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Zenon Moszner

On stability of some functional equations and topology of their target spaces

Dedicated to the Memory of Bogdan Choczewski, my Colleague

Abstract. Some remarks about the coherence of the stability of several functional equations with topology of their target spaces are given. The equations in question are: homogeneity (first of all) and quadratic ones, as well as those of Drygas, Jensen and Schröder. Moreover, we prove, by method different than those used in earlier papers, the superstability of the following equations: Dhombres', Lobachevski's and Mikusiński's and those of cosine, sine and of homomorphisms.

1. Stability and comleteness

1.1. Introduction

The coherence of the stability of functional equations with the completeness of pertinent target spaces was first discussed in the paper by G.L. Forti and J. Schwaiger [7] and by W. Jabłoński and J. Schwaiger [10].

We start with reminding the following Forti-Schwaiger theorem [7]:

Let A be an abelian group containing an element of infinite order, let Y be a normed space and assume that for all function $f\colon A \to Y$ such that $f(x+y) - f(x) - f(y)$ is bounded there exists an additive function $h\colon A \to Y$ for which $f(x) - h(x)$ is bounded (i.e., the b-stability of Cauchy equation). Then Y is complete.

The stability of the Cauchy equation of this kind has been called *b-stability* in my paper [11], in which there is also proved that all assumptions of this theorem are essential and some further remarks are collected.

1.2. The homogeneity equation

In the first section of the present paper we are concerned with the homogeneity equation

$$h(\alpha x) = \phi(\alpha) h(x) \qquad \text{for } \alpha \in A,\ x \in X, \tag{1}$$

where A is a group, $\phi\colon A \to \mathbb{R} \setminus \{0\}$ is a given homomorphism, the operation $(\alpha, x) \to \alpha x\colon A \times X \to X$ is an action of A on a set X and h maps a set X into a normed space Y.

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W. Jabłoński and J. Schwaiger proved in [10] (Theorem 6) the following theorem.

Theorem 1.2.1
Assume that

(a) *A is a group isomorphic to $H \times A'$ with some subgroup H of $\mathbb{K} \in \{\mathbb{R}, \mathbb{C}\}$,*

(b) *H contains an element z_0 of modulus different from 1,*

(c) *the operation $(\alpha, x) \to \alpha x \colon A \times X \to X$ is an action of A on a set X,*

(d) *there exists an $x_0 \in X$ such that the stabilizer $A(x_0) = \{\alpha \in A : \ \alpha x_0 = x_0\}$ is trivial,*

(e) *Y is a normed space,*

(f) *for any homomorphism $\phi \colon A \to \mathbb{R} \setminus \{0\}$ the homogeneity equation (1) is stable in the following sense: for every function $f \colon X \to Y$ such that for some positive ε and δ*

$$|f(\alpha x) - \phi(\alpha) f(x)| \leq \varepsilon |\phi(\alpha)| + \delta \qquad \text{for } \alpha \in A,\ x \in X \tag{2}$$

there exists a solution $h \colon X \to Y$ of (1) such that $f - h$ is bounded.

Then Y is a Banach space, i.e., a complete normed space.

The first remark contains some corrections to the paper [10].

Remark 1.2.2

1) On p.129_{2-1} in place of "$f|_{Ax} = 0$ if $X \ni x \neq x_0$" read "$f|_{Ax} = 0$ if $Ax \neq Ax_0$" (the condition $x \neq x_0$ in [10] is not sufficient for the correct definition of f as it may happen that $Ax = Ax_0$ for some $x \neq x_0$, e.g., if $x = \alpha x_0$ for $\alpha \neq 1$).

2) On p.130^6 instead "ϕ into $\mathbb{R} \setminus \{0\}$" there should be "$\phi$ into $\mathbb{K} \setminus \{0\}$" (in the proof of Theorem 1 one has $\phi((z, \alpha')) = z$ and if $\mathbb{K} = \mathbb{C}$ there might be $z \in \mathbb{C}$).

3) On p.130_1 in place of y_p read y_m.

4) The estimation in the case 3(b), on p.131^6 in the proof of Theorem 6, is not true since $f(x) = f((z, \alpha')x_0) = 0$ for $|z| < 1$. However

$$|f((z_1, \alpha_1')x) - \phi((z_1, \alpha_1'))f(x)| = |zz_1||y_n| \leq |z_1||y_n| \leq M|z_1|$$

and the estimation holds with $\varepsilon := \max\{M, r\}$.

In Remarks 1.2.3–1.2.5, 1.2.7–1.2.8 we formulate comments to Theorem 1.2.1.

Remark 1.2.3
The assumption (b) is essential. In fact, if $H = \{1\}$ and $A = \{1\}$, then for arbitrary set X and for every normed (not necessarily complete) space Y the assumptions (a)–(e) are evidently satisfied. We shall show that the supposition (f) is also satisfied. Indeed, if $\phi \equiv 1$, then every function from X to Y is a solution of (1) and if there exists α_0 such that $\phi(\alpha_0) \neq 1$, then every function f fulfilling (2) is bounded:

$$|f(x)| \leq |1 - \phi(\alpha_0)|^{-1}(\varepsilon|\phi(\alpha_0)| + \delta) \qquad \text{for } x \in X,$$

whence $f(x) - 0$ is also bounded.

Remark 1.2.4
The supposition (d) is essential. In fact, for every normed space Y (not necessarily complete) if $\alpha x = x$ for all $\alpha \in A$, $x \in X$, then we have the same situation as we had in Remark 1.2.3.

Remark 1.2.5
Theorem 1.2.1 fails to be true if the stability in the supposition (f) is replaced by the *Ulam-Hyers stability*, i.e., by the condition: for every $\varepsilon > 0$ there exists $\delta > 0$ such that for every function f satisfying

$$|f(\alpha x) - \phi(\alpha)f(x)| \leq \delta \qquad \text{for } \alpha \in A,\ x \in X \tag{3}$$

there exists a solution h of (1) for which

$$|f(x) - h(x)| \leq \varepsilon \qquad \text{for } \alpha \in A,\ x \in X. \tag{4}$$

Namely, if X is a real vector space and the operation $\alpha x\colon (\mathbb{R} \setminus \{0\}) \times X \to X$ is the multiplication by scalars, then equation (1) is stable in the Ulam-Hyers sense for every normed space Y.

To see this, take a selector S of the family of orbits of the operation αx. The function $h(x) = \phi(\alpha)f(x_1)$ for $x = \alpha x_1$, $\alpha \in \mathbb{R} \setminus \{0\}$ and for $x_1 \in S$, is then a solution of (1) satisfying (4) for the function f as in (3) with $\delta = \varepsilon$.

This argument proves also that Theorem 1.2.1 became false if one would accept $\varepsilon = 0$.

Conclusion 1.2.6
The stability in Theorem 1.2.1 (see (f)*) of the equation* (1) *is not equivalent to the Ulam-Hyers stability of this equation.*

Remark 1.2.7
Theorem 1.2.1 would become false if in (f) one replaced the universal quantifier "for any homomorphism ϕ ..." by the existential quantifier "there exists a homomorphism ϕ ...". Indeed, for $\phi(x) \equiv 1$ every function $h\colon X \to Y$ constant on the orbits Ax for $x \in X$ is a solution of equation (1): $h(\alpha x) = h(x)$. Let S be a selector of the family of orbits of the action αx. Put $h(\alpha x_1) = f(x_1)$ for $x_1 \in S$, $\alpha \in A$ and $f\colon X \to Y$ the function satisfying $|f(\alpha x) - f(x)| \leq \varepsilon + \delta$. Then $|f(\alpha x_1) - f(x_1)| \leq \varepsilon + \delta$, thus $|f(\alpha x_1) - h(\alpha x_1)| \leq \varepsilon + \delta$, and the function $f - h$ is bounded, no matter whether the normed space Y is complete or not.

The universal quantifier "for any homomorphism ϕ ..." in the supposition (f) of Theorem 1.2.1 suggests the following

Question
For which groups A the supposition (f) of Theorem 1.2.1 is satisfied?

Remark 1.2.8
Theorem 1.2.1 need not be true if Y is a metric space. Indeed, if $A = (\mathbb{R} \setminus \{0\}, \cdot)$, $X = Y = \mathbb{R}$, the operation αx in (c) is the ordinary multiplication and the metric on Y is defined by $\rho(x, y) = |\arctan x - \arctan y|$, then equation (1) is stable (the space Y is bounded) and Y is not complete.

We are going to present a theorem that works in the case of metric spaces and is a generalization of Theorem 1.2.1. To this end we introduce the notion of a G-space.

