FOLIA 128

Annales Universitatis Paedagogicae Cracoviensis

Studia Mathematica XII (2013)

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On Levinson's inequality

Abstract. We give a very simple proof of the classical Levinson inequality and generalize the result by Mercer.

1. Introduction

In 1964 Norman Levinson ([3]) used the Taylor expansion to prove the following inequality.

Theorem 1.1 ([3])
Suppose that $f\colon [0,c] \to \mathbb{R}$ *has a nonnegative third derivative, for* $i = 1,\dots,n$ $p_i > 0$, $0 \leqslant x_i \leqslant \frac{c}{2}$, $y_i = c - x_i$ *and* $\sum_{i=1}^n p_i = 1$, *then the inequality*

$$\sum_{i=1}^n p_i f(x_i) - f(\overline{x}) \leqslant \sum_{i=1}^n p_i f(y_i) - f(\overline{y}) \tag{1}$$

holds. Here $\overline{x} = \sum_{i=1}^n p_i x_i$ *and* $\overline{y} = \sum_{i=1}^n p_i y_i$ *denote the weighted arithmetic means.*

The same year Tiberiu Popoviciu generalized it by showing that for (1) to hold it is enough that f be 3-convex ([6]), and Peter S. Bullen ([2]) gave an alternative proof using mathematical induction. There are also other versions of this inequality, see e.g. [5]. By rescaling axes, the classic Levinson inequality can be restated in the following way.

Theorem 1.2 ([2])
If $f\colon [a,b] \to \mathbb{R}$ *is 3-convex,* $a \leqslant x_i,\ y_i \leqslant b$, $x_i + y_i = c$, $p_i > 0$ *for* $i = 1,\dots,n$, $\sum_{i=1}^n p_i = 1$ *and*

$$\max(x_1,\dots,x_n) \leqslant \min(y_1,\dots,y_n), \tag{2}$$

then (1) *holds.*

As we can see, both versions assume that x's and y's add up to the same number. Recently, Mercer made a significant improvement in [4].

AMS (2000) Subject Classification: 26D15.

Theorem 1.3
If $f\colon [a,b]\to\mathbb{R}$ *satisfies* $f'''\geqslant 0$, $p_i>0$ *for* $i=1,\dots,n$, $\sum_{i=1}^n p_i=1$, *and* $a\leqslant x_i$, $y_i\leqslant b$ *are such that* (2) *holds and*

$$\sum_{i=1}^n p_i(x_i-\overline{x})^2=\sum_{i=1}^n p_i(y_i-\overline{y})^2, \tag{3}$$

then (1) *holds.*

It is natural to try to get similar result for 3-convex function. In this paper we shall give a very simple proof of Theorem 1.2 and generalize the Mercer's result in the following way.

Theorem 1.4
Let I *be an open subinterval of* $\mathbb{R}$ *(bounded or unbounded),* $f\colon I\to\mathbb{R}$ *be a 3-convex function and* $X,Y\colon(\Omega,\mu)\to I$ *be two random variables satisfying*

V1) $\mathbf{E}[X^2]$, $\mathbf{E}[Y^2]$, $\mathbf{E}[f(X)]$, $\mathbf{E}[f(Y)]$, $\mathbf{E}[f'(X)]$, $\mathbf{E}[f'(Y)]$, $\mathbf{E}[Xf'(X)]$, $\mathbf{E}[Yf'(Y)]$ *are finite,*

V2) $\operatorname{ess\,sup} X\leqslant\operatorname{ess\,inf} Y$,

V3) $\mathbf{Var}\,X=\mathbf{Var}\,Y$,

then

$$\mathbf{E}[f(X)]-f(\mathbf{E}[X])\leqslant\mathbf{E}[f(Y)]-f(\mathbf{E}[Y]). \tag{4}$$

We also provide some conditions for (4) to hold if the condition V3) is not satisfied.

2. Definitions and basic properties

Definition 2.1
For a function $f\colon I\to\mathbb{R}$ its n^{th} *divided difference* is defined inductively by the formula

$$[x_0,\dots,x_n;f]=\begin{cases} f(x_0) & \text{if } n=0,\\ \dfrac{[x_0,\dots,x_{n-2},x_n;f]-[x_0,\dots,x_{n-2},x_{n-1};f]}{x_n-x_{n-1}} & \text{if } n>0,\end{cases}$$

where x_i's are pairwise distinct.

