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## Underlying functors on fibered manifolds

*To Andrzej Zajtz, on the occasion of his 70th birthday*

**Abstract.** For a product preserving bundle functor on the category of fibered manifolds we describe subordinated functors and we introduce the concept of the underlying functor. We also show that there is an affine bundle structure on product preserving functors on fibered manifolds.

### Introduction

Let  $\mathcal{M}f$  be the category of all manifolds and all smooth maps,  $\mathcal{FM}$  be the category of fibered manifolds and all fiber preserving maps and  $\mathcal{FM}_m$  be the category of fibered manifolds over  $m$ -dimensional bases and fibered manifold morphisms with local diffeomorphisms as base maps. It is well known that the product preserving bundle functors on  $\mathcal{M}f$  coincide with Weil functors and their natural transformations are in bijection with the algebra homomorphisms, [6]. In particular, for every product preserving bundle functor  $F$  on  $\mathcal{M}f$  there exists a Weil algebra  $A$  such that  $F$  is a Weil functor of the form  $F = T^A$ . Further, W.M. Mikulski [9] has clarified that all product preserving bundle functors on  $\mathcal{FM}$  are of the form  $T^\mu$ , where  $\mu: A \rightarrow B$  is a homomorphism of Weil algebras. Finally, I. Kolář and W.M. Mikulski have characterized all fiber product preserving functors on  $\mathcal{FM}_m$  in terms of Weil algebras, [7].

Recently it has been also pointed out that one can introduce an affine bundle structure on product preserving bundles. The first general result from this field is the paper [4] by I. Kolář, who described the affine structure on product preserving bundles on  $\mathcal{M}f$ . In particular, he introduced the underlying  $k$ -th order Weil functor  $T^{A_k}$  for every  $r$ -th order Weil functor  $T^A$  and proved that  $T^A M \rightarrow T^{A_{r-1}} M$  is an affine bundle. Further, in [2] we introduced the general concept of a subordinated functor and we showed that there is an affine structure on the fiber product preserving functors on  $\mathcal{FM}_m$ .

The aim of this paper is to define underlying functors for every product preserving bundle functor on  $\mathcal{FM}$  and to describe affine bundle structure on

such functors. In Section 1 we recall some properties of product preserving functors on  $\mathcal{FM}$  and we describe natural transformations between such functors. In Section 2 we study subordinated functors on  $\mathcal{FM}$  and in Section 3 we introduce the concept of an underlying functor on  $\mathcal{FM}$ . Finally, in Section 4 we show that there is an affine bundle structure on product preserving functors on  $\mathcal{FM}$ .

All manifolds and maps are assumed to be infinitely differentiable. Unless otherwise specified, we use the terminology and notation from the book [6].

## 1. Product preserving bundle functors on fibered manifolds

First we recall the result by W.M. Mikulski, who has characterized all product preserving bundle functors defined on the category  $\mathcal{FM}$  of fibered manifolds in terms of Weil algebras, [9]. A product preserving bundle functor  $F$  on  $\mathcal{FM}$  determines a homomorphism of Weil algebras  $\mu: A \rightarrow B$  in the following way. Denote by  $i, j: \mathcal{M}f \rightarrow \mathcal{FM}$  two canonical bundle functors defined by  $i(M) = \text{id}_M: M \rightarrow M$ ,  $i(f) = (f, f)$ ,  $j(M) = \text{pt}_M: M \rightarrow \text{pt}$ ,  $j(f) = (f, \text{id}_{\text{pt}})$ , where  $\text{pt}$  denote one element manifold. Then  $t_M = (\text{id}_M, \text{pt}_M): i(M) \rightarrow j(M)$  is the identity natural transformation. Applying functor  $F$ , we obtain two product preserving functors  $F \circ i$ ,  $F \circ j$  on  $\mathcal{M}f$  and a natural transformation  $F \circ t: F \circ i \rightarrow F \circ j$ . By the theory of product preserving bundle functors [6], there exist two Weil algebras  $A$  and  $B$  such that  $F \circ i$  and  $F \circ j$  are Weil functors of the form  $F \circ i = T^A$ ,  $F \circ j = T^B$  and we have a Weil algebra homomorphism  $\mu: A \rightarrow B$  such that  $F \circ t = \mu$ .