Definition
Let $(G, \cdot)$ be a semigroup and let Z be a nonempty set with a fixed element θ. Assume that we are given a semigroup action on Z, that is we have a function $\cdot\colon G \times Z \to Z$ which satisfies the following conditions:

$$(g_1 g_2)z = g_1(g_2 z) \quad \text{for } g_1, g_2 \in G,\ z \in Z \qquad \text{and} \qquad 1z = z \quad \text{for } z \in Z,$$

if there exists neutral element 1 in G. Let moreover $g\theta = \theta$ for $g \in G$ and $0z = \theta$ for $z \in Z$ if there exists the absorbing element 0 in G. Then the pair $(Z, (G, \cdot))$ satisfying these conditions we will call a *G-space*.

We have however the following generalization of Theorem 1.2.1.

Theorem 1.2.9
Assume that

1) *the operation $\alpha x\colon A \times X \to X$ is an action of a group A on some set X,*

2) *$(Y, (\mathbb{K}, \cdot))$ is a $\mathbb{K}$-space with a metric ρ on Y such that*

$$\rho(\lambda a, \lambda b) \le |\lambda| \rho(a, b) \qquad \text{for } \lambda \in \mathbb{K},\ a, b \in Y, \tag{5}$$

3) *there exists a homomorphism $\phi\colon A \to \mathbb{K}^\star := \mathbb{K} \setminus \{0\}$, such that, for some $x_0 \in X$, a stabilizer $A(x_0)$ of the operation αx is contained in the kernel $K(|\phi|) = \{x \in A : \ |\phi(x)| = 1\}$ of the homomorphism $|\phi(x)|\colon A \to \mathbb{K}^\star$, $K(|\phi|) \neq A$ and for which equation (1) postulated for $x \in Ax_0$, $\alpha \in A$ is max-stable, i.e., for every function $f\colon X \to Y$ such that*

$$\rho[f(\alpha x), \phi(\alpha) f(x)] \le \max(|\phi(\alpha)|, 1) \qquad \text{for } \alpha \in A,\ x \in Ax_0 \tag{6}$$

there exists some solution $h\colon Ax_0 \to Y$ of (1) for which $\rho[f(x), h(x)]$ is bounded for x in Ax_0.

Then Y is complete.

Remark 1.2.10
By (5) we have for $0 \neq \lambda \in \mathbb{K}$ and $a, b \in Y$

$$|\lambda|\rho(a,b) = |\lambda|\rho(\lambda^{-1}\lambda a, \lambda^{-1}\lambda b) \leq |\lambda||\lambda|^{-1}\rho(\lambda a, \lambda b) = \rho(\lambda a, \lambda b) \leq |\lambda|\rho(a,b),$$

whence $\rho(\lambda a, \lambda b) = |\lambda|\rho(a,b)$ (the equality being valid for $\lambda = 0$ too).

The proof of Theorem 1.2.9 is analogous to that of Theorem 6 in [10]. It is given here for the convenience of readers and because of the comments that conclude this section.

Proof of Theorem 1.2.9. Since $K(|\phi|) \neq A$, there exists an $\alpha_0 \in A$ such that $r := |\phi(\alpha_0)| \neq 1$. One can assume without loss of generality that $r > 1$. Let y_n be a Cauchy sequence of elements of Y. We may assume that $\rho(y_{n+m}, y_n) \leq r^{-n}$ for $n, m \in \mathbb{N}$. We put

$$g(\alpha x_0) = \begin{cases} |\phi(\alpha)| y_n & \text{if } r^n \leq |\phi(\alpha)| < r^{n+1}, \\ 0 & \text{if } |\phi(\alpha)| < 1. \end{cases}$$

The function g is well defined since $\alpha x_0 = \beta x_0$ implies $\beta^{-1}\alpha x_0 = x_0$, in turn $\beta^{-1}\alpha \in A(x_0) \subset K(|\phi|)$, whence $|\phi(\beta^{-1}\alpha)| = 1$ and $|\phi(\alpha)| = |\phi(\beta)|$. Fix an $x \in Ax_0$ and put $L := \rho[g(\alpha x), \phi(\alpha)g(x)]$. Furthermore, let $M := \sup_{k\in\mathbb{N}} \rho(y_k, 0)$.

Hence $x = \alpha' x_0$. Denote $z = |\phi(\alpha)|$ and $z' = |\phi(\alpha')|$ and consider four cases possible.

(1) If $z \geq 1$ and $z' \geq 1$, then $r^n \leq z < r^{n+1}$ and $r^m \leq z' < r^{m+1}$ for some $n, m \in \mathbb{N}$. Thus $r^{n+m} \leq zz' < r^{n+m+2}$. Thus, in view of Remark 1.2.10, we get

$$L = \rho[g(\alpha\alpha' x_0), \phi(\alpha)g(\alpha' x_0)] = \rho(zz' y_{n+m+\sigma}, zz' y_m)$$

with some $\sigma \in \{0, 1\}$. Thus

$$L = zz'\rho(y_{n+m+\sigma}, y_m) \leq r^{m+1} z r^{-m} = rz.$$

(2) If $z' \geq 1$ and $z < 1$, then $r^n \leq z' < r^{n+1}$ for some integer n and we consider two subcases. If $zz' < 1$ we have $g(\alpha x) = 0$ and $g(x) = z' y_n$ for some $n \in \mathbb{N}$, thus $L = zz'\rho(0, y_n) \leq M$. If $zz' \geq 1$, then we take an $r^m \leq zz' < r^{m+1}$ for some integer m with $0 \leq m \leq n$, thus $L = zz'\rho(y_m, y_n) \leq r^{m+1}r^{-m} = r$.

(3) If $z' < 1$ and $z \geq 1$, then $g(x) = 0$ and we have two subcases. If $zz' < 1$, then $L = \rho(0, \phi(\alpha)\cdot 0) = 0$. If $zz' \geq 1$ let $m \in \mathbb{N}_0$ be such that $r^m \leq zz' < r^{m+1}$. Thus $L = zz'\rho(y_m, 0) \leq Mz$.

(4) In the case $z < 1$ and $z' < 1$ we have $zz' < 1$ and $L = \rho(0,0) = 0$.

Thus is proved that $L \leq \max\{r, M\} \max\{|\phi(\alpha)|, 1\}$. Hence a function $f = \varepsilon^{-1} g$, where $\varepsilon = \max\{r, M\}$, satisfies (6). Hence by 3) there exists a $\beta > 0$ and a function $h\colon Ax_0 \to Y$ such that $h(\alpha x) = \phi(\alpha)h(x)$ for $x \in Ax_0$, $\alpha \in \mathbb{K}$ and $\rho[f(x), h(x)] \leq \beta$ for $x \in Ax_0$. Therefore

$$\begin{aligned} r^n \rho[y_n, \varepsilon h(x_0)] &= |\phi(\alpha_0^n)|\rho[y_n, \varepsilon h(x_0)] = \rho[\phi(\alpha_0^n) y_n, \phi(\alpha_0^n)\varepsilon h(x_0)] \\ &= \rho[g(\alpha_0^n x_0), \varepsilon h(\alpha_0^n x_0)] = \rho[\varepsilon f(\alpha_0^n x_0), \varepsilon h(\alpha_0^n x_0)] \\ &\leq \varepsilon\beta, \end{aligned}$$

thus $y_n \to \varepsilon h(x_0)$.

Remark 1.2.11

Theorem 1.2.9 is a generalization of Theorem 1.2.1. In fact, the assumption (a) (with not very clear structure of A') implies that

(a') there exists an homomorphism $\psi: A \to \mathbb{K}$. Indeed, if $i = (i_1, i_2)$ is an isomorphism from A onto $H \times A'$, then i_1 is an homomorphism from A onto H.

Therefore the assumption (b) has the form

(b') there exists a $\alpha_0 \in A$ such that $|\psi(\alpha_0)| =: r \neq 1$.

If the function $f: Ax_0 \to Y$ satisfies (6) with $\phi = \psi$, then the function $f^\star: X \to Y$ defined by

$$f^\star(x) = \begin{cases} f(x) & \text{for } x \in Ax_0, \\ 0 & \text{for } x \notin Ax_0 \end{cases}$$

fulfils (2) with $\phi = \psi$ and $\varepsilon = \delta = 1$. Thus the assumption 3) of Theorem 1.2.9 is satisfied for $\phi = \psi$.

Remark 1.2.12

Assume that A is a group, $\alpha x: A \times X \to X$ is an action of A on a set X and Y is a space with metric ρ. Then the ρ-*stability* of (1), i.e., the condition

for every function $f: X \to Y$ such that for some positive ε, δ

$$\rho[f(\alpha x), \phi(\alpha) f(x)] \leq \varepsilon |\phi(\alpha)| + \delta \qquad \text{for } \alpha \in A,\ x \in X \tag{7}$$

there exists some solution $h: X \to Y$ of (1) such that $\rho[f(x), h(x)]$ is bounded,

evidently implies the max-stability of this equation.

