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Approximation by some combinations of the Poisson integrals for Hermite and Laguerre expansions

Abstract. The aim of this paper is the study of a rate of convergence of some combinations of the Poisson integrals for Hermite and Laguerre expansions. We are able to achieve faster convergence for our modified operators over the Poisson integrals. We prove also the Voronovskaya type theorem for these new operators.

1. Introduction

In recent years, several classical positive linear operators, for instance the Szász-Mirakyan, Baskakov-Durrmeyer operators, have been investigated intensively. There have been proposed some modifications of these operators, which have a better rate of convergence than the classical operators (see [1, 8]).

This work presents a new modification of the Poisson integrals for Hermite and Laguerre expansions. We obtain certain positive linear operators, which have better error estimation than the Poisson integrals studied in [2, 4]. Moreover, this modification makes it possible to state the Voronovskaya type formula for these Poisson integrals.

For the function $f \in L^p(e^{-z^2})$, $p \geq 1$ (f is defined on $\mathbb{R}$), the Poisson integral for Hermite polynomial expansion is defined by

$$F(f)(x,y) = F(f;x,y) = \int_0^1 T(x,r)A(f;r,y)\,dr, \qquad x>0,\ y\in\mathbb{R}, \tag{1}$$

where

$$T(x,r) = \frac{x\exp(\frac{x^2}{2\ln r})}{(2\pi)^{\frac{1}{2}}r(-\ln r)^{\frac{3}{2}}},$$

$$A(f;r,x) = \int_{-\infty}^{\infty} P(r,x,z)f(z)e^{-z^2}\,dz, \qquad 0<r<1,\ x>0,$$

$$P(r,x,z) = \sum_{n=0}^{\infty}\frac{r^nH_n(x)H_n(z)}{\sqrt{\pi}2^n n!} = \frac{1}{\sqrt{\pi(1-r^2)}}\exp\Big(\frac{-r^2x^2+2rxz-r^2z^2}{1-r^2}\Big),$$

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and H_n is the n-th Hermite polynomial, $n = 0, 1, 2, \ldots$ (see, for instance, [6]). We note that

$$H_0(z) = 1, \qquad H_1(z) = 2z, \qquad H_2(z) = 4z^2 - 2,$$
$$H_3(z) = 8z^3 - 12z, \qquad H_4(z) = 16z^4 - 48z^2 + 12.$$

The Poisson integral of a function $f \in L^p(z^\alpha e^{-z})$, $p \geq 1$, $\alpha > -1$ (f is defined on $\mathbb{R}_+ = [0, \infty)$) for Laguerre polynomial expansion is defined by

$$G(f)(x, y) = G(f; x, y) = \int_0^1 T\Big(\frac{x}{\sqrt{2}}, r\Big) B(f; r, y)\, dr, \qquad x > 0,\ y \in \mathbb{R}_+, \tag{2}$$

where

$$B(f; r, x) = \int_0^\infty K(r, x, z) f(z) z^\alpha e^{-z}\, dz, \qquad 0 < r < 1,\ x > 0,$$

$$K(r, x, z) = \sum_{n=0}^\infty \frac{r^n n!}{\Gamma(n + \alpha + 1)} L_n^\alpha(x)\, L_n^\alpha(z)$$
$$= \frac{(rxz)^{-\frac{\alpha}{2}}}{1 - r} \exp\Big(\frac{-r(x + z)}{1 - r}\Big) I_\alpha\Big(\frac{2(rxz)^{\frac{1}{2}}}{1 - r}\Big),$$

I_α is the modified Bessel function given by

$$I_\alpha(s) = \sum_{n=0}^\infty \frac{s^{\alpha + 2n}}{2^{\alpha + 2n} n! \Gamma(\alpha + n + 1)}$$