On the other hand, consider an arbitrary homomorphism of Weil algebras  $\mu: A \rightarrow B$ . Then  $\mu$  induces a bundle functor  $T^\mu$  on  $\mathcal{FM}$  in the following way. First,  $\mu$  determines two bundle functors  $T^A$  and  $T^B$  on  $\mathcal{M}f$  and a natural transformation (denoted by the same symbol)  $\mu: T^A \rightarrow T^B$ . If  $p: Y \rightarrow M$  is a fibered manifold, then  $T^B p: T^B Y \rightarrow T^B M$  and we can construct the induced bundle  $T^\mu Y$  as the pull back  $T^\mu Y = \mu_M^* T^B Y$  with respect to  $\mu_M: T^A M \rightarrow T^B M$ . In other words,

$$\begin{aligned} T^\mu Y &= T^A M \times_{T^B M} T^B Y \\ &= \{(U, V) \in T^A M \times T^B(Y); \mu_M(U) = T^B p(V)\}. \end{aligned} \quad (1)$$

Given a fibered manifold morphism  $f: Y \rightarrow \overline{Y}$  over a base map  $\underline{f}: M \rightarrow \overline{M}$ , we have  $T^B f: T^B Y \rightarrow T^B \overline{Y}$  and we can construct the induced map

$$T^\mu f = T^A \underline{f} \times_{T^B \underline{f}} T^B f: T^\mu Y \rightarrow T^\mu \overline{Y}.$$

This defines a product preserving bundle functor  $T^\mu$  on  $\mathcal{FM}$ . W.M. Mikulski has deduced that every product preserving bundle functor  $F$  on  $\mathcal{FM}$  is naturally equivalent to  $T^\mu$  for some Weil algebra homomorphism  $\mu: A \rightarrow B$ , [9] (see also [3] for a simplified proof).

Consider two algebra homomorphisms  $\mu: A \rightarrow B$ ,  $\nu: C \rightarrow D$  and two bundle functors  $T^\mu$  and  $T^\nu$  on  $\mathcal{FM}$ . W.M. Mikulski has also proved that natural transformations  $T^\mu \rightarrow T^\nu$  are in bijection with couples of algebra homomorphisms  $f_1: A \rightarrow C$ ,  $f_2: B \rightarrow D$  such that

$$\nu \circ f_1 = f_2 \circ \mu. \quad (2)$$

The algebra homomorphisms  $f_1$  and  $f_2$  induce natural transformations (denoted by the same symbols)  $f_1: T^A \rightarrow T^C$  and  $f_2: T^B \rightarrow T^D$ . Then we can construct a map

$$(f_{1M}, f_{2Y}): T^A M \times T^B Y \rightarrow T^C M \times T^D Y.$$

We have

PROPOSITION 1

*Natural transformations  $T^\mu \rightarrow T^\nu$  are of the form  $(f_{1M}, f_{2Y})$ .*

*Proof.* Consider an element  $(U, V) \in T^\mu Y \subset T^A M \times T^B Y$ . Then

$$(f_{1M}(U), f_{2Y}(V)) \in T^C M \times T^D Y \supset T^\nu Y.$$

By (2),  $\nu_M(f_{1M}(U)) = f_{2M}(\mu_M(U))$ . Further, the following diagram commutes

$$\begin{array}{ccc} T^B Y & \xrightarrow{f_{2Y}} & T^D Y \\ T^B p \downarrow & & \downarrow T^D p \\ T^B M & \xrightarrow{f_{2M}} & T^D M \end{array}$$

Thus, we have  $\nu_M(f_{1M}(U)) = f_{2M}(\mu_M(U)) = f_{2M}(T^B p(V)) = T^D p(f_{2Y}(V))$ , which yields  $(f_{1M}(U), f_{2Y}(V)) \in T^\nu(Y)$ .

By [6], the order of a bundle functor on  $\mathcal{FM}$  is determined by three numbers  $(q, s, r)$  and is based on the concept of  $(q, s, r)$ -jet,  $s \geq q \leq r$ . Consider two fibered manifold morphisms  $f, g: Y \rightarrow \bar{Y}$  with base maps  $\underline{f}, \underline{g}: M \rightarrow \bar{M}$ . We say that  $f$  and  $g$  determine the same  $(q, s, r)$ -jet at  $y \in Y$ ,  $j_y^{q,s,r} f = j_y^{q,s,r} g$ , if

$$j_y^q f = j_y^q g, \quad j_y^s(f|Y_x) = j_y^s(g|Y_x) \quad \text{and} \quad j_x^r \underline{f} = j_x^r \underline{g}, \quad x = p(y),$$

where  $p: Y \rightarrow M$  is a fibered manifold projection. A bundle functor  $F$  on  $\mathcal{FM}$  is said to be of the order  $(q, s, r)$ , if  $j_y^{q,s,r} f = j_y^{q,s,r} g$  implies  $Ff|F_y Y = Fg|F_y Y$ . In such a case the integer  $r$  is called the base order of  $F$ ,  $s$  is called the fiber order of  $F$  and  $q$  is called the total order of  $F$ , see [2].