Conversely, the assumptions 1) and 2) in the Theorem 1.2.9 are satisfied and if the homomorphism ϕ and the operation $\cdot$ in the equation (1) are such that there exists a stabilizer $A(x_0) \subset K(|\phi|)$ and $K(|\phi|) \neq A$, then the max-stability of (1) implies the stability of this equation in virtue of the below Theorem 1.2.13, since if our equation is max-stable, then the space Y is complete by Theorem 1.2.9.

The following theorem is to some extend inverse to Theorem 1.2.9.

Theorem 1.2.13

Assume the suppositions 1) *and* 2) *of Theorem 1.2.9.*

A) *If the kernel* $K(|\phi|) \neq A$ *for the homomorphism* $\phi: A \to \mathbb{K}^\star$ *occurring in the equation* (1) *and the space* Y *is complete, then the equation* (1) *is* ρ*-stable.*

B) *Equation* (1) *is* ρ*-stable also if* $K(|\phi|) = A$ *provided the metric in* Y *is such that*

$$\rho[(1-\lambda)a, 0] \leq \rho(a, \lambda a) \qquad \textit{for } a \in Y,\ \lambda \in \mathbb{K} \textit{ and } |\lambda| = 1. \tag{8}$$

Proof. A) Assume that there exists an $\alpha_0 \in A$ such that $r = |\phi(\alpha_0)| > 1$. By (2) we have $\rho[f(\alpha_0 x), \phi(\alpha_0)f(x)] \le \varepsilon r + \delta =: \beta$ and by induction

$$\rho[f(\alpha_0^n x), \phi(\alpha_0^n)f(x)] \le \beta(r-1)^{-1}(r^n - 1).$$

Then there exists $\lim_{n\to\infty} \phi(\alpha_0^n)^{-1} f(\alpha_0^n x) =: h(x)$ by the classical Hyers argument – usually called the "direct method". Since

$$\rho[f(\alpha_0^n \alpha x), \phi(\alpha)f(\alpha_0^n x)] \le \varepsilon|\phi(\alpha)| + \delta \qquad \text{for } \alpha \in K,\ x \in X$$

the function h is a solution of (1). Moreover, in view of (1), we get

$$\rho[f(\alpha_0^n x), \phi(\alpha_0^n)f(x)] \le r^n \varepsilon + \delta$$

what implies $\rho[f(x), h(x)] \le \varepsilon$. The proof of A) is finished.

B) Assume that the function $f\colon X \to Y$ satisfies (7) with $|\phi| = 1$ and put $\mu := \varepsilon + \delta$. Notice that the two stabilizers $A(x_1)$ and $A(x_2)$ of some orbit **O** of action αx are conjugate since

$$x_2 = \gamma x_1 \implies A(x_2) = \gamma^{-1} A(x_1)\gamma.$$

We have thus for an orbit **O** two cases:

a) there exists an stabilizer $A(x_0)$ of the orbit O contained in $K(\phi)$,

b) any stabilizer of **O** is not contained in $K(\phi)$.

Ad a) The function $h^\star(x) = h(\alpha x_0) = \phi(\alpha)f(x_0)$ for $x = \alpha x_0$ is well defined and it is a solution of (1) for $(\alpha, x) \in A \times \mathbf{O}$. Moreover we have

$$\rho[f(x), h^\star(x)] = \rho[f(\alpha x_0), h^\star(\alpha x_0)] = \rho[f(\alpha x_0), \phi(\alpha)f(x_0)] \le \mu \qquad \text{for } x = \alpha x_0.$$

Ad b) Let x be in **O**. The stabilizer $A(x)$ is not contained in $K(\phi)$. Thus there exists α_0 such that $\alpha_0 x = x$ and $\phi(\alpha_0) \ne 1$. Since $\{\phi(\alpha_0)^k\}_{k\in\mathbb{Z}}$ is a multiplicative group on the unit circle with at least two elements, there exists an $n \in \mathbb{N}$ such that $|1 - \phi(\alpha_0^n)| \ge 1$. Therefore

$$\begin{aligned}\rho[f(x), 0] &\le |1 - \phi(\alpha_0^n)|\rho[f(x), 0] = \rho[\{1 - \phi(\alpha_0^n)\}f(x), 0] \\ &\le \rho[f(x), \phi(\alpha_0^n)f(x)] = \rho[f(\alpha_0^n x), \phi(\alpha_0^n)f(x)] \\ &\le \mu.\end{aligned}$$

Consequently, for every f there exists a solution h of (1) on $A \times X$ ($h = h^\star$ in the case a) and $h = 0$ in the case b)) such that $\rho[f(x), h(x)] \le \mu$ for $x \in X$.

Remark 1.2.14
If $\mathbb{K} = \mathbb{R}$, then applying (8) with $\lambda = -1$, we get

$$\rho(2a, 0) \le \rho(a, -a) \qquad \text{for } a \in Y.$$

Thus, if ρ satisfies (5), then (see Remark 1.2.10)

$$\rho(a, -a) \le \rho(a, 0) + \rho(0, -a) = 2\rho(a, 0) = \rho(2a, 0),$$

and we obtain $\rho(2a, 0) = \rho(a, -a)$ for $a \in Y$.

Remark 1.2.15
If the equation (1) is max-stable, then the constant, which bounds $\rho[f(x), h(x)]$ on Ax_0, may depend "a priori" on f. If for the homomorphism ϕ in (1) there exists $\alpha_0 \in A$ such that $r := |\phi(\alpha_0)| \neq 1$, then this function $\rho[f(x), h(x)]$ is bounded by 1 on Ax_0. Really, let be the equation (1) max-stable, $f\colon X \to Y$ the function satisfying (6) and $h\colon X \to Y$ the solution of (1) such that $\rho[f(x), h(x)] \leq \beta$ for some $\beta > 0$. We have for $n \in \mathbb{N}$

$$\begin{aligned} r^n \rho[f(x), h(x)] &= |\phi(\alpha_0^n)|\rho[f(x), h(x)] = \rho[\phi(\alpha_0^n)f(x), \phi(\alpha_0^n)h(x)] \\ &\leq \rho[\phi(\alpha_0^n)f(x), f(\alpha_0^n x)] + \rho[f(\alpha_0^n x), h(\alpha_0^n x)] \leq \max(r^n, 1) + \beta \\ &\leq r^n + \beta. \end{aligned}$$

We may suppose, without loss of generality, that $r > 1$. Thus for $n \to \infty$ we obtain $\rho[f(x), h(x)] \leq 1$ for $x \in Ax_0$. This means that there exist the homogeneity equations for which the bound of $\rho[f(x), h(x)]$ not depend on f.

By analogous argument we obtain the inequality $\rho[f(x), h(x)] \leq \varepsilon$ for the ρ-stability.

Finally, the function h spoken in Theorem 1.2.13 is unique. For, let h_1 and h_2 be solutions of (1) such that $\rho[f(x), h_1(x)] \leq \varepsilon$ and $\rho[f(x), h_2(x)] \leq \varepsilon$. Then $\rho[h_1(x), h_2(x)] \leq 2\varepsilon$ and

$$r^n \rho[h_1(x), h_2(x)] = \rho[\phi(\alpha_0^n)h_1(x), \phi(\alpha_0^n)h_2(x)] = \rho[h_1(\alpha_0^n x), h_2(\alpha_0^n x)] \leq 2\varepsilon$$

for $n \in \mathbb{N}$. This implies $\rho[h_1(x), h_2(x)] = 0$, whence $h_1(x) = h_2(x)$.

Remark 1.2.16
Let $h\colon X \to Y$ be a solution of (1). For every function $f\colon X \to Y$, if $\rho[f(x), h(x)] \leq \varepsilon$ for $x \in X$, then $\rho[\phi(\alpha)f(x), \phi(\alpha)h(x)] \leq \varepsilon|\phi(\alpha)|$ and $\rho[f(\alpha x), h(\alpha x)] \leq \varepsilon$. Thus

$$\rho[f(\alpha x), \phi(\alpha)f(x)] \leq \varepsilon|\phi(\alpha)| + \varepsilon \qquad \text{for } x \in X,\ \alpha \in A. \tag{9}$$

Conclusion 1.2.17
If the equation (1) *is ρ-stable and the function f satisfies* (7), *then f fulfils* (9).

The following result is a particular case of Theorem 1.2.1.