(see [3]), and L_n^α is the n-th Laguerre polynomial, $n = 0, 1, 2, \ldots$ (see, for instance, [6]). We note that

$$L_0^\alpha(s) = 1,$$
$$L_1^\alpha(s) = 1 + \alpha - s,$$
$$L_2^\alpha(s) = \frac{1}{2}[(\alpha + 1)(\alpha + 2) - 2(\alpha + 2)s + s^2],$$
$$L_3^\alpha(s) = \frac{1}{6}[(\alpha + 1)(\alpha + 2)(\alpha + 3) - 3(\alpha + 2)(\alpha + 3)s + 3(\alpha + 3)s^2 - s^3],$$
$$L_4^\alpha(s) = \frac{1}{24}[(\alpha + 1)(\alpha + 2)(\alpha + 3)(\alpha + 4) - 4(\alpha + 2)(\alpha + 3)(\alpha + 4)s + 6(\alpha + 3)(\alpha + 4)s^2 - 4(\alpha + 4)s^3 + s^4].$$

Muckenhoupt obtained the following results.

Theorem 1.1 ([4])
If $f \in L^p(e^{-z^2})$, then $F(f; x, \cdot) \in L^p(e^{-z^2})$ for $x > 0$ and

(a) $\|F(f; x, \cdot)\|_p \leq \|f\|_p$, $1 \leq p \leq \infty$,

(b) $\|F(f; x, \cdot) - f\|_p \to 0$ *as* $x \to 0^+$ *for* $1 \leq p < \infty$,

(c) $\lim_{x\to 0^+} F(f;x,y) = f(y)$ *almost everywhere,* $1 \le p \le \infty$,

(d) $\frac{\partial^2 F}{\partial x^2} + \frac{\partial^2 F}{\partial y^2} - 2y\frac{\partial F}{\partial y} = 0$ *in* $\Omega = \{(x,y):\ x>0,\ y\in\mathbb{R}\}$,

where $\|f\|_p$ *denotes the norm in* $L^p(e^{-z^2})$ *of a function* f *defined on* $\mathbb{R}$.

Theorem 1.2 ([4])
If $f \in L^p(z^\alpha e^{-z})$, *then* $G(f;x,\cdot) \in L^p(z^\alpha e^{-z})$ *and*

(a) $\|G(f;x,\cdot)\|_p \le \|f\|_p$, $1 \le p \le \infty$,

(b) $\|G(f;x,\cdot) - f\|_p \to 0$ *as* $x \to 0^+$ *for* $1 \le p < \infty$,

(c) $\lim_{x\to 0^+} G(f;x,y) = f(y)$ *almost everywhere in* $[0,\infty)$, $1 \le p \le \infty$,

(d) $\frac{\partial^2 G}{\partial x^2} + y\frac{\partial^2 G}{\partial y^2} + (\alpha+1-y)\frac{\partial G}{\partial y} = 0$ *in* $\Omega = \{(x,y):\ x>0,\ y\ge 0\}$,

where $\|f\|_p$ *denotes the norm in* $L^p(z^\alpha e^{-z})$ *of a function* f *defined on* $\mathbb{R}_+$.

In the paper [2] some estimations of the rate of convergence of the integrals F and G were given. The following theorems were proved.

Theorem 1.3 ([2])
If $f \in C(\mathbb{R}) \cap L^p(e^{-z^2})$, *then*

$$|F(f;x,y) - f(y)| \le 3\omega(f,\mu_x(y))$$

for $x>0$ *and* $y\in\mathbb{R}$, *where*

$$\mu_x(y) = \Big(y^2\big(1 - 2e^{-\sqrt{2}x} + e^{-2x}\big) + \frac{1}{2}\big(1-e^{-2x}\big)\Big)^{\frac{1}{2}}$$

and $\omega(f,\delta)$ *is the classical modulus of continuity of function* f.

