_1(0) \end{cases} \qquad (\beta_2)$$

or

$$\begin{cases} \text{(i) } F_1(p) = F_1(0) + (nm-n-m)F_1(0)p + c\mu^2[M(p)-p] + a(p) + D(p,p) \\ \text{(ii) } K_1(p) = \mu[M(p)-p] - nK_1(0)p + K_1(0) \end{cases} \qquad (\beta_3)$$

where $\mu \neq 0$ is an arbitrary real constant; $b_1 : \mathbb{R} \to \mathbb{R}$ is an additive mapping with $b_1(1)$ an arbitrary real constant; the mappings $a\colon \mathbb{R} \to \mathbb{R}$ and $D\colon \mathbb{R} \times I \to \mathbb{R}$ are as described in the Modified Form of Result 2.2; the mappings $M\colon I \to \mathbb{R}$ and $\ell\colon I \to \mathbb{R}$ are as described in Theorem 3.1.

Proof. Let us define the mappings $F_2\colon I \to \mathbb{R}$ and $K_2\colon I \to \mathbb{R}$ as

$$F_2(p) = F_1(p) - (nm-n-m)F_1(0)p - F_1(0) \qquad (4.1)$$

and

$$K_2(p) = K_1(p) + nK_1(0)p - K_1(0) \qquad (4.2)$$

for all $p \in I$. Then, from (4.1) and (4.2), we have (1.1) and

$$F_2(1) = F_1(1) - (n-1)(m-1)F_1(0), \quad K_2(1) = K_1(1) + (n-1)K_1(0). \qquad (4.3)$$

Also, from (FE3), (4.1) and (4.2), the functional equation (FE4) follows for all $(p_1, \ldots, p_n) \in \Gamma_n$, $(q_1, \ldots, q_m) \in \Gamma_m$; $n \geq 3$, $m \geq 3$, being fixed integers.

From (4.1), (4.2) and (FE4), we obtain

$$\begin{aligned} &\sum_{i=1}^{n}\sum_{j=1}^{m} F_1(p_iq_j) - (nm-n-m)F_1(0) - nmF_1(0) \\ &\quad = \sum_{i=1}^{n} F_1(p_i) - (nm-n-m)F_1(0) - nF_1(0) \\ &\qquad + \sum_{j=1}^{m} F_1(q_j) - (nm-n-m)F_1(0) - mF_1(0) \\ &\qquad + c\Big[\sum_{i=1}^{n} K_1(p_i) + nK_1(0) - nK_1(0)\Big] \times \Big[\sum_{j=1}^{m} K_1(q_j) + nK_1(0) - mK_1(0)\Big]. \end{aligned}$$

This gives

$$\begin{aligned} \sum_{i=1}^{n}\sum_{j=1}^{m} F_1(p_iq_j) = \sum_{i=1}^{n} F_1(p_i) + \sum_{j=1}^{m} F_1(q_j) + c\sum_{i=1}^{n} K_1(p_i)\sum_{j=1}^{m} K_1(q_j) \\ + c(n-m)K_1(0)\sum_{i=1}^{n} K_1(p_i) \end{aligned}$$

which is (FE3). So, by Theorem 3.1, the solutions (α_1) to (α_3) follow. The required solutions (β_1) to (β_3) follow respectively from solutions (α_1) to (α_3) by making use of (4.1) and (4.2). The details are omitted for the sake of brevity.

5. On the functional equation (FE2)

In this section, by making use of Theorem 4.1, we prove:

Theorem 5.1
Let $F_1\colon I\to\mathbb{R}$, $K_1\colon I\to\mathbb{R}$ and $L_1\colon I\to\mathbb{R}$ be mappings which satisfy the functional equation (FE2) *for all $(p_1,\dots,p_n)\in\Gamma_n$, $(q_1,\dots,q_m)\in\Gamma_m$; $n\ge 3$, $m\ge 3$ being fixed integers. Then, for all $p\in I$, any general solution (F_1,K_1,L_1) of* (FE2) *is of the form*

$$\begin{cases} F_1(p)=F_1(0)+(nm-n-m)F_1(0)p+a(p)+D(p,p)\\ K_1(p)=\bar{A}_1(p)+K_1(0)\\ L_1 \textit{ an arbitrary real-valued mapping}\end{cases}\tag{S$_1$}$$