Corollary 1.2.18
Assume that $A = (\mathbb{R}\setminus\{0\}, \cdot)$, $X = \mathbb{R}$, the operation $\cdot$ is the ordinary multiplication, $\phi(\alpha) = \alpha$ and conditions (e) *and* (f) *are satisfied. Then Y is a Banach space.*

Remark 1.2.19
The assumptions of Theorem 1.2.1 are seemingly more general than those of Corollary 1.2.18. However, in virtue of Theorem 1.2.1 its assumptions imply however the stability of the equation

$$h(\alpha x) = \alpha h(x) \qquad \text{for } \alpha \in \mathbb{R} \setminus \{0\},\ x \in \mathbb{R},$$

where αx is the ordinary multiplication, because of the following

Conclusion 1.2.20
Assume that $A = (\mathbb{R} \setminus \{0\}, \cdot)$ *(*$A = (\mathbb{C} \setminus \{0\}, \cdot)$*),* $X = \mathbb{R}$ *(*$X = \mathbb{C}$*), the operation* αx *is the ordinary multiplication,* $\phi(\alpha) = \alpha$ *and* (e) *is satisfied. Then the statements: "condition* (f) *is satisfied" and "*Y *is a real (complex) Banach space" are equivalent.*

1.2.1. Applications

Theorem 1.2.1 does not suit direct for applications because of the requirement "for all homomorphism" in its assumption (f). In order to get completeness of a normed space it is thus more convenient to apply Corollary 1.2.18 or Theorem 1.2.13.

Example 1.2.21
Let V be a Banach space. The normed space (V^S), where S is an arbitrary set, of the functions $f\colon S \to V$, such that $\sup_{s\in S} |f|_s < \infty$ (where $f_s = f(s)$), with the usual addition and multiplication by scalars and with $|f| = \sup_{s\in S} |f_s|$, is complete, since the equation of homogeneity from $\mathbb{R}$ to (V^S) is stable.

Indeed, if $f\colon \mathbb{R} \to (V^S)$ and $|f(\alpha x) - \alpha f(x)| = \sup_{s\in S} |f_s(\alpha x) - \alpha f_s(x)| \le \varepsilon|\alpha| + \delta$ for $x \in \mathbb{R}$, $\alpha \in \mathbb{R} \setminus \{0\}$ with some positive ε and δ, then $|f_s(\alpha x) - \alpha f_s(x)| \le \varepsilon|\alpha| + \delta$ for $x \in \mathbb{R}$, $\alpha \in \mathbb{R} \setminus \{0\}$ and $s \in S$. By Theorem 1.2.13, there exists a homogeneous function $h_s\colon G \to V$ such that $|f_s(x) - h_s(x)| \le \varepsilon$ for $x \in \mathbb{R}$ and $s \in S$. The function $h = (h_s)\colon \mathbb{R} \to V$ is a homogeneous function and $|f(x) - h(x)| = \sup_{s\in S} |f_s(x) - g_s(x)| \le \varepsilon$ for $x \in G$.

The following lemma will be helpful in the next example

Lemma 1.2.22
Assume that $(G, +)$ *is a groupoid,* Y *is a normed space and* $f\colon G \to Y$ *is a function such that* $|f(2x) - 2f(x)| \le \beta$ *for* $x \in G$ *and for some* $\beta > 0$*. Then either* $|f(x)| \le \beta$ *for* $x \in G$ *or the function* f *is unbounded.*

Proof. Assume that $|f(x)| > \beta$ for some $x \in G$. Then there exists a $\gamma > 0$ such that $|f(x)| = \beta + \gamma$. This equality, when combined with the inequality

$$\beta \ge |f(2x) - 2f(x)| \ge |2f(x)| - |f(2x)|$$

gives

$$|f(2x)| + \beta \ge |2f(x)| = 2|f(x)| = 2\beta + 2\gamma.$$

Consequently, $|f(2x)| \ge \beta + 2\gamma$. By induction, we deduce that $|f(2^n x)| \ge \beta + 2^n\gamma$ for every $n \in \mathbb{N}$. Thus the function f is unbounded.

Example 1.2.23
Let V be a Banach space. The space $(V^{\mathbb{N}})$ of the sequences $x = (x_n)_{n\in\mathbb{N}}$ such that $|x_1| + \sum_{n=1}^{\infty} |x_n - x_{n+1}| < \infty$ with the norm

$$|x| = |x_1| + \sum_{n=1}^{\infty} |x_n - x_{n+1}|$$

and with the ordinary addition and multiplication by scalars is a complete normed space, since the equation of homogeneity (1), postulated on $\mathbb{R} \times (V^{\mathbb{N}})$, is stable.

Indeed, if $f\colon \mathbb{R} \to (V^{\mathbb{N}})$ and

$$\begin{aligned}|f(\alpha x) - \alpha f(x)| \\ &= |f_1(\alpha x) - \alpha f_1(x)| + \sum_{n=1}^{\infty} |f_n(\alpha x) - \alpha f_n(x) - f_{n+1}(\alpha x) + \alpha f_{n+1}(x)| \\ &\le \varepsilon|\alpha| + \delta\end{aligned}$$

for $x, \alpha \in \mathbb{R}$, where $f = (f_n)$ and $f_n\colon \mathbb{R} \to V$, then $|f_1(\alpha x) - \alpha f_1(x)| \le \varepsilon|\alpha| + \delta$ and by Theorem 1.2.13 there exists a solution $g_1\colon \mathbb{R} \to V$ of the equation

$$g(\alpha x) - \alpha g(x) = 0 \tag{10}$$

such that $|f_1(x) - g_1(x)| \le \varepsilon$ for $x \in \mathbb{R}$. Since

$$|f_n(\alpha x) - \alpha f_n(x) - f_{n+1}(\alpha x) + \alpha f_{n+1}(x)| \le \varepsilon|\alpha| + \delta,$$

thus by Theorem 1.2.13 there exists a solution g_{n+1} of (10) such that $|f_{n+1}(x) - f_n(x) - g_{n+1}(x)| \le \varepsilon$ for $x \in \mathbb{R}$ and $n = 1, 2, \ldots$. Applying the latter inequality for $n = 1$, we obtain

$$|f_2(x) - (g_1(x) + g_2(x))| \le |f_1(x) - g_1(x)| + |f_2(x) - f_1(x) - g_2(x)| \le 2\varepsilon$$

and by induction $|f_n(x) - (g_1(x) + \ldots + g_n(x))| \le n\varepsilon$. The function $h_n(x) = g_1(x) + \ldots + g_n(x)$ being a solution of (10), thus the function $h = (h_n)$ is a solution of the equation $k(\alpha x) = \alpha k(x)$ for $k\colon \mathbb{R} \to (V^{\mathbb{N}})$ and for $p = (p_n) = f - h = (f_n - h_n)$ we have $|p_n(x)| \le n\varepsilon$. Therefore $|f(2x) - 2f(x)| \le 2\varepsilon + \delta$, and $|p(2x) - 2p(x)| \le 2\alpha + \delta$.

Fix an $m \in \mathbb{N}$ and consider the normed space V^m of all sequences $x = (x_1, \ldots, x_m)$, where $x_n \in V$ for $n = 1, \ldots, m$, equipped with the standard addition and multiplication by scalars and with the norm given by

$$|x|_m = |x_1| + \sum_{n=1}^{m-1} |x_n - x_{n+1}|.$$

We have for the function $P(x) = (p_1(x), \ldots, p_m(x))$

$$|P(x)|_m = |p_1(x)| + \sum_{n=1}^{m-1} |p_n(x) - p_{n+1}(x)|.$$

Thus this function is bounded and $|P(2x) - 2P(x)|_m \le 2\alpha + \delta$. In view of Lemma 1.2.22 there is $|P(x)|_m \le 2\alpha + \delta$ for every $m \in \mathbb{N}$. This inequality yields

$$|p_1(x)| + \sum_{n=1}^{\infty} |p_n(x) - p_{n+1}(x)| \le 2\alpha + \delta,$$

whence $|f(x) - h(x)| \le 2\alpha + \delta$ for $x \in \mathbb{R}$. Therefore the equation of homogeneity from $\mathbb{R}$ to $(V^{\mathbb{N}})$ is stable.

We continue with examples showing the coherence of stability with completeness of the target space for other functional equations.

1.3. The Schröder's equation

From Theorem 1.2.9 we get

Corollary 1.3.1
Let $X = \mathbb{R}$ ($X = \mathbb{C}$), $\lambda \in \mathbb{R}$ ($\lambda \in \mathbb{C}$) and $0 \neq |\lambda| \neq 1$. Assume that the supposition 2) in Theorem 1.2.9 is satisfied. The Schröder's equation

$$h(\lambda * x) = \lambda h(x), \tag{11}$$

where $*$ is the ordinary multiplication, is b-stable (i.e., for every function $f\colon X \to Y$ there exists a solution h of (11) such that $\rho[f(x), h(x)]$ is bounded provided so is $\rho[f(\lambda * x), \lambda f(x)]$) if and only if Y is complete.