Theorem 1.4 ([2])
If $f \in C(\mathbb{R}_+) \cap L^p(z^\alpha e^{-z})$, *then*

$$|G(f;x,y) - f(y)| \le 3\omega(f,\mu_{\alpha,x}(y))$$

for $x>0$ *and* $y\ge 0$, *where*

$$\begin{aligned}\mu_{\alpha,x}(y) &\\ = \Big(&y^2 - 2y^2e^{-x} + y^2e^{-\sqrt{2}x} + 2(\alpha+2)ye^{-x} - 2(\alpha+2)ye^{-\sqrt{2}x}\\ &- 2(\alpha+1)y + 2(\alpha+1)ye^{-x} + (\alpha+2)(\alpha+1)\big(1-2e^{-x}+e^{-\sqrt{2}x}\big)\Big)^{\frac{1}{2}}.\end{aligned}$$

Since

$$\lim_{x\to 0^+}\frac{\mu_x^2(y)}{x} = y^2(2\sqrt{2}-2)+1,$$

$$\lim_{x\to 0^+}\frac{\mu_{\alpha,x}^2(y)}{x} = (2-\sqrt{2})(y^2 + (\alpha+1)(\alpha+2) + 2y(\alpha+2)(\sqrt{2}-1) - 2y(\alpha+1),$$

we conclude that

$$|F(f;x,y)-f(y)|\le K_1(y)\omega(f,\sqrt{x})$$

and

$$|G(f;x,y)-f(y)|\le K_2(y,\alpha)\omega(f,\sqrt{x}),$$

where $K_1(y)$, $K_2(y,\alpha)$ are positive constants.

2. The new integrals H and L

In the present section we will propose some combinations of the operators F and G. Our new operators have a better rate of convergence than the Poisson integrals given by (1) and (2).

We define the operator $H(f)$ by

$$H(f)(x,y)=H(f;x,y)=2F(f;x,y)-F(f;2x,y),\qquad x>0,\ y\in\mathbb{R}. \tag{3}$$

It is easy to observe that operator $H(f)$ is linear and positive.

Let $e_i(z)=z^i$, $i=1,2,3,4$, $e_0(z)\equiv 1$ and $\psi_y(z)=z-y$. From

$$\int_0^1 T(x,r)r^n\,dr=\exp\big(-(2n)^{\frac{1}{2}}x\big),\qquad n=0,1,2,\dots$$

(see [4]) and by (3) we get the following lemmas.

Lemma 2.1

Let $x>0$. For each $y\in\mathbb{R}$ it follows

$$\begin{aligned}
H(e_0;x,y)&=H_0(y)=1,\\
H(e_1;x,y)&=\frac{1}{2}H_1(y)\big(2e^{-\sqrt{2}x}-e^{-2\sqrt{2}x}\big),\\
H(e_2;x,y)&=\frac{1}{4}H_2(y)\big(2e^{-2x}-e^{-4x}\big)+\frac{1}{2},\\
H(e_3;x,y)&=\frac{1}{8}H_3(y)\big(2e^{-\sqrt{6}x}-e^{-2\sqrt{6}x}\big)+\frac{3}{4}H_1(y)\big(2e^{-\sqrt{2}x}-e^{-2\sqrt{2}x}\big),\\
H(e_4;x,y)&=\frac{1}{16}H_4(y)\big(2e^{-2\sqrt{2}x}-e^{-4\sqrt{2}x}\big)+\frac{3}{4}H_2(y)\big(2e^{-2x}-e^{-4x}\big)+\frac{3}{4}.
\end{aligned}$$

Lemma 2.2

Let $x>0$. For each $y\in\mathbb{R}$ we have

$$\begin{aligned}
H(\psi_y;x,y)&=\frac{1}{2}H_1(y)\big(2e^{-\sqrt{2}x}-e^{-2\sqrt{2}x}-1\big),\\
H(\psi_y^2;x,y)&=\frac{1}{4}H_2(y)\big(2e^{-2x}-e^{-4x}-1\big)-yH_1(y)\big(2e^{-\sqrt{2}x}-e^{-2\sqrt{2}x}-1\big),\\
H(\psi_y^4;x,y)&=\frac{1}{16}H_4(y)\big(2e^{-2\sqrt{2}x}-e^{-4\sqrt{2}x}-1\big)
\end{aligned}$$