Definition 2.2
A function $f\colon I\to\mathbb{R}$ is called *n-convex* if

$$[x_0,\dots,x_n;f]\geq 0$$

for any choice of arguments, and *n-concave* if the inverse inequality holds.

The 1-convex (resp. concave) functions are the increasing (resp. decreasing) functions, while the 2-convex (resp. concave) functions are convex (resp. concave) functions, i.e. the functions satisfying the inequality

$$f(tx + (1-t)y) \leqslant \text{ (resp. } \geqslant) \, tf(x) + (1-t)f(y)$$

for all $x, y \in I$ and $0 < t < 1$.

The following property follows immediately from definition.

Property 2.3
The function g is convex if and only if its divided difference $h(x, y) = \frac{g(x)-g(y)}{x-y}$, $x \neq y$ increases in both variables.

This implies immediately

Property 2.4
If g is convex, then its right and left derivatives exist in the interior of its domain and for $x < y$ the inequalities

$$g'_-(x) \leqslant g'_+(x) \leqslant g'_-(y) \leqslant g'_+(y)$$

hold.

Another useful property of symmetric sum follows also from Property 2.3.

Property 2.5
If g is convex, then the symmetric sum $g(a+x) + g(a-x)$ increases for $x > 0$.

Boas and Widder gave the following characterization of n-convex functions.

Property 2.6 ([1])
If a function g is n-convex, then it is $n-2$ times differentiable and $g^{(n-2)}$ is convex.

At the end of this section let us remind the Leibniz Rule.

Lemma 2.7 (Differentiation under integral)
Let I be an open subinterval of $\mathbb{R}$ and (Ω, μ) be a measure space. Suppose a function $h\colon I \times \Omega \to \mathbb{R}$ satisfies the following conditions:

L1) *for every $t \in I$ the function $h(t, \cdot)$ is integrable,*

L2) *for every $t \in I$ the derivative h_t exists μ a.e.*

L3) *there exists an integrable function $\theta\colon \Omega \to \mathbb{R}$ such that for every t $|h_t(t, \omega)| \leqslant \theta(\omega)$.*

Then for every $t \in I$

$$\frac{d}{dt} \int_{\Omega} h(t, \omega) \, \mu(d\omega) = \int_{\Omega} h_t(t, \omega) \, \mu(d\omega).$$

3. Simple proof of Theorem 1.2

We can rewrite (1) as

$$f\Big(c-\sum_{i=1}^{n}p_ix_i\Big)-f\Big(\sum_{i=1}^{n}p_ix_i\Big)\leqslant\sum_{i=1}^{n}p_if(c-x_i)-\sum_{i=1}^{n}p_if(x_i),\tag{5}$$

which is the Jensen inequality for the function $g(x)=f(c-x)-f(x)$. The function g is differentiable by Property 2.6. To show its convexity, note that

$$g'(x)=-[f'(c-x)+f'(x)]=-\Big[f'\Big(\frac{c}{2}+\Big(\frac{c}{2}-x\Big)\Big)+f'\Big(\frac{c}{2}-\Big(\frac{c}{2}-x\Big)\Big)\Big].$$

As x increases, $\frac{c}{2}-x$ decreases, and by Property 2.5, the expression in square brackets decreases. Thus g' increases, so g is convex and we are done.

4. Proof of Theorem 1.4

For $0\leqslant t\leqslant 1$ define two new random variables

$$X_t=tX+(1-t)\mathbf{E}[X]\quad\text{and}\quad Y_t=tY+(1-t)\mathbf{E}[Y]$$

and consider the function

$$U(t,\omega)=f(Y_t(\omega))-f(\mathbf{E}[Y])-[f(X_t(\omega))-f(\mathbf{E}[X])].\tag{6}$$

Our goal will be to show that $V(t)=\int_\Omega U(t,\omega)\,\mu(d\omega)$ is a nonnegative, increasing, convex function of t.

Let d be any number satisfying $\operatorname{ess\,sup}X\leqslant d\leqslant\operatorname{ess\,inf}Y$. The proof is split into several steps.