Proof. Let $A = (\mathbb{Z}, +)$, $\phi(\alpha) = \lambda^\alpha$, $\alpha x = \lambda^\alpha * x$ and let $\rho[f(\alpha x), \phi(\alpha) f(x)] = \rho[f(\lambda^\alpha * x), \lambda^\alpha f(x)]$ be bounded for $x \in X$, $\alpha \in A$. Then for $\alpha = 1$ $\rho[f(\lambda * x), \lambda f(x)]$ is bounded too, and by assumption that (11) is b-stable, there exists a solution h of (11) such that $\rho[f(x), h(x)]$ is bounded. This h satisfies (1) since $h(\lambda x) = h(\lambda^\alpha * x) = \lambda^\alpha h(x) = \phi(\alpha) h(x)$. Since all assumptions of Theorem 1.2.9 are satisfied, thus Y is complete.

The converse implication is true in virtue of Theorem 1.2.13 since the stability of equation $h(\lambda^\alpha * x) = \lambda^\alpha h(x)$ for $x \in X$, $\alpha \in A = \mathbb{Z}$ implies the b-stability of equation (11) (see the beginning of the proof of Theorem 1.2.13).

1.4. The quadratic equation

Theorem 1.4.1
The quadratic functional equation

$$h(k+p) + h(k-p) = 2h(k) + 2h(p), \tag{12}$$

where $h\colon \mathbb{Z} \to Y$ and Y is a normed space, is b-stable (i.e., for every function $f\colon \mathbb{Z} \to Y$ for which $|f(k+p) + f(k-p) - 2f(k) - 2f(p)|$ is bounded there exists a solution h of (12) such that $|f(k) - h(k)|$ is bounded) if and only if Y is complete.

Proof. Sufficiency follows via the "direct method" (see [11]).

Necessity. Let y_n be a Cauchy sequence of elements of Y and assume that $|y_{n+m} - y_n| \leq 4^{-n}$ for $n, m \in \mathbb{N}_0$. We show that for the function f given by $f(k) = k^2 y_{|k|}$ for $k \in \mathbb{Z}$ the expression

$$\begin{aligned} L = L(k,p) &= f(k+p) + f(k-p) - 2f(k) - 2f(p) \\ &= (k+p)^2 y_{|k+p|} + (k-p)^2 y_{|k-p|} - 2k^2 y_{|k|} - 2p^2 y_{|p|} \qquad \text{for } k, p \in \mathbb{Z} \end{aligned}$$

is bounded. Assume without loss of generality that $k \geq p$. The following four cases are possible:

1) $k, p \geq 0$. We have

$$\begin{aligned} |L| &= |2k^2(y_{k+p} - y_k) + 2p^2(y_{k+p} - y_p) - (k-p)^2(y_{k+p} - y_{k-p})| \\ &\leq 2k^2 4^{-k} + 2p^2 4^{-p} + (k-p)^2 4^{-k+p} \\ &\leq 5. \end{aligned}$$

2) $k, k+p \le 0$; $p < 0$. We have

$$\begin{aligned}|L| &= |2k^2(y_{k-p} - y_k) + 2p^2(y_{k-p} - y_{-p}) + (k+p)^2(y_{k+p} - y_{k-p})| \\ &\le 2k^2 4^{-k} + 2p^2 4^p + (k+p)^2 4^{k+p} \\ &\le 5.\end{aligned}$$

3) $k \ge 0$; $p, k+p < 0$. We have

$$\begin{aligned}|L| &= |(k+p)^2(y_{-k-p} - y_{k-p}) + 2k^2(y_{k-p} - y_k) + 2p^2(y_{k-p} - y_{-p})| \\ &\le (k+p)^2 4^{k+p} + 2k^2 4^{-k} + 2p^2 4^p \\ &\le 5.\end{aligned}$$

4) $k, p < 0$. We have

$$\begin{aligned}|L| &= |2k^2(y_{-k-p} - y_{-k}) + 2p^2(y_{-k-p} - y_{-p}) + (k-p)^2(y_{k-p} - y_{-k-p})| \\ &\le 2k^2 4^k + 2p^2 4^p + (k-p)^2 4^{p-k} \\ &\le 5.\end{aligned}$$

Since equation (12) is b-stable, there exists a solution h of (12) and some positive M such that $|f(k) - h(k)| \le M$ for $k \in \mathbb{Z}$. For $n \in \mathbb{N}_0$ we have $h(n) = n^2 h(1)$ (see [6] p.89, Theorem 10.1) and $|n^2 y_n - n^2 h(1)| \le M$, thus $y_n \to h(1)$ if $n \to \infty$.

1.5. The Drygas' equation

A theorem analogous to Theorem 1.4.1 is true for the Drygas' equation

$$h(k+p) + h(k-p) = 2h(k) + h(p) + h(-p)$$

for the function h from $\mathbb{Z}$ to normed space Y. Its proof is practically the same as that of Theorem 1.4.1 since the general solution of Drygas' equation is of the form $q + a$, where q is a solution of the equation (12) and a is an additive function, whence $q(n) + a(n) = n^2 q(1) + na(1)$ for $n \in \mathbb{N}_0$. Moreover, the Drygas' equation is b-stable if h is the function from the abelian group to the Banach space (see [13]) and

$$\begin{aligned}&f(k+p) + f(k-p) - 2f(k) - f(p) - f(-p) \\ &\quad = (k+p)^2 y_{|k+p|} + (k-p)^2 y_{|k-p|} - 2k^2 y_{|k|} - 2p^2 y_{|p|} \\ &\quad = L(k,p)\end{aligned}$$

for the function $f(k) = k^2 y_{|k|}$ for $k \in \mathbb{Z}$ and the function $L(k,p)$ as in the proof of Theorem 1.4.1.

1.6. The Jensen's equation

We have an analogous theorem for Jensen's equation

$$h\Big(\frac{x+y}{2}\Big) = \frac{h(x) + h(y)}{2}, \tag{13}$$

where $h\colon \mathbb{R} \to Y$ and Y is a normed space.

If Y is complete, the "direct method" is applied for the proof that equation (13) is b-stable.

We proceed as follows. Let y_n be a Cauchy sequence of elements of Y and assume that $|y_{n+m} - y_n| \leq 2^{-n}$ for $n, m \in \mathbb{N}_0$. For the function $f: \mathbb{R} \to Y$

$$f(x) := ([x] + 1 - x)f([x]) + (x - [x])f([x] + 1),$$

where $[x]$ is the entire part of x, $f(n) := ny_n$ for $n \in \mathbb{N}_0$, $f(n) := -f(-n)$ for $n \in \mathbb{Z} \setminus \mathbb{N}_0$, the function $f(x+y) - f(x) - f(y)$ is bounded (see [7]). This implies that the expression $f(\frac{x+y}{2}) - f(\frac{x}{2}) - f(\frac{y}{2})$ is bounded. Thus, taking $y = x$, we conclude that the difference $f(x) - 2f(\frac{x}{2})$ is bounded. Consequently, the difference

$$f\Big(\frac{x+y}{2}\Big) - \frac{f(x)+f(y)}{2} = \Big[f\Big(\frac{x+y}{2}\Big) - f\Big(\frac{x}{2}\Big) - f\Big(\frac{y}{2}\Big)\Big] + \frac{1}{2}\Big[2f\Big(\frac{x}{2}\Big) - f(x)\Big] + \frac{1}{2}\Big[2f\Big(\frac{y}{2}\Big) - f(y)\Big]$$

is bounded too. If the equation (13) is b-stable, then there exists a solution h of this equation such that $|f(x) - h(x)| \leq M$ for $x \in \mathbb{R}$ and some positive M. Since $h(x) = a(x) + b$, where $a: \mathbb{R} \to Y$ is an additive function and $b \in Y$ (see e.g. [6], p.11), we obtain

$$|f(n) - h(n)| = |ny_n - na(1) - b| \leq M \qquad \text{for } n \in \mathbb{N},$$

thus $y_n \to a(1)$ for $n \to \infty$. Thus Y actually is a complete space.

Remark 1.6.1
We have the same result for the equation

$$h(x+y) + h(x-y) = 2h(x)$$

for $h: \mathbb{N} \to Y$ since

$$f(x+y) + f(x-y) - 2f(x) = [f(x+y) + f(x-y) - f(2x)] + [f(2x) - 2f(x)].$$

Here the property of 2-divisibility in the domains of the functions under consideration is not necessary.

Remark 1.6.2
The stability (adequately defined) of the equation considered above and of the Cauchy equation (see [9]) for a complete target space is proved by the "direct method" and vice versa for these equations this stability implies completeness of target space. However the equation (1) is b-stable for an arbitrary target space (even not complete) if every stabilizer of the action αx is trivial (this can be proved without use of the "direct method" – see the end of Remark 1.2.5). In addition, if $K(|\phi|) \neq A$, then b-stability, when the target space is complete, can be proved by the "direct method" too (see the proof of Theorem 1.2.13).