$$
\begin{aligned}
&- \frac{1}{2} y H_3(y)\big(2e^{-\sqrt{6}x} - e^{-2\sqrt{6}x} - 1\big) \\
&+ \frac{3}{4} H_2(y)\big(1 + 2y^2\big)\big(2e^{-2x} - e^{-4x} - 1\big) \\
&- H_1(y)\big(3y + 2y^3\big)\big(2e^{-\sqrt{2}x} - e^{-2\sqrt{2}x} - 1\big).
\end{aligned}
$$

Applying Lemma 2.2 and the L'Hospital's Rule we obtain the result.

LEMMA 2.3
For every fixed $y \in \mathbb{R}$ we have

$$
\lim_{x\to 0+} \frac{H(\psi_y; x, y)}{x^2} = -4y, \qquad \lim_{x\to 0+} \frac{H(\psi_y^2; x, y)}{x^2} = 4, \qquad \lim_{x\to 0+} \frac{H(\psi_y^4; x, y)}{x^2} = 0.
$$

According to the arguments of the paper [2], the results of Lemma 2.2 and Lemma 2.3 we shall state the estimation of the rate of approximation of f by $H(f)$.

THEOREM 2.4
If $f \in C(\mathbb{R}) \cap L^p(e^{-z^2})$, then

$$
|H(f; x, y) - f(y)| \le 3\omega(f, \mu_x^*(y)) \le K(y)\omega(f, x)
$$

for $x > 0$ and $y \in \mathbb{R}$, where $\mu_x^*(y) = (H(\psi_y^2; x, y))^{\frac{1}{2}}$ and $K(y)$ is a positive constant.

REMARK 2.5
Drawing a comparison between Theorem 1.3 and Theorem 2.4 we notice that the approximation order of the operator $H(f)$ is essentially better than that of $F(f)$.

We shall define the operator $L(f)$ for the function $f \in L^p(z^\alpha e^{-z})$:

$$
L(f)(x, y) = L(f; x, y) = 2G(f; x, y) - G(f; 2x, y), \qquad x > 0,\ y \in \mathbb{R}_+. \tag{4}
$$

The operator $L(f)$ is linear and positive and by simple calculation we get the following lemmas.

LEMMA 2.6
Let $x > 0$ and $\alpha > -1$. For each $y \in \mathbb{R}_+$ it follows

$$
\begin{aligned}
L(e_0; x, y) &= L_0^\alpha(y) = 1, \\
L(e_1; x, y) &= L_1^\alpha(y)\big(1 + e^{-2x} - 2e^{-x}\big) + y L_0^\alpha(y), \\
L(e_2; x, y) &= -2L_2^\alpha(y)\big(1 + e^{-2\sqrt{2}x} + 2e^{-\sqrt{2}x}\big) \\
&\quad + 2(\alpha+2)L_1^\alpha(y)\big(1 + e^{-2x} - 2e^{-x}\big) + y^2 L_0^\alpha(y), \\
L(e_3; x, y) &= 6L_3^\alpha(y)\big(1 + e^{-2\sqrt{3}x} - 2e^{-\sqrt{3}x}\big) \\
&\quad - 6(\alpha+3)L_2^\alpha(y)\big(1 + e^{-2\sqrt{2}x} - 2e^{-\sqrt{2}x}\big) \\
&\quad + 3(\alpha+3)(\alpha+2)L_1^\alpha(y)\big(1 + e^{-2x} - 2e^{-x}\big) + y^3 L_0^\alpha(y),
\end{aligned}
$$

$$\begin{aligned} L(e_4;x,y) = &-24L_4^\alpha(y)\big(1+e^{-4x}-2e^{-2x}\big)\\ &+24(\alpha+4)L_3^\alpha(y)\big(1+e^{-2\sqrt{3}x}-2e^{-\sqrt{3}x}\big)\\ &-12(\alpha+4)(\alpha+3)L_2^\alpha(y)\big(1+e^{-2\sqrt{2}x}-2e^{-\sqrt{2}x}\big)\\ &+4(\alpha+4)(\alpha+3)(\alpha+2)L_1^\alpha(y)\big(1+e^{-2x}-2e^{-x}\big)+y^4L_0^\alpha(y). \end{aligned}$$