4.1. Behavior near the endpoints of the interval $I=(a,b)$

The functions f and f' are continuous, thus bounded on every compact subinterval of (a,b). In order to obtain some global bounds we need to know how they behave near the ends.

Remark 4.1
Since a function g is convex or concave in the interval (a,b), its monotonicity cannot change more that once. If it is not bounded near a, we conclude that there exists $a'>a$ such that g preserves sign and $|g|$ decreases in (a,a'). Similarly, or g is bounded near b or there exists $b'<b$ such that g preserves sign and $|g|$ increases in (b',b).

Since f is 3-convex, its derivative is convex and its monotonicity changes at most once. Thus the convexity of f changes at most once, and consequently the Remark 4.1 applies to both f and f'.

4.2. Absolute integrability

Let g be any of f, f' or $\mathrm{Id}_\mathbb{R}\cdot f'$, where $\mathrm{Id}_\mathbb{R}$ is the identity function on $\mathbb{R}$. It follows from the results of the subsection 4.1. that g is bounded from above or from below on $(a, d]$. Since

$$\int\limits_{\Omega} g(X)\,\mu(d\omega) = \int\limits_{\{\omega\colon g(X(\omega))>0\}} g(X)\,\mu(d\omega) + \int\limits_{\{\omega\colon g(X(\omega))<0\}} g(X)\,\mu(d\omega)$$

we conclude that two integrals in this formula are finite, thus so is the third one, therefore

$$\int\limits_{\Omega} |g(X)|\,\mu(d\omega) = \int\limits_{\{\omega\colon g(X(\omega))>0\}} g(X)\,\mu(d\omega) - \int\limits_{\{\omega\colon g(X(\omega))<0\}} g(X)\,\mu(d\omega)$$

is also finite.

The same holds for $g(Y)$.

4.3. Leibniz Rule

We shall show now that the function U defined by formula (6) satisfies the Leibniz Rule. Consider $f(X_t)$. If f is bounded near a, then clearly the assumption L1) is satisfied. If not, then choose a' as in step 4.1. in such a way that $a' < \mathbf{E}[X]$. Let $M = \max_{x\in[a',d]} |f(x)|$. Then we have

$$|f(X_t(\omega))| \leqslant \begin{cases} M & \text{if } X_t(\omega) \in [a', d], \\ |f(X(\omega))| & \text{otherwise, since } X_t(\omega) \geqslant X(\omega). \end{cases}$$

Thus $|f(X_t)| \leqslant \max(M, f(X))$ and we are done with L1).

By Property 2.6 condition L2) is also satisfied. To show that L3) holds note that

$$\left|\frac{d}{dt} f(X_t)\right| = |(X - \mathbf{E}[X])f'(X_t)| \leqslant |X||f'(X_t)| + |\mathbf{E}[X]||f'(X_t)|.$$

If f' is bounded, then the right-hand side is not greater that $M|X| + N$ for some $M, N > 0$, otherwise the same reasoning as above applied to f' gives the bound $\max(|Xf'(X)| + |\mathbf{E}[X]||f'(X)|, (\max(|a'|, |d|) + |\mathbf{E}[X]|)M)$.

The same reasoning can be applied to $f(Y_t)$, so we can apply the Leibniz Rule to the function U.

4.4. Final step

Clearly, $U(0, \omega) = 0$, so $V(0) = 0$. The function U is differentiable in t for all ω and

$$U_t(t, \omega) = (Y(\omega) - \mathbf{E}[Y])f'(Y_t(\omega)) - (X(\omega) - \mathbf{E}[X])f'(X_t(\omega)). \tag{7}$$

Applying the Leibniz Rule we obtain

$$V'(0) = \int\limits_{\Omega} U_t(0, \omega)\,\mu(d\omega) = 0.$$

In virtue of Property 2.6, the convexity of the derivative of a 3-convex functions implies that $f''_-(\operatorname{ess\,sup} X) \leqslant f''_+(\operatorname{ess\,inf} Y)$ ($f''_\pm$ denotes here the right and left derivatives of f'). Let A be an arbitrary number from the interval $[f''_-(\operatorname{ess\,sup} X), f''_+(\operatorname{ess\,inf} Y)]$.