Consider a bundle functor  $F = T^\mu$  determined by  $\mu: A \rightarrow B$  and denote by  $N_A$  and  $N_B$  the ideals of all nilpotent elements of  $A$  and  $B$ , respectively. The nilpotency implies  $\mu(N_A) \subset N_B$ . For  $t \geq 1$  we have  $N_B^t \subset N_B$ , which yields  $\mu(N_A)N_B^t \subset N_B$ . So there exists the smallest integer  $t$  such that  $\mu(N_A)N_B^t = 0$ .

By A. Cabras and I. Kolář [1], the smallest integer  $t$  satisfying  $\mu(N_A)N_B^t = 0$  is called the order of  $\mu$  and is denoted by  $\text{ord}(\mu)$ . A. Cabras and I. Kolář have also deduced that the order  $(q, s, r)$  of a bundle functor  $T^\mu$  on  $\mathcal{FM}$  is of the form

$$q = \text{ord}(\mu), \quad s = \text{ord}(B), \quad r = \max(\text{ord}(A), \text{ord}(\mu)),$$

see [1]. Obviously, we have  $\mu(N_A)N_B^s \subset N_B^{s+1} = 0$ , which implies the condition  $s \geq q \leq r$ .

## 2. Subordinated functors on fibered manifolds

Let  $A = \mathbb{R} \times N_A$  be a Weil algebra of order  $r$ , where  $N_A$  is the ideal of all nilpotent elements of  $A$ . I. Kolář has recently introduced the underlying algebra of order  $k$  as the factor algebra  $A_k = A/N_A^{k+1}$ . The corresponding Weil functor  $T^{A_k}$  is said to be the underlying  $k$ -th order functor of  $T^A$ , [4]. In [2] we have introduced the more general concept of a subordinated functor. In general,  $G$  is called a subordinated functor of a functor  $F$ , if there exists a surjective natural transformation

$$t: F \rightarrow G.$$

In such a case we also say that  $G$  is dominated by  $F$ . A Weil algebra  $\tilde{A}$  is said to be dominated by  $A$ , if the Weil functor  $T^{\tilde{A}}$  is dominated by  $T^A$ . By [2],  $\tilde{A}$  is dominated by  $A$  if and only if we have an algebra epimorphism  $A \rightarrow \tilde{A}$ . This yields

$$\tilde{A} = A/I$$

for some ideal  $I \subset A$ . Clearly, for  $I = N_A^{k+1}$  we obtain the particular concept of the underlying algebra  $A_k$  from [4]. In [2] we have also proved

LEMMA 1

*Every  $k$ -th order Weil algebra  $\tilde{A}$ , which is dominated by  $A$ , is also dominated by  $A_k$ . So there is an epimorphism  $\varphi: A_k \rightarrow \tilde{A}$ .*

PROPOSITION 2

*Let  $F = T^\mu$  and  $G = T^\nu$  be two bundle functors on  $\mathcal{FM}$  determined by algebra homomorphisms  $\mu: A \rightarrow B$  and  $\nu: C \rightarrow D$ . The functor  $T^\nu$  is dominated by  $T^\mu$  if and only if the following conditions are satisfied:*

- (i)  $C = A/I$  is dominated by  $A$ ,
- (ii)  $D = B/J$  is dominated by  $J$ ,
- (iii) the ideals  $I \subset A$  and  $J \subset B$  satisfy  $\mu(I) \subset J$ .

*Proof.* Suppose first that  $G$  is dominated by  $F$ . Then there are algebra epimorphisms  $f_1: A \rightarrow C$  and  $f_2: B \rightarrow D$  such that

$$\nu \circ f_1 = f_2 \circ \mu. \quad (3)$$

Hence the Weil algebra  $C$  is dominated by  $A$  and  $D$  is dominated by  $B$ , so that we may write  $C = A/I$  and  $D = B/J$  for some ideals  $I \subset A$  and  $J \subset B$ . Further, we have  $f_2(\mu(I)) = \nu(f_1(I)) = \nu(0) = 0$  which reads  $\mu(I) \subset \ker(f_2) = J$ .