LEMMA 2.7
Let $x>0$ and $\alpha>-1$. For each $y\in\mathbb{R}_+$ we have

$$\begin{aligned} L(\psi_y;x,y) &= L_1^\alpha(y)\big(1+e^{-2x}-2e^{-x}\big),\\ L(\psi_y^2;x,y) &= -2L_2^\alpha(y)\big(1+e^{-2\sqrt{2}x}-2e^{-\sqrt{2}x}\big)\\ &\quad+2(\alpha+2-y)L_1^\alpha(y)\big(1+e^{-2x}-2e^{-x}\big),\\ L(\psi_y^4;x,y) &= -24L_4^\alpha(y)\big(1+e^{-4x}-2e^{-2x}\big)\\ &\quad+24(\alpha+4-y)L_3^\alpha(y)\big(1+e^{-2\sqrt{3}x}-2e^{-\sqrt{3}x}\big)\\ &\quad-12\big((\alpha+4)(\alpha+3)-2y(\alpha+3)+y^2\big)L_2^\alpha(y)\big(1+e^{-2\sqrt{2}x}-2e^{-\sqrt{2}x}\big)\\ &\quad+4\big((\alpha+4)(\alpha+3)(\alpha+2)-3y(\alpha+3)(\alpha+2)\\ &\quad+3y^2(\alpha+2)-y^3\big)L_1^\alpha(y)\big(1+e^{-2x}-2e^{-x}\big). \end{aligned}$$

From Lemma 2.7 we obtain the result.

LEMMA 2.8
For every fixed $y\in\mathbb{R}_+$,

$$\lim_{x\to0^+}\frac{L(\psi_y;x,y)}{x^2}=L_1^\alpha(y),\qquad \lim_{x\to0^+}\frac{L(\psi_y^2;x,y)}{x^2}=2y,\qquad \lim_{x\to0^+}\frac{L(\psi_y^4;x,y)}{x^2}=0.$$

Using the arguments of the paper [2] and the above lemmas we can state the following estimation.

THEOREM 2.9
If $f\in C(\mathbb{R}_+)\cap L^p(z^\alpha e^{-z})$, then

$$|L(f;x,y)-f(y)|\le 3\omega(f,\mu_{\alpha,x}^*(y))\le K_1(y)\omega(f,x)$$

for $x>0$ and $y\in\mathbb{R}_+$, where $\mu_{\alpha,x}^*(y)=(L(\psi_y^2;x,y))^{\frac{1}{2}}$ and $K_1(y)$ is a positive constant.

REMARK 2.10
Drawing a comparison between Theorem 2.9 and Theorem 1.4 we notice, similarly to the results for $H(f)$, that the approximation order of $L(f)$ is essentially better than that of $G(f)$.

3. The Voronovskaya type theorem

The Voronovskaya theorem for some Poisson integrals for Hermite and Laguerre expansions was established by Toczek and Wachnicki in [7] and also studied by Özarslan and Duman in [5]. In this section we present the Voronovskaya type theorem for the operators $H(f)$ and $L(f)$.