We have

$$\begin{aligned}\frac{U_t(u,\omega)-U_t(v,\omega)}{u-v} &= (Y(\omega)-\mathbf{E}[Y])\frac{f'(Y_u(\omega))-f'(Y_v(\omega))}{u-v}\\ &\quad -(X(\omega)-\mathbf{E}[X])\frac{f'(X_u(\omega))-f'(X_v(\omega))}{u-v}\\ &= (Y(\omega)-\mathbf{E}[Y])^2\frac{f'(Y_u(\omega))-f'(Y_v(\omega))}{Y_u(\omega)-Y_v(\omega)}\\ &\quad -(X(\omega)-\mathbf{E}[X])^2\frac{f'(X_u(\omega))-f'(X_v(\omega))}{X_u(\omega)-X_v(\omega)}\\ &= (Y(\omega)-\mathbf{E}[Y])^2\Big[\frac{f'(Y_u(\omega))-f'(Y_v(\omega))}{Y_u(\omega)-Y_v(\omega)}-A\Big]\\ &\quad +(X(\omega)-\mathbf{E}[X])^2\Big[A-\frac{f'(X_u(\omega))-f'(X_v(\omega))}{X_u(\omega)-X_v(\omega)}\Big]\\ &\quad +A[(Y(\omega)-\mathbf{E}[Y])^2-(X(\omega)-\mathbf{E}[X])^2].\end{aligned} \tag{8}$$

Property 2.3 applied to f' shows that the expressions in square brackets in (8) are nonnegative a.e., so

$$\frac{U_t(u,\omega)-U_t(v,\omega)}{u-v} \geqslant A[(Y(\omega)-\mathbf{E}[Y])^2-(X(\omega)-\mathbf{E}[X])^2] \quad \text{a.e.}$$

Integrating and applying once more the Leibniz Rule one get

$$\frac{V'(u)-V'(v)}{u-v} \geqslant A(\mathbf{Var}\,Y-\mathbf{Var}\,X)=0. \tag{9}$$

This means V' is increasing, and we conclude that V is nonnegative and convex in the unit interval. In particular $V(0) \leqslant V(1)$, which is the Levinson inequality.

5. Further results

Let us take a closer look at the proof of Theorem 1.4. We have some freedom in selection of A: this can be any number from the gap between $f''(X)$ and $f''(Y)$. The critical inequality (9) holds not only if $\mathbf{Var}\,Y = \mathbf{Var}\,X$ but also in case when $A(\mathbf{Var}\,Y - \mathbf{Var}\,X) \geqslant 0$. This means that if the gap between X's and Y's is in the area where the function f is strictly convex/concave (i.e. $A > (<)0$), then the assumption V3) of Theorem 1.4 can be replaced by a weaker condition $\mathbf{Var}\,Y \geqslant (\leqslant)\,\mathbf{Var}\,X$.

If f changes convexity between X's and Y's, then we can choose $A = 0$ and the assumption V3) becomes obsolete. In fact, in this case the Levinson inequality follows immediately from the Jensen inequality, because the left-hand side of (4) is nonpositive, while its right-hand side is nonnegative. Thus we obtain the following version of Theorem 1.4.

Theorem 5.1
Let I be an open subinterval of $\mathbb{R}$ (bounded or unbounded), $f\colon I \to \mathbb{R}$ be a 3-convex function and $X, Y\colon (\Omega, \mu) \to I$ be two random variables satisfying

(1) $\mathbf{E}[X^2]$, $\mathbf{E}[Y^2]$, $\mathbf{E}[f(X)]$, $\mathbf{E}[f(Y)]$, $\mathbf{E}[f'(X)]$, $\mathbf{E}[f'(Y)]$, $\mathbf{E}[Xf'(X)]$, $\mathbf{E}[Yf'(Y)]$ *are finite,*

(2) $\operatorname{ess\,sup} X \leqslant \operatorname{ess\,inf} Y$,

(3) $f''_+(\operatorname{ess\,sup} X) > 0$ *and* $\mathbf{Var}\, X \leqslant \mathbf{Var}\, Y$, *or* $f''_-(\operatorname{ess\,inf} Y) < 0$ *and* $\mathbf{Var}\, X \geqslant \mathbf{Var}\, Y$, *or* $f''_+(\operatorname{ess\,sup} X) \leqslant 0 \leqslant f''_-(\operatorname{ess\,inf} Y)$,

then

$$\mathbf{E}[f(X)] - f(\mathbf{E}[X]) \leqslant \mathbf{E}[f(Y)] - f(\mathbf{E}[Y])$$

If X and Y are discrete random variables, then we obtain the following corollary.