On the other hand, consider a bundle functor  $F = T^\mu$ ,  $\mu: A \rightarrow B$  and two ideals  $I \subset A$ ,  $J \subset B$  satisfying  $\mu(I) \subset J$ . Clearly, we can define an algebra homomorphism

$$\nu: A/I \rightarrow B/J \quad \text{by } \nu(a + I) = \mu(a) + J \quad (4)$$

such that (3) is true. So the functor  $T^\nu$  is dominated by  $T^\mu$ .

Let  $T^A$  and  $T^{\tilde{A}}$  be two Weil functors such that  $T^{\tilde{A}}$  is dominated by  $T^A$ . By [2], if the order of  $T^A$  is  $r$ , then the order of  $T^{\tilde{A}}$  is at most  $r$ . A similar result is true also for bundle functors defined on  $\mathcal{FM}$ .

**PROPOSITION 3**

*Let  $T^\mu$  and  $T^\nu$  be two bundle functors on  $\mathcal{FM}$ ,  $\mu: A \rightarrow B$  and  $\nu: C \rightarrow D$ . Denote by  $(q, s, r)$  the order of  $T^\mu$  and by  $(\bar{q}, \bar{s}, \bar{r})$  the order of  $T^\nu$ . If  $T^\nu$  is dominated by  $T^\mu$ , then  $\bar{q} \leq q$ ,  $\bar{s} \leq s$  and  $\bar{r} \leq r$ .*

*Proof.* By [1], the order  $(q, s, r)$  is given by  $q = \text{ord}(\mu)$ ,  $s = \text{ord}(B)$ ,  $r = \max(\text{ord}(A), \text{ord}(\mu))$ . The condition  $\bar{s} \leq s$  follows from the fact that the Weil algebra  $D$  is dominated by  $B$ . Denote by  $N_A$ ,  $N_B$ ,  $N_C$  and  $N_D$  the ideals of nilpotent elements. From the epimorphisms  $f_1: A \rightarrow C$  and  $f_2: B \rightarrow D$  it follows  $N_C = f_1(N_A)$  and  $N_D = f_2(N_B)$ . So we have

$$\begin{aligned} \nu(N_C)N_D^q &= \nu(f_1(N_A))f_2(N_B^q) = f_2(\mu(N_A))f_2(N_B^q) = f_2(\mu(N_A)N_B^q) \\ &= 0, \end{aligned}$$

which yields  $\bar{q} \leq q$ . Finally, the algebra  $C$  is dominated by  $A$ , so that  $\text{ord}(C) \leq \text{ord}(A)$ , which implies  $\bar{r} \leq r$ .

**EXAMPLE 1**

Write  $J^{q,s,r}(Y, \bar{Y})$  for the space of all  $(q, s, r)$ -jets of the  $\mathcal{FM}$ -morphisms of  $Y$  into  $\bar{Y}$ . Denoting by  $\mathbb{R}^{k,\ell} = \mathbb{R}^k \times \mathbb{R}^\ell \rightarrow \mathbb{R}^k$  the product fibered manifold, we can define a bundle functor  $T_{k,\ell}^{q,s,r}$  of fibered velocities of dimension  $(k, \ell)$  and order  $(q, s, r)$  by

$$T_{k,\ell}^{q,s,r} = J_{(0,0)}^{q,s,r}(\mathbb{R}^{k,\ell}, Y).$$

Clearly, the functor  $T_{k,\ell}^{q,s,r}$  has a subordinated functor  $T_{k,\ell}^{\bar{q},\bar{s},\bar{r}}$  for every  $\bar{q} \leq q$ ,  $\bar{s} \leq s$  and  $\bar{r} \leq r$ . Moreover, A. Cabras and I. Kolář have deduced, [1], that

for every product preserving bundle functor  $T^\mu$  on  $\mathcal{FM}$  of the order  $(q, s, r)$  there exists a velocities functor  $T_{k,\ell}^{q,s,r}$  and a surjective natural transformation  $T_{k,\ell}^{q,s,r} \rightarrow T^\mu$ . Hence every bundle functor  $T^\mu$  of the order  $(q, s, r)$  is dominated by a velocities functor  $T_{k,\ell}^{q,s,r}$ .