2. Superstability

2.1. The sine equation

P.W. Cholewa proved in [5] that the sine functional equation

$$f(x+y)f(x-y) = f^2(x) - f^2(y), \tag{14}$$

where f is a function from an abelian group G uniquely divisible by 2 into $\mathbb{C}$, is *superstable*, i.e., every unbounded function $f\colon G \to \mathbb{C}$ such that

$$|f(x+y)f(x-y) - f^2(x) + f^2(y)| \le \delta \qquad \text{for } x, y \in G,\ \delta > 0, \tag{15}$$

is a solution of (14). In this proof is essential that $\mathbb{C}$ forms a field. In fact, it is proved in [5] that *every unbounded function f from G to A (where A is a normed, commutative algebra A with the multiplicative norm and a unity), satisfying* (15) *and which has at least one invertible value, is a solution of* (14). Note that according to a generalization of Mazur-Gelfand theorem [14] , an algebra A over $\mathbb{C}$, which is a field (i.e., every element not equal to zero has an inverse), is isomorphic with $\mathbb{C}$ whenever there exists a non-trivial linear functional defined on A.

We present the following result in this spirit.

Theorem 2.1.1
Let G be a commutative group, uniquely 2-divisible and let A be a finite-dimensional normed commutative algebra without the zero divisors. Then equation (14) *for $f\colon G \to A$ is superstable, i.e., every unbounded function satisfying* (15) *is a solution of* (14).

Proof. This is a modification of the proof in [5].

Since f is unbounded, then there exists a sequence $x_n \in G$ such that $\lim_{n\to\infty} |f(x_n)| = \infty$ and $\mu_n := f(x_n) \neq 0$. The sequence $|\mu_n|^{-1}\mu_n$ is bounded and A has finite dimension. Thus we can assume that $\lim_{n\to\infty} |\mu_n|^{-1}\mu_n = \varepsilon$ for some $\varepsilon \in A$, $|\varepsilon| = 1$. Putting $x = y = \frac{1}{2}x_n$ in (15) we obtain $|\mu_n f(0)| \le \delta$, thus

$$\big||\mu_n|^{-1}\mu_n f(0)\big| = |\mu_n|^{-1}|\mu_n f(0)| \le \delta|\mu_n|^{-1}.$$

Letting $n \to \infty$ we obtain $|\varepsilon f(0)| = 0$. This yields $\varepsilon f(0) = 0$, in turn $f(0) = 0$ since A has no zero divisors.

By (15) we have

$$\begin{aligned}
&|f(x+y)\mu_n + f(x-y)\mu_n - f(x)[f(y+x_n) - f(y-x_n)]| \\
&\quad\le \Big|f(x+y)\mu_n - f^2\Big(\frac{x+y+x_n}{2}\Big) + f^2\Big(\frac{x+y-x_n}{2}\Big)\Big| \\
&\qquad + \Big|f(x-y)\mu_n - f^2\Big(\frac{x-y+x_n}{2}\Big) + f^2\Big(\frac{x-y-x_n}{2}\Big)\Big| \\
&\qquad + \Big|f^2\Big(\frac{x+y+x_n}{2}\Big) - f^2\Big(\frac{x-y-x_n}{2}\Big) - f(x)f(y+x_n)\Big| \\
&\qquad + \Big|f(x)f(y-x_n) + f^2\Big(\frac{x-y+x_n}{2}\Big) - f^2\Big(\frac{x+y-x_n}{2}\Big)\Big| \\
&\quad\le 4\,\delta
\end{aligned}$$

for $x, y \in G$, thus

$$|\mu_n|^{-1}|f(x+y)\mu_n + f(x-y)\mu_n - f(x)[f(y+x_n) - f(y-x_n)]| \le 4\delta|\mu_n|^{-1}.$$

On letting $n \to \infty$ we obtain

$$g(x,y) := \lim_{n\to\infty} |\mu_n|^{-1} f(x)[f(y+x_n) - f(y-x_n)] = [f(x+y) + f(x-y)]\varepsilon. \quad (16)$$

Therefore taking in (16) $x = \frac{u+v}{2}$ and $y = \frac{u-v}{2}$, we get

$$f(u)\varepsilon + f(v)\varepsilon = g\Big(\frac{u+v}{2}, \frac{u-v}{2}\Big) \qquad \text{for } u, v \in G.$$

Put $x = 0$ in (16). Then $f(y) = -f(-y)$ for $y \in G$. Thus we obtain

$$\begin{aligned} \varepsilon f(x+y) &= \varepsilon f(x+y) + \varepsilon f(0) = g\Big(\frac{x+y}{2}, \frac{x+y}{2}\Big), \\ \varepsilon f(x-y) &= \varepsilon f(x-y) + \varepsilon f(0) = g\Big(\frac{x-y}{2}, \frac{x-y}{2}\Big), \\ \varepsilon f(x) - \varepsilon f(y) &= \varepsilon f(x) + \varepsilon f(-y) = g\Big(\frac{x-y}{2}, \frac{x+y}{2}\Big). \end{aligned}$$

Hence by the definition (16) of the function g

$$\begin{aligned} \varepsilon^2 f(x+y) f(x-y) &= g\Big(\frac{x+y}{2}, \frac{x+y}{2}\Big) g\Big(\frac{x-y}{2}, \frac{x-y}{2}\Big) \\ &= g\Big(\frac{x+y}{2}, \frac{x-y}{2}\Big) g\Big(\frac{x-y}{2}, \frac{x+y}{2}\Big) \\ &= \varepsilon[f(x) + f(y)]\varepsilon[f(x) - f(y)] \\ &= \varepsilon^2[f^2(x) - f^2(y)] \end{aligned}$$

and

$$\varepsilon^2[f(x+y)f(x-y) - f^2(x) + f^2(y)] = 0.$$

Since $\varepsilon^2 \neq 0$ and A has no zero divisors we have arrived at the equality

$$f(x+y)f(x-y) - f^2(x) + f^2(y) = 0.$$

Therefore the function f is a solution of (14).

Remark 2.1.2

For a normed space X the supposition that it has a finite dimension is equivalent to the assumption that every bounded subset of X is compact in X. The assumption in Theorem 2.1.1 that A is of finite dimension is then of the topological nature.

It is essential in the above proof that for every function $f: G \to A$ satisfying (15) if there exists a sequence $x_n \in G$ such that $\lim_{n\to\infty} |f(x_n)| = \infty$ and $f(x_n) \neq 0$, then there exists a convergent subsequence of the sequence $|f(x_n)|^{-1} f(x_n)$. This condition is not true for each infinite-dimensional commutative algebra A, if G is the additive group of A. Indeed, if A has infinite dimension, then there exists a sequence $a_n \in A$ such that $|a_n| = 1$ and there does not exists any convergent subsequence of a_n ([1], p.127–128). The function $f(x) = x$ satisfies (15) for every $\delta > 0$, for $x_n = na_n$ we have $0 \neq |f(x_n)| \to \infty$ for $n \to \infty$ and there is no a convergent subsequence of $|f(x_n)|^{-1} f(x_n) = a_n$.

Remark 2.1.3
The assumption that A has no zero divisors is essential in Theorem 2.1.1. For, let M be the algebra of diagonal 2×2-matrices with ordinary addition and multiplication of matrices and with the norm

$$\left\| \begin{bmatrix} a & 0 \\ 0 & b \end{bmatrix} \right\| := \max\{|a|, |b|\} \qquad \text{for } a, b \in \mathbb{R}.$$

If $G = (\mathbb{R}, +)$, $\delta > 0$ and the function $f\colon G \to M$ is given by

$$f(x) = \begin{bmatrix} x & 0 \\ 0 & \sqrt{\delta} \end{bmatrix},$$

then f is unbounded and satisfies (15) abut it is not a solution of (14).

2.2. The equation of homomorphisms

J.A. Baker proved in [2] (Theorem 1) that if S is a semigroup then for every function $f\colon S \to \mathbb{C}$ such that $|f(xy) - f(x)f(y)| \le \delta$ for $x, y \in S$ and for some positive δ we have

$$|f(x)| \le \frac{1 + \sqrt{1 + 4\delta}}{2} \quad \text{for } x \in S \qquad \text{or} \qquad f(xy) = f(x)f(y) \quad \text{for } x, y \in S.$$

His proof also works when $\mathbb{C}$ is replaced by an arbitrary normed algebra with the multiplicative norm in place of $\mathbb{C}$.

We have here the case of so called *uniform superstability*, since the constant which bounds the bounded solution of the inequality $|f(xy) - f(x)f(y)| \le \delta$ does not depend on f.

Remark 2.2.1
The superstability of the sine equation (14) is not uniform. Indeed, the bounded function $f_n\colon \mathbb{R} \to \mathbb{R}$

$$f_n(x) = n \sin x + \frac{1}{n} \qquad \text{for } n \in \mathbb{N}$$

satisfies inequality (15) with $\delta = 3$, it is not a solution of (14) and the family of functions $\{f_n(x)\}_{n\in\mathbb{N}}$ is not commonly bounded.

By the method used in the proof of Theorem 2.1.1 we obtain also the following theorem.