THEOREM 3.1
Let $y \in \mathbb{R}$. If $f \in L^\infty(\mathbb{R})$ and f is a continuous, differentiable function in the neighbourhood of the point $y \in \mathbb{R}$ and $f''(y)$ exists, then

$$\lim_{x\to 0^+} \frac{H(f;x,y) - f(y)}{x^2} = -4yf'(y) + 2f''(y).$$

Proof. We remark that

$$f(t) = f(y) + (t-y)f'(y) + \frac{1}{2}(t-y)^2 f''(y) + \phi(t,y)(t-y)^2$$

and ϕ is a bounded and continuous function of the variable t and

$$\lim_{t\to y} \phi(t,y) = 0.$$

In the sequel

$$H(f;x,y) = f(y) + f'(y)H(\psi_y;x,y) + \frac{1}{2}f''(y)H(\psi_y^2;x,y) + H(\phi(t,y)\psi_y^2;x,y).$$

By Lemma 2.3 we have

$$\lim_{x\to 0^+} \frac{H(f;x,y) - f(y)}{x^2} = -4yf'(y) + 2f''(y) + \lim_{x\to 0^+} \frac{H(\phi(t,y)\psi_y^2;x,y)}{x^2}.$$

We prove that

$$\lim_{x\to 0^+} \frac{H(\phi(t,y)\psi_y^2;x,y)}{x^2} = 0. \tag{5}$$

Let $\varepsilon > 0$. There exists a number $\delta > 0$ such that $|\phi(t,y)| < \frac{\varepsilon}{2}$ for $|t-y| < \delta$. Moreover, there is a number $M > 0$ such that $|\phi(t,y)| < M$ for $t \in \mathbb{R}$. Hence

$$\begin{aligned} &H(\phi(t,y)\psi_y^2;x,y) \\ &= \int\limits_{|z-y|<\delta} \int\limits_0^1 (2T(x,r) - T(2x,r))P(r,y,z)\phi(z,y)\psi_y^2(z)e^{-z^2}\,dr\,dz \\ &\quad + \int\limits_{|z-y|\geq\delta} \int\limits_0^1 (2T(x,r) - T(2x,r))P(r,y,z)\phi(z,y)\psi_y^2(z)e^{-z^2}\,dr\,dz \\ &= I_1 + I_2 \end{aligned}$$

and

$$\begin{aligned}|I_1| &\leq \varepsilon \int\limits_{|z-y|<\delta} \int\limits_0^1 (2T(x,r) - T(2x,r))P(r,y,z)\psi_y^2(z)e^{-z^2}\,dr\,dz \\ &\leq \varepsilon H(\psi_y^2; x, y).\end{aligned}$$

If $|z-y| \geq \delta$, then $\frac{(z-y)^2}{\delta^2} \geq 1$ and

$$\begin{aligned}|I_2| &\leq \frac{M}{\delta^2} \int\limits_{|z-y|\geq\delta} \int\limits_0^1 (2T(x,r) - T(2x,r))P(r,y,z)\psi_y^4(z)e^{-z^2}\,dr\,dz \\ &\leq \frac{M}{\delta^2} H(\psi_y^4; x, y).\end{aligned}$$

It follows that

$$\frac{|H(\phi(t,y)\psi_y^2; x, y)|}{x^2} \leq \varepsilon \frac{H(\psi_y^2; x, y)}{x^2} + \frac{M}{\delta^2}\frac{H(\psi_y^4; x, y)}{x^2}.$$

By Lemma 2.3 we have (5) and the proof is completed.

Corollary 3.2
If the function f verifies the assumptions of Theorem 3.1, then

$$H(f; x, y) - f(y) = O(x^2) \qquad \text{as } x \to 0^+.$$

Using the same method we obtain the following result.

Theorem 3.3
Let $y \in \mathbb{R}_+$. If $f \in L^\infty(\mathbb{R}_+)$ and f is a continuous, differentiable function in the certain neighbourhood of the point $y \in \mathbb{R}_+$, and $f''(y)$ exists, then

$$\lim_{x\to 0^+} \frac{L(f; x, y) - f(y)}{x^2} = (\alpha + 1 - y)f'(y) + yf''(y)$$

and

$$L(f; x, y) - f(y) = O(x^2) \qquad \text{as } x \to 0^+.$$

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