## EXAMPLE 2

If  $A = B$  and  $\mu = \text{id}_A$ , then  $T^\mu Y = T^A M \times_{T^A M} T^A Y = T^A Y \rightarrow Y$ . Clearly, the order of  $T^\mu$  is  $(r, r, r)$ , where  $r = \text{ord}(A)$ . Further, if  $C = D$  and  $\nu = \text{id}_C$ , then  $T^\nu Y = T^C Y \rightarrow Y$ . If the Weil algebra  $C$  is dominated by  $A$ , then the functor  $T^\nu$  is dominated by  $T^\mu$ .

## EXAMPLE 3

Consider a product preserving functor  $T^\mu$ ,  $\mu: A \rightarrow B$  and write  $A_m = A/N_A^{m+1}$  for the underlying algebra of order  $m$ . Further, for  $\ell \geq k$  we can define a factor algebra  $B_{k,\ell}^\mu = B/\langle \mu(N_A)N_B^k, N_B^{\ell+1} \rangle$ . If  $m \geq k$ , then we have  $\mu(N_A^{m+1}) \subset \mu(N_A)N_B^k$ , so that there is an induced algebra homomorphism (4)

$$\mu_{k,\ell,m}: A_m \rightarrow B_{k,\ell}^\mu, \quad \mu_{k,\ell,m}(a + N_A^{m+1}) = \mu(a) + \langle \mu(N_A)N_B^k, N_B^{\ell+1} \rangle.$$

By [8], the functor  $T^{\mu_{k,\ell,m}}$  is dominated by  $T^\mu$ .

## EXAMPLE 4

Consider a product preserving functor  $T^\mu$ ,  $\mu: A \rightarrow B$  and write  $I = N_A^{m+1}$ ,  $J = N_B^{\ell+1}$ ,  $A_m = A/I$ ,  $B_\ell = B/J$ . If  $m \geq \ell$ , then we have  $\mu(I) \subset J$ . By Proposition 2, for  $m \geq \ell$  there exists an induced algebra homomorphism (4)

$$\mu_{\ell,m}: A_m \rightarrow B_\ell, \quad \mu_{\ell,m}(a + N_A^{m+1}) = \mu(a) + N_B^{\ell+1}$$

such that the functor  $T^{\mu_{\ell,m}}$  is dominated by  $T^\mu$ . One evaluates directly that if the order of  $T^\mu$  is  $(q, s, r)$ , then the order of  $T^{\mu_{\ell,m}}$  is  $(q, \ell, m)$ .

## EXAMPLE 5

Consider a product preserving functor  $T^\mu$ ,  $\mu: A \rightarrow B$ . Write  $I = 0$  and let  $J \subset B$  be an arbitrary ideal. Then  $A = A/I$  and the Weil algebra  $D = B/J$  is dominated by  $B$ . Clearly, the condition  $\mu(I) \subset J$  from Proposition 2 is satisfied for an arbitrary ideal  $J \subset B$ . So we can define an induced algebra homomorphism (4)

$$\nu: A \rightarrow D = B/J \quad \text{by } \nu(a) = \mu(a) + J.$$

By Proposition 2, the functor  $T^\nu$  is dominated by  $T^\mu$ .

## EXAMPLE 6

Write  $J = N_B$  in Example 5. Then we have  $D = B/N_B = \mathbb{R}$  and the induced Weil algebra homomorphism (4)  $\nu: A \rightarrow D = \mathbb{R}$  is of the form  $\nu(r, n) = r$ . So the subordinated functor  $T^\nu$  is of the form  $T^\nu Y = T^A M \times_M Y$ . Clearly, the

order of  $T^\nu$  is  $(0, 0, \text{ord}(A))$  and the surjective natural transformation  $T^\mu \rightarrow T^\nu$  is of the form

$$(T^A M \times_{T^B M} T^B Y) \rightarrow (T^A M \times_M Y), \quad (U, V) \mapsto (U, q_Y^B(V)),$$

where  $q_Y^B: T^B Y \rightarrow Y$  is the bundle projection. Moreover, one can verify directly that the fiber order of an arbitrary functor  $T^\alpha$  on  $\mathcal{FM}$  is zero if and only if  $T^\alpha Y = T^A M \times_M Y$ .