Theorem 2.2.2
Let $(G, \cdot)$ be a commutative semigroup and let $(A, \cdot)$ be a groupoid equipped with

- *the multiplication $(\lambda, a) \to \lambda a\colon \mathbb{R}_+ \times A \to A$ such that*

$$\lambda(ab) = (\lambda a)b = a(\lambda b) \qquad \textit{for } a, b \in A,\ \lambda \in R_+,$$

- (H) *an element $0 \in A$ such that $\lambda 0 = 0$ for $\lambda \in \mathbb{R}_+$ and $a^2 \neq 0$ for every $a \in A$, $a \neq 0$,*

– *a metric* ρ *satisfying the condition*

$$\rho(\lambda a, \lambda b) \le \lambda\rho(a, b) \qquad \text{for } a, b \in A,\ \lambda > 0. \tag{17}$$

Moreover, assume that A *is cancellable on the left (on the right) by the element* $\neq 0$*, the groupoid operation* $\cdot$ *in* A *is continuous and that the unit sphere* $S(0;1)$ *is compact in* A*. Then for every unbounded function* $f\colon G \to A$ *such that* $\rho[f(xy), f(x)f(y)]$ *is bounded we have* $f(xy) = f(x)f(y)$*.*

Proof. Assume that f is an unbounded function such that $\rho(f(xy), f(x)f(y))$ is bounded. Then there exists a sequence $x_n \in G$ such that $\lim_{n\to\infty} \rho[f(x_n), 0] = \infty$ and $f(x_n) \neq 0$ for $n \in \mathbb{N}$. Put $\lambda_n := \{\rho[f(x_n), 0]\}^{-1}$ for $n \in \mathbb{N}$. Since

$$\rho[\lambda_n f(x_n), 0] = \rho[\lambda_n f(x_n), \lambda_n 0] \le \lambda_n \rho[f(x_n), 0] = 1 \qquad \text{for } n \in \mathbb{N}$$

we have $\lambda_n f(x_n) \in S$. Thus we can assume that $\lim_{n\to\infty} \lambda_n f(x_n) = \varepsilon \neq 0$ for some $\varepsilon \in S$. Hence $\varepsilon^2 \neq 0$, too. Assume that an unbounded function $f\colon G \to A$ satisfies $\rho[f(xy), f(x)f(y)] \le \delta$ for some positive δ and $x, y \in G$. From the inequality

$$\rho[f(x)f(y), f(y)f(x)] \le \rho[f(x)f(y), f(xy)] + \rho[f(yx), f(y)f(x)] \le 2\delta, \qquad x, y \in G$$

we have

$$\rho[f(xy), f(y)f(x)] \le \rho[f(xy), f(x)f(y)] + \rho[f(x)f(y), f(y)f(x)] \le 3\delta, \qquad x, y \in G.$$

Since $\rho[f(x_n x), f(x_n)f(x)] \le \delta$ the inequality

$$\rho[\lambda_n f(x_n x), \lambda_n f(x_n)f(x)] \le \lambda_n \rho[f(x_n x), f(x_n)f(x)] \le \delta\lambda_n$$

follows from (17). Letting $n \to \infty$ we get $\varepsilon f(x) = \lim_{n\to\infty} \lambda_n f(x_n x)$ for $x \in G$. Analogously we obtain $f(x)\varepsilon = \lim_{n\to\infty} \lambda_n f(x_n x)$ (because $\rho[f(x_n x), f(x)f(x_n)] \le 3\delta$). So, we have proved that $\varepsilon f(x) = f(x)\varepsilon$ for $x \in G$.

Multiplying by λ_n^2 the inequality

$$\begin{aligned} &\rho[f(x_n xy)f(x_n), f(x_n x)f(x_n y)] \\ &\qquad \le \rho[f(x_n xy)f(x_n), f(x_n^2 xy)] + \rho[f(x_n^2 xy), f(x_n x)f(x_n y)] \\ &\qquad \le 2\delta \end{aligned}$$

we obtain in virtue of (17), when $n \to \infty$, that $\rho[\varepsilon f(xy)\varepsilon, \varepsilon f(x)\varepsilon f(y)] = 0$. Thus $f(xy) = f(x)f(y)$.

Remark 2.2.3

If A has the absorbing element 0, i.e., $0a = a0 = 0$ for $a \in A$, then $\lambda 0 = \lambda(0 \cdot 0) = (\lambda 0)0 = 0$. Furthermore, if $a^2 = 0$ for some $a \in A$, then $aa = 0 = a0$, so by the cancellation law, we get $a = 0$. Hence $a^2 \neq 0$ for $a \neq 0$. Thus the assumption (H) of Theorem 2.2.2 is satisfied.

If $\lambda(\mu a) = (\lambda\mu)a$ and $1a = a$ for $a \in A$ and $\lambda, \mu \in \mathbb{R}_+$, then we get by (17), as in Remark 1.2.10, the inequality $\lambda\rho(a, b) \le \rho(\lambda a, \lambda b)$. Thus (17) is equivalent to $\rho(\lambda a, \lambda b) = \lambda\rho(a, b)$.

All suppositions of Theorem 2.2.2 are satisfied if A is a finite-dimensional normed algebra without the zero divisors.

Remark 2.2.4
The assumption that A is cancellable is essential in Theorem 2.2.2. Indeed, if $G = (\mathbb{R}, \cdot)$, $\delta > 0$, $\varepsilon > 0$, $|\varepsilon - \varepsilon^2| = \delta$ and the function $f\colon G \to M$ is given by

$$f(x) = \begin{bmatrix} x & 0 \\ 0 & \varepsilon \end{bmatrix},$$

then the function f is unbounded, satisfies $|f(xy) - f(x)f(y)| \le \delta$ and it is not a solution of $f(xy) = f(x)f(y)$.

Remark 2.2.5
Explanation of the role of the compactness of $S(0;1)$, given in Remark 2.1.2 on Theorem 2.1.1, remains valid for Theorem 2.2.2.

2.3. The Lobachevski's equation

P. Găvruta proved in [8] that the Lobachevski's equation

$$g^2\Big(\frac{x+y}{2}\Big) = g(x)g(y) \tag{18}$$

for $g\colon G \to \mathbb{C}$, where G is an abelian group uniquely 2-divisible, is superstable, i.e., for every function $f\colon G \to \mathbb{C}$ such that

$$\Big|f^2\Big(\frac{x+y}{2}\Big) - f(x)f(y)\Big| \le \delta \qquad \text{for } x, y \in G,\ \delta > 0 \tag{19}$$

we have either

$$|f(x)| \le \frac{|f(0)| + \sqrt{|f(0)|^2 + 4\delta}}{2} \qquad \text{for } x \in G \tag{20}$$

or the function f is a solution of (18).

Remark 2.3.1
The constant which bounds the function f in (20) depends on f. For $g\colon \mathbb{R} \to \mathbb{C}$ and natural metric in $\mathbb{C}$ the superstable equation of Lobachevski is not uniformly superstable. Indeed, for $\delta > 0$ and $a \ge 0$ the function

$$f_a(x) = \begin{cases} \sqrt{a^2 + \delta} & \text{for } x \in \mathbb{R} \setminus \{0\}, \\ a & \text{for } x = 0 \end{cases}$$

satisfies (19), it is not a solution of the Lobachevski's equation and the family of functions $\{f_a\}_{a \ge 0}$ is not commonly bounded.

We have the following result on the superstability of equation (18).

Theorem 2.3.2
Let G be a commutative semigroup, uniquely 2-divisible and with the neutral element 0 and let A be a finite-dimensional commutative normed algebra without the zero divisors. Then all unbounded function $f\colon G \to A$ satisfying (19) is a solution of (18).

Proof. This is a modification of the proof of Theorem 2 in [8].

Assume that $f\colon G \to A$ is an unbounded function satisfying (19). If $f(0) = 0$, then $|f^2(x)| \le \delta$. If f is unbounded, then there exists a sequence x_n of element of G such that $|f(x_n)| \to \infty$ and $\mu_n := f(x_n) \neq 0$. Since the sequence $|\mu_n|^{-1}\mu_n$ is bounded and the dimension of A is finite, then we can assume that $\lim_{n\to\infty} |\mu_n|^{-1}\mu_n = \varepsilon$ for some $\varepsilon \in A$, $\varepsilon \neq 0$. We have

$$|\mu_n|^{-2}\mu_n^2 = |\mu_n|^{-1}\mu_n|\mu_n|^{-1}\mu_n \to \varepsilon^2 \qquad \text{for } n \to \infty$$

and $||\mu_n|^{-2}\mu_n^2| \le \delta|\mu_n|^{-2}$, so that $|\mu_n|^{-2}\mu_n^2 \to 0$ for $n \to \infty$. We obtain $\varepsilon^2 = 0$, thus A has a zero divisor ε – a contradiction. Therefore the function f is bounded.