or

$$\begin{cases} F_1(p)=F_1(0)+(nm-n-m)F_1(0)p+a(p)+D(p,p)\\ K_1 \textit{ an arbitrary real-valued mapping}\\ L_1(p)=\bar{A}_2(p)+L_1(0)\end{cases}\tag{S$_2$}$$

or

$$(\beta_1)(\mathrm{i}),(\beta_1)(\mathrm{ii}) \quad \textit{and} \quad L_1(p)=c[b_1(p)-nK_1(0)p]+\bar{A}_3(p)+L_1(0)\tag{S$_3$}$$

or

$$(\beta_2)(\mathrm{i}),(\beta_2)(\mathrm{ii}) \quad \textit{and} \quad L_1(p)=c[p\ell(p)-nK_1(0)p]+\bar{A}_3(p)+L_1(0)\tag{S$_4$}$$

or

$$(\beta_3)(\mathrm{i}),(\beta_3)(\mathrm{ii}) \quad \textit{and} \quad L_1(p)=c\{\mu[M(p)-p]-nK_1(0)p\}+\bar{A}_3(p)+L_1(0)\tag{S$_5$}$$

where $\bar{A}_i\colon\mathbb{R}\to\mathbb{R}$ $(i=1,2,3)$, $b_1\colon\mathbb{R}\to\mathbb{R}$ are additive mappings with $\bar{A}_1(1)=-nK_1(0)$, $\bar{A}_2(1)=-mK_1(0)$, $\bar{A}_3(1)=-mL_1(0)+cnK_1(0)$; $\mu\ne 0$, $c\ne 0$ and $b_1(1)$ are arbitrary real constants; the mappings $a\colon\mathbb{R}\to\mathbb{R}$ and $D\colon\mathbb{R}\times I\to\mathbb{R}$ are as described in the Modified Form of Result 2.2; the mappings $M\colon I\to\mathbb{R}$ and $\ell\colon I\to\mathbb{R}$ are as described in Theorem 3.1.

Proof. The functional equation (FE2) has, indeed, two types of general solutions (F_1,K_1,L_1). The first type of these general solutions (F_1,K_1,L_1) are those whose first two components F_1 and K_1 do not form a general solution (F_1,K_1) of (FE3). There are two such solutions (S_1) and (S_2). The second type of general solutions (F_1,K_1,L_1) are those whose first two components F_1 and K_1 do form a general solution (F_1,K_1) of (FE3). There are three such solutions (S_3) to (S_5) and each such solution (F_1,K_1,L_1) may be written in the form $(\beta,L_1(p))$, where $\beta=(\beta_t(\mathrm{i}),\beta_t(\mathrm{ii})), t=1,2,3$. Both types of these solutions can be obtained by making use of Result 2.4.

From (2.7), for all $(p_1,\dots,p_n)\in\Gamma_n$, $\sum_{i=1}^n K_1(p_i)=0$. So, referring to (FE2), the mapping L_1 is arbitrary and also (FE2) reduces to the functional equation $\sum_{i=1}^n\sum_{j=1}^m F_1(p_iq_j)=\sum_{i=1}^n F_1(p_i)+\sum_{j=1}^m F_1(q_j)$ valid for all $(p_1,\dots,p_n)\in\Gamma_n$,

$(q_1, \ldots, q_m) \in \Gamma_m$; $n \geq 3$, $m \geq 3$ being fixed integers. By Modified Form of Result 2.2, it follows that

$$F_1(p) = F_1(0) + (nm - n - m)F_1(0)p + a(p) + D(p,p) \tag{5.1}$$

for all $p \in I$. Equation (5.1), together with (2.7), constitute the solution (S_1) of (FE2). Similarly, one can obtain the solution (S_2) of (FE2). Now, referring to (2.9), we have

$$\sum_{j=1}^{m} L_1(q_j) = c\Bigg[\sum_{j=1}^{m} K_1(q_j) + (n-m)K_1(0)\Bigg]. \tag{5.2}$$

Making use of (5.2) in (FE2), we get the functional equation (FE3) whose solutions are (β_1), (β_2) and (β_3). Making use of the respective forms of $K_1(p)$ appearing in (β_1)(ii), (β_2)(ii) and (β_3)(ii), the corresponding forms of $L_1(p)$ can be obtained.

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