EXAMPLE 7

Write  $J = N_B^{\ell+1}$  in Example 5. Then we have  $D = B_\ell = B/N_B^{\ell+1}$ . By Proposition 2, there is an induced algebra homomorphism (4)

$$\mu_\ell: A \rightarrow B_\ell, \quad \mu_\ell(a) = \mu(a) + N_B^{\ell+1}$$

such that the functor  $T^{\mu_\ell}$  is dominated by  $T^\mu$ . Denote by  $f: B \rightarrow B_\ell$  the algebra epimorphism given by  $f(b) = b + N_B^{\ell+1}$  and suppose that the order of  $T^\mu$  is  $(q, s, r)$ . We can write

$$\begin{aligned} \mu_\ell(N_A)N_D^t &= f(\mu(N_A))(N_B + N_B^{\ell+1})^t = (\mu(N_A) + N_B^{\ell+1})(N_B + N_B^{\ell+1})^t \\ &= \mu(N_A)N_B^t. \end{aligned}$$

This yields  $\text{ord}(\mu_\ell) = \text{ord}(\mu) = q$ . Thus, we have proved: If the order of  $T^\mu$  is  $(q, s, r)$ , then the order of  $T^{\mu_\ell}$  is  $(q, \ell, r)$ .

In what follows we shall write  $T^\nu \prec T^\mu$  if the functor  $T^\nu$  is dominated by  $T^\mu$ . Consider now the functors  $T^{\mu_{k,\ell,m}}$ ,  $T^{\mu_{\ell,m}}$  and  $T^{\mu_\ell}$  from Examples 3, 4 and 7, respectively.

PROPOSITION 4

Let  $T^\mu$  be a functor on  $\mathcal{FM}$  of the order  $(q, s, r)$ , which is determined by an algebra homomorphism  $\mu: A \rightarrow B$ . Then we have:

$$\begin{aligned} T^{\mu_{k,\ell,m}} \prec T^{\mu_\ell} \prec T^\mu & \quad \text{for any } \ell \geq k \leq m, \\ T^{\mu_{k,\ell,m}} \prec T^{\mu_{\ell,m}} \prec T^{\mu_\ell} \prec T^\mu & \quad \text{for any } \ell \geq k \leq m, m \geq \ell. \end{aligned}$$

*Proof.* Clearly, all the functors  $T^{\mu_{k,\ell,m}}$ ,  $T^{\mu_{\ell,m}}$  and  $T^{\mu_\ell}$  are dominated by  $T^\mu$ . First we prove  $T^{\mu_{k,\ell,m}} \prec T^{\mu_\ell}$ . Consider a diagram

$$\begin{array}{ccc} A & \xrightarrow{\mu} & B \\ \parallel & & \downarrow f_1 \\ A & \xrightarrow{\mu_\ell} & B_\ell \\ f_2 \downarrow & & \downarrow f_3 \\ A_m & \xrightarrow{\mu_{k,\ell,m}} & B_{k,\ell}^\mu \end{array}$$

where  $f_1$  and  $f_2$  are algebra epimorphisms defined by  $f_1(b) = b + B_B^{\ell+1}$  and  $f_2(a) = a + N_A^{m+1}$ . Write  $J = \langle \mu(N_A)N_B^k, N_B^{\ell+1} \rangle$ . The inclusion  $N_B^{\ell+1} \subset J$  defines an epimorphism  $f_3: B_\ell \rightarrow B/J$ . Further, we have

$$\mu_{k,\ell,m}(f_2(a)) = \mu_{k,\ell,m}(a + N_A^{m+1}) = \mu(a) + J$$

and

$$f_3(\mu_\ell(a)) = f_3(\mu(a) + N_B^{\ell+1}) = \mu(a) + J.$$

Hence the diagram commutes and we have  $T^{\mu_{k,\ell,m}} \prec T^{\mu_\ell}$ . Using an analogous diagram chasing, we prove directly the remaining relations.

### 3. Underlying functors on fibered manifolds

In [2] we have introduced the concept of an underlying functor on  $\mathcal{M}f$  and on  $\mathcal{FM}_m$ , which can be modified also for bundle functors on  $\mathcal{FM}$ . Let  $F$  be a bundle functor on  $\mathcal{FM}$ .

DEFINITION

A bundle functor  $F_a^f$  is said to be the underlying functor of  $F$  with the fiber order  $a$ , if:

- (1)  $F_a^f$  is dominated by  $F$ ,
- (2) The fiber order of  $F_a^f$  is  $a$ ,
- (3) Every functor  $\tilde{F}$  on  $\mathcal{FM}$  with the fiber order  $a$ , which is dominated by  $F$ , is also dominated by  $F_a^f$ .