Corollary 5.2
Let $f\colon I \to \mathbb{R}$ be a 3-convex function, $p_1, \ldots, p_m$ and $q_1, \ldots, q_n$ be positive real numbers such that $p_1 + \ldots + p_m = q_1 + \ldots + q_n = 1$. For $x_1, \ldots, x_m, y_1, \ldots, y_n \in I$ such that

$$\max_{1\leqslant i\leqslant m} x_i \leqslant \min_{1\leqslant i\leqslant n} y_i, \tag{10}$$

let $\overline{x} = \sum_{i=1}^m p_i x_i$ and $\overline{y} = \sum_{i=1}^n q_i y_i$. If

$$f''_+(\max x_i) > 0 \quad \text{and} \quad \sum_{i=1}^m p_i (x_i - \overline{x})^2 \leqslant \sum_{i=1}^n q_i (y_i - \overline{y})^2,$$

or

$$f''_-(\min x_i) < 0 \quad \text{and} \quad \sum_{i=1}^m p_i (x_i - \overline{x})^2 \geqslant \sum_{i=1}^n q_i (y_i - \overline{y})^2,$$

or

$$f''_+\big(\max_{1\leqslant i\leqslant m} x_i\big) \leqslant 0 \leqslant f''_-\big(\min_{1\leqslant i\leqslant n} y_i\big),$$

then

$$\sum_{i=1}^m p_i f(x_i) - f(\overline{x}) \leqslant \sum_{i=1}^n q_i f(y_i) - f(\overline{y}).$$

The function V used in the proof of Theorem 1.4 is convex, so we can apply the Hermite-Hadamard inequality.

Corollary 5.3
If the assumptions of Theorem 1.4 or 5.1 are satisfied and $F(x) = \int f(x)\,dx$, then

$$\begin{aligned} 0 &\leqslant \mathbf{E}\Big[f\Big(\frac{Y + \mathbf{E}[Y]}{2}\Big)\Big] - f(\mathbf{E}[Y]) - \mathbf{E}\Big[f\Big(\frac{X + \mathbf{E}[X]}{2}\Big)\Big] + f(\mathbf{E}[X]) \\ &\leqslant \mathbf{E}\Big[\frac{F(Y) - F(\mathbf{E}[Y])}{Y - \mathbf{E}[Y]}\Big] - f(\mathbf{E}[Y]) - \mathbf{E}\Big[\frac{F(X) - F(\mathbf{E}[X])}{X - \mathbf{E}[X]}\Big] + f(\mathbf{E}[X]) \\ &\leqslant \frac{1}{2}(\mathbf{E}[f(Y)] - f(\mathbf{E}[Y]) - \mathbf{E}[f(X)] + f(\mathbf{E}[X])). \end{aligned}$$

Its discrete version reads as follows

Corollary 5.4
Under the assumptions of Corollary 5.2 the following inequalities are valid:

$$\begin{aligned}0 &\leqslant \sum_{i=1}^{n} q_i f\Big(\frac{y_i+\overline{y}}{2}\Big) - f(\overline{y}) - \sum_{i=1}^{m} p_i f\Big(\frac{x_i+\overline{x}}{2}\Big) + f(\overline{x})\\ &\leqslant \sum_{i=1}^{n} q_i \frac{\int_{\overline{y}}^{y_i} f(t)\,dt}{y_i-\overline{y}} - f(\overline{y}) - \sum_{i=1}^{m} p_i \frac{\int_{\overline{x}}^{x_i} f(t)\,dt}{x_i-\overline{x}} + f(\overline{x})\\ &\leqslant \frac{1}{2}\Big[\sum_{i=1}^{n} q_i f(y_i) - f(\overline{y}) - \sum_{i=1}^{m} p_i f(x_i) + f(\overline{x})\Big].\end{aligned}$$