Roughly speaking, the underlying functor  $F_a^f$  is “the greatest” subordinated functor among all subordinated functors of  $F$  with the fiber order  $a$ . Replacing the fiber order with the base order (or with the total order), we obtain the underlying functor  $F_a^b$  with the base order  $a$  (or the underlying functor  $F_a^t$  with the total order  $a$ ). Clearly, the concept of a subordinated functor is quite general. On the other hand, the definition of an underlying functor depends on the order. As the order of a bundle functor on  $\mathcal{FM}$  depends on three integers, we have three types of underlying functors on  $\mathcal{FM}$ . We prove

PROPOSITION 5

Let  $F = T^\mu$ ,  $\mu: A \rightarrow B$  be a product preserving functor on  $\mathcal{FM}$  and let  $T^{\mu_\ell}$  be the functor from Example 7. Then the underlying functor of  $F$  with the fiber order  $\ell$  is of the form  $F_\ell^f = T^{\mu_\ell}$ .

*Proof.* By Example 7,  $T^{\mu_\ell}$  is dominated by  $T^\mu$  and has the fiber order  $\ell$ . Consider an arbitrary functor  $T^\nu$ ,  $\nu: C = A/I \rightarrow D = B/J$  of the fiber order  $\ell$ , which is dominated by  $T^\mu$ . Obviously,  $\text{ord}(D) = \ell$ . By Lemma 1 there is an epimorphism  $\varphi: B_\ell \rightarrow D$ ,  $\varphi(b + N_B^{\ell+1}) = b + J$ . Further, consider a diagram



$$\begin{array}{ccc}
 A & \xrightarrow{\mu_\ell} & B_\ell \\
 f_1 \downarrow & & \downarrow \varphi \\
 C = A/I & \xrightarrow{\nu} & D = B/J
 \end{array}$$

where  $f_1: A \rightarrow A/I$  is an epimorphism defined by  $f_1(a) = a + I$ . We can write

$$\nu(f_1(a)) = \nu(a + I) = \mu(a) + J$$

and

$$\varphi(\mu_\ell(a)) = \varphi(\mu(a) + N_B^{\ell+1}) = \mu(a) + J.$$

So the diagram commutes and the functor  $T^\nu$  is dominated by  $T^{\mu_\ell}$ .

Denote by  $\mathcal{T}$  the class of functors  $T^\nu$ ,  $\nu: C \rightarrow D$  of the order  $(k, \ell, m)$ ,  $\ell \geq k \leq m$ , which are dominated by  $F = T^\mu$  and satisfy the condition

$$\text{ord}(C) \geq \text{ord}(D). \quad (5)$$

For example, the functor  $T^{\mu_\ell, m}$  from Example 4 is an element of  $\mathcal{T}$ . Further, if the algebra  $D$  is dominated by  $C$ , then (5) is true.

#### LEMMA 2

Let  $T^\nu$  be a functor of the order  $(k, \ell, m)$  from the class  $\mathcal{T}$ . Then we have  $\text{ord}(C) = m \geq \ell$ .

*Proof.* The relation  $\text{ord}(C) \geq \text{ord}(D) = \ell \geq k$  implies  $\text{ord}(C) = m$ .

#### PROPOSITION 6

Let  $F = T^\mu$ ,  $\mu: A \rightarrow B$  be a product preserving functor on  $\mathcal{FM}$  and let  $T^{\mu_\ell, m}$  be the functor from Example 4. On the class  $\mathcal{T}$  of subordinated functors of  $F$ , the underlying functor with the base order  $m$  is of the form  $F_m^b = T^{\mu_\ell, m}$ .

*Proof.* By Example 4, the functor  $T^{\mu_\ell, m}$  has the order  $(q, \ell, m)$  and is dominated by  $F$ . Consider now an arbitrary functor  $T^\nu \in \mathcal{T}$ , where  $\nu: C \rightarrow D$ . Then we have  $C = A/I$  and  $D = B/J$ , where  $\text{ord}(D) = \ell$ . Further, Lemma 2 implies  $\text{ord}(C) = m$ . By Lemma 1, there are epimorphisms  $\varphi_1: A_m \rightarrow C$  and  $\varphi_2: B_\ell \rightarrow D$ . We can write

$$\nu(\varphi_1(a + N_A^{m+1})) = \nu(a + I) = \mu(a) + J$$

and

$$\varphi_2(\mu_{\ell, m}(a + N_A^{m+1})) = \varphi_2(\mu(a) + N_B^{\ell+1}) = \mu(a) + J.$$

Thus, the functor  $T^\nu$  is dominated by  $T^{\mu_\ell, m}$ .