Note that the rightmost inequality can be rewritten in a nice symmetric form

$$\sum_{i=1}^{m} p_i\Big(\frac{f(x_i)+f(\overline{x})}{2} - \frac{\int_{\overline{x}}^{x_i} f(t)\,dt}{x_i-\overline{x}}\Big) \leqslant \sum_{i=1}^{n} q_i\Big(\frac{f(y_i)+f(\overline{y})}{2} - \frac{\int_{\overline{y}}^{y_i} f(t)\,dt}{y_i-\overline{y}}\Big), \quad (11)$$

while the leftmost inequality is

$$\sum_{i=1}^{m} p_i\Big(\frac{\int_{\overline{x}}^{x_i} f(t)\,dt}{x_i-\overline{x}} - f\Big(\frac{x_i+\overline{x}}{2}\Big)\Big) \leqslant \sum_{i=1}^{n} q_i\Big(\frac{\int_{\overline{y}}^{y_i} f(t)\,dt}{y_i-\overline{y}} - f\Big(\frac{y_i+\overline{y}}{2}\Big)\Big). \quad (12)$$

6. Applications

As an application we provide generalization and refinement of the famous Ky-Fan inequality: if $0 < x_i \leqslant \frac{1}{2}$ and $y_i = 1 - x_i$, then

$$\frac{A(y_1,\ldots,y_i)}{A(x_1,\ldots,x_i)} \leqslant \frac{G(y_1,\ldots,y_i)}{G(x_1,\ldots,x_i)}.$$

Levinson in [3] noticed, that this inequality (in weighted version) follows from his inequality by taking $f(x) = \log x$. The logarithmic function is concave, so if $\sum_{i=1}^{m} p_i(x_i-\overline{x})^2 \geqslant \sum_{i=1}^{n} q_i(y_i-\overline{y})^2$ and (10) holds, then Corollary 5.4 yields

$$\frac{\sum_{i=1}^{n} q_i y_i}{\sum_{i=1}^{m} p_i x_i} \leqslant \frac{\prod_{i=1}^{n} A^{q_i}(y_i,\overline{y})}{\prod_{i=1}^{m} A^{p_i}(x_i,\overline{x})} \leqslant \frac{\prod_{i=1}^{n} I^{q_i}(y_i,\overline{y})}{\prod_{i=1}^{m} I^{p_i}(x_i,\overline{x})} \leqslant \frac{\prod_{i=1}^{n} G^{q_i}(y_i,\overline{y})}{\prod_{i=1}^{m} G^{p_i}(x_i,\overline{x})} \leqslant \frac{\prod_{i=1}^{n} y_i^{q_i}}{\prod_{i=1}^{m} x_i^{p_i}},$$

where $A(x,y) = \frac{x+y}{2}$, $I(x,y) = e^{-1}(\frac{x^x}{y^y})^{\frac{1}{x-y}}$ and $G(x,y) = \sqrt{xy}$ are the arithmetic, identric and geometric means.

Consider the function $f(x) = -\frac{1}{x}$ for $x > 0$.

If x's and y's are the same as in the previous example, then, by (11)

$$\sum_{i=1}^{m} p_i\Big(\frac{1}{H(x_i,\overline{x})} - \frac{1}{L(x_i,\overline{x})}\Big) \geqslant \sum_{i=1}^{n} q_i\Big(\frac{1}{H(y_i,\overline{y})} - \frac{1}{L(y_i,\overline{y})}\Big),$$

where $H(x,y)=\frac{2xy}{x+y}$ and $L(x,y)=\frac{x-y}{\log x-\log y}$ denote the harmonic and the logarithmic means. In this case Corollary 5.4 gives

$$\frac{1}{\overline{x}}-\frac{1}{\overline{y}}\geqslant\sum_{i=1}^{m}\frac{p_i}{L(x_i,\overline{x})}-\sum_{i=1}^{n}\frac{q_i}{L(y_i,\overline{y})}.$$

In particular, setting $y_1=\ldots=y_n>\max x_i$ we obtain the inequality

$$\frac{1}{\overline{x}}\geqslant\sum_{i=1}^{m}\frac{p_i}{L(x_i,\overline{x})}$$

valid for all positive x's.

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Received: May 24, 2013; final version: September 2, 2013;
available online: November 18, 2013.