Write  $\mu: A \rightarrow B$ ,  $\nu: C \rightarrow D$  and suppose that the functor  $T^\nu$  is dominated by  $T^\mu$ . Obviously, if  $\mu$  is an epimorphism, then  $\nu$  is an epimorphism too. This implies, that if  $\mu$  is an epimorphism, then every functor  $T^\nu$  of the order  $(k, \ell, m)$ , which is dominated by  $T^\mu$ , is an element of the class  $\mathcal{T}$ . Thus, we have

COROLLARY 1

Let  $F = T^\mu$ ,  $\mu: A \rightarrow B$  be a product preserving functor on  $\mathcal{FM}$ . If  $\mu$  is an epimorphism, then the underlying functor of  $F$  with the base order  $m$  is of the form  $F_m^b = T^{\mu_\ell, m}$  for some  $\ell \leq m$ .

#### 4. Affine bundle structure on product preserving functors on $\mathcal{FM}$

Let  $F = T^\mu$  be a functor with the fiber order  $b$ , which is determined by a homomorphism  $\mu: A \rightarrow B$ . By Proposition 5, the underlying functor with the fiber order  $(b-1)$  is of the form  $F_{b-1}^f = T^{\mu_{b-1}}$ . We have

PROPOSITION 7

$T^\mu Y \rightarrow T^{\mu_{b-1}} Y$  is an affine bundle, whose associated vector bundle is the pull back of  $TY \otimes N_B^b$  over  $T^{\mu_{b-1}} Y$ .

*Proof.* By [4],  $T^B Y \rightarrow T^{B-1} Y$  is an affine bundle, whose associated vector bundle is the pull back of  $TY \otimes N_B^b$  over  $T^{B-1} Y$ . Consider now the expression of  $T^\mu Y$  in the form (1). For  $(U, V) \in T^A M \times T^B Y$  and  $v \in TY \otimes N_B^b$  we can define the addition by

$$(U, V) + v := (U, V + v) \in T^A M \times T^B Y.$$

We have  $T^B p: T^B Y \rightarrow T^B M$ , so that  $T^B p(V + v) \in T^B M$ . As  $T^B M \rightarrow T^{B-1} M$  is also an affine bundle, we can write  $T^B p(V + v) = T^B p(V) + w$  for some  $w \in TM \otimes N_B^b$ . Clearly,  $w$  is of the form  $w = (Tp \otimes \text{id}_{N_B^b})(v)$ . Further,  $w = 0$  iff  $v \in VY \otimes N_B^b$ . In such a case we have  $T^B p(V + v) = T^B p(V) = \mu_M(U)$ . So for  $(U, V) \in T^\mu Y$  and  $v \in VY \otimes N_B^b$ , the sum  $(U, V) + v$  is an element of  $T^\mu Y$ .

As a particular case we obtain the result by I. Kolář and W.M. Mikulski from [8], who have proved that  $T^{\mu_{q,s,r}} Y \rightarrow T^{\mu_{q,s-1,r}} Y$  is an affine bundle.

#### References

- [1] A. Cabras, I. Kolář, *Prolongation of projectable tangent valued forms*, Arch. Math. (Brno) **38** (2002), 243-257.
- [2] M. Doupovec, *On the underlying lower order bundle functors*, to appear in Czechoslovak Math. J.

- [3] M. Doupovec, I. Kolář, *On the jets of fibered manifold morphisms*, Cahiers Topo. Geom. Diff. Categoriqes **40** (1999), 21-30.
- [4] I. Kolář, *Affine structure on Weil bundles*, Nagoya Math. J. **158** (2000), 99-106.
- [5] I. Kolář, *A general point of view to nonholonomic jet bundles*, Cahiers Topo. Geom. Diff. Categoriqes **44-2** (2003), 149-160.
- [6] I. Kolář, P.W. Michor, J. Slovák, *Natural Operations in Differential Geometry*, Springer-Verlag, Berlin, 1993.
- [7] I. Kolář, W.M. Mikulski, *On the fiber product preserving bundle functors*, Differential Geometry and Its Applications **11** (1999), 105-115.
- [8] I. Kolář, W.M. Mikulski, *Contact elements on fibered manifolds*, Czechoslovak Math. J. **53(128)** (2003), no. 4, 1017-1030.
- [9] W.M. Mikulski, *Product preserving bundle functors on fibered manifolds*, Arch. Math. (Brno) **32** (1996), 307-316.

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