Assume now that f(0) $\neq 0$ and put $F(x) := |f(0)|^{-1}f(x)$ and $\gamma := \delta|f(0)|^{-2}$. Then $|F(0)| = 1$ and

$$\Big|F^2\Big(\frac{x+y}{2}\Big) - F(x)F(y)\Big| \le \gamma \qquad \text{for } x, y \in G. \tag{21}$$

If f is unbounded, then so is F. Therefore there exists a sequence x_n of elements of G such that

$$\lim_{n\to\infty} |F(x_n)| = \infty \qquad \text{and} \qquad \Lambda_n := F(x_n) \neq 0.$$

Since A is finite-dimensional and the sequence $|\Lambda_n|^{-1}\Lambda_n$ is bounded, there exists a convergent subsequence of this sequence. Without loss of generality we may assume that the sequence $|\Lambda_n|^{-1}\Lambda_n$ is convergent to an $\varepsilon \in A$, $\varepsilon \neq 0$. By (21)

$$\Big||\Lambda_n|^{-1}\Lambda_n F(x) - |\Lambda_n|^{-1}F^2\Big(\frac{x+x_n}{2}\Big)\Big| \le |\Lambda_n|^{-1}\gamma$$

whence

$$\varepsilon F(x) = \lim_{n\to\infty} |\Lambda_n|^{-1}F^2\Big(\frac{x+x_n}{2}\Big) \qquad \text{for } x \in G. \tag{22}$$

There exists a function $K\colon G \to \mathbb{R}$ such that

$$\Big||\Lambda_n|^{-1}F^2\Big(\frac{x+x_n}{2}\Big)\Big| \le K(x) \qquad \text{for } n \in \mathbb{N},\ x \in G.$$

Because of

$$\Big|F\Big(\frac{x+x_n}{2}\Big)F\Big(\frac{y+x_n}{2}\Big) - F^2\Big(\frac{x+y+2x_n}{4}\Big)\Big| \le \gamma$$

we have

$$\begin{aligned}
\Big|F^2\Big(\frac{x+x_n}{2}\Big)&F^2\Big(\frac{y+x_n}{2}\Big) - F^4\Big(\frac{x+y+2x_n}{4}\Big)\Big| \\
&\le \Big|F\Big(\frac{x+x_n}{2}\Big)F\Big(\frac{y+x_n}{2}\Big) - F^2\Big(\frac{x+y+2x_n}{4}\Big)\Big| \\
&\quad \times \Big|F\Big(\frac{x+x_n}{2}\Big)F\Big(\frac{y+x_n}{2}\Big) + F^2\Big(\frac{x+y+2x_n}{4}\Big)\Big| \\
&\le \gamma\Big[\gamma + 2K\Big(\frac{x+y}{2}\Big)|\Lambda_n|\Big].
\end{aligned}$$

Dividing this inequality by $|\Lambda_n|^2$ and letting $n \to \infty$, in view of (22), we obtain that the function $\varepsilon^2 F$ is a solution of (18). Since A has no zero divisors, F and so $f = |f(0)|F$ satisfy (18).

Remark 2.3.3
The assumption that A has no zero divisors is essential in Theorem 2.3.2. For, if $G = (\mathbb{R}, +)$, $\delta > 0$ and the function $f\colon G \to M$ is given by

$$f(x) = \begin{bmatrix} e^x & 0 \\ 0 & h(x) \end{bmatrix},$$

where $h(0) = 0$, $h(x) = \sqrt{\delta}$ for $x \neq 0$, then this function is unbounded and satisfies (19) but it is not a solution of (18).

Remark 2.3.4
Theorem 2.3.2 and its proof bring no information on the value of the constant which may bound the bounded solution of (19). The superstability in this case we propose to call *undetermined superstability.* When $A = \mathbb{C}$, the constant in (20) is given by P. Găvruta in [8].

It is possible to put $K(f) = \sup_{x\in G} |f(x)|$ for the function f in the class C of bounded solutions of the inequality occurring in the definition of superstability. If $L := \sup_{f\in C} K(f) < \infty$, then we have uniform superstability, otherwise we have only the superstability. Note that $L = \infty$ for the sine equation (see Remark 2.2.1) and for the Lobachevski's equation (see Remark 2.3.1).

2.4. The cosine equation

For the cosine D'Alembert equation

$$g(x + y) + g(x - y) = 2g(x)g(y) \tag{23}$$

for g from an abelian group G to $\mathbb{C}$, we have $L < \infty$, thus the uniform superstability. Namely, J.A. Baker proved in [2] (and P. Găvruta in [8]) that for any function $f\colon G \to \mathbb{C}$ satisfying

$$|f(x + y) + f(x - y) - 2f(x)f(y)| \leq \delta \qquad \text{for } x, y \in G,\ \delta > 0 \tag{24}$$

we either

$$|f(x)| \leq \frac{1 + \sqrt{1 + 2\delta}}{2} \qquad \text{for } x \in G$$

or f is a solution of (23).

We supply the following result, together with the proof which is a modification of that of Theorem 1 from [8].

Theorem 2.4.1
Let G be an abelian group and let A be a finite-dimensional normed algebra with the unity e *and without the zero divisors. Then any unbounded function $f\colon G \to A$ satisfying* (24) *is a solution of* (23).

Proof. If the function f is unbounded, then there exists a sequence $x_n \in G$ such that $|f(x_n)| \to \infty$ and $\mu_n := f(x_n) \neq 0$. The sequence $|\mu_n|^{-1}\mu_n$ is bounded and A is finite-dimensional, thus we can assume that $|\mu_n|^{-1}\mu_n \to \varepsilon$ for some $\varepsilon \in A$. We have $|\varepsilon| = 1$, thus $\varepsilon \neq 0$. It results from (24) that

$$|2\mu_n f(x) - f(x_n + x) - f(x_n - x)| \leq \delta \qquad \text{for } n \in \mathbb{N},$$

what implies

$$\varepsilon f(x) = \lim_{n\to\infty} \frac{1}{2}|\mu_n|^{-1}[f(x_n + x) + f(x_n - x)]. \tag{25}$$

Putting $y = 0$, $x = x_n$ in (24) we have $|2\mu_n - 2\mu_n f(0)| \leq \delta$, thus $|\mu_n|^{-1}|2\mu_n - 2\mu_n f(0)| \leq \delta|\mu_n|^{-1}$. On letting $n \to \infty$ one sees that $\varepsilon[\mathrm{e} - f(0)] = \varepsilon - \varepsilon f(0) = 0$. Therefore $f(0) = \mathrm{e}$ because A has no zero divisors. Putting $x = 0$ in (24) we obtain

$$|f(-y) - f(y)| = |f(y) + f(-y) - 2f(0)f(y)| \leq \delta$$

whence, for $y = x_n - x$

$$|\mu_n|^{-1}|f(x - x_n) - f(x_n - x)| \leq \delta|\mu_n|^{-1}$$

and therefore

$$\lim_{n\to\infty} |\mu_n|^{-1}|f(x - x_n) - f(x_n - x)| = 0.$$

We have as above

$$f(x)\varepsilon = \lim_{n\to\infty} \frac{1}{2}|\mu_n|^{-1}[f(x_n + x) + f(x - x_n)]$$

and, in virtue of (25),

$$\varepsilon f(x) = \lim_{n\to\infty} \frac{1}{2}|\mu_n|^{-1}[f(x_n + x) + f(x_n - x) + f(x - x_n) - f(x_n - x)] = f(x)\varepsilon$$

for $x \in G$. Putting

$$\begin{aligned} A_n &= [f(x + x_n) + f(x - x_n)][f(y + x_n) + f(y - x_n)], \\ B_n &= [f(x + y + x_n) + f(x + y - x_n) + f(x - y + x_n) + f(x - y - x_n)]f(x_n) \end{aligned}$$

we calculate the limits

$$\begin{aligned} 2\varepsilon^2 f(x)f(y) &= \lim_{n\to\infty} \frac{1}{2}|\mu_n|^{-2}A_n \\ \varepsilon^2[f(x + y) + f(x - y)] &= \lim_{n\to\infty} \frac{1}{2}|\mu_n|^{-2}B_n. \end{aligned} \tag{26}$$

From (24) we have the series of inequalities

$$\begin{aligned} &|2f(x + x_n)f(y + x_n) - f(x + y + 2x_n) - f(x - y)| \leq \delta, \\ &|2f(x - x_n)f(y + x_n) - f(x - y - 2x_n) - f(x + y)| \leq \delta, \\ &|2f(x + x_n)f(y - x_n) - f(x - y + 2x_n) - f(x + y)| \leq \delta, \\ &|2f(x - x_n)f(y - x_n) - f(x + y - 2x_n) - f(x - y)| \leq \delta, \\ &|2f(x + y + x_n)f(x_n) - f(x + y + 2x_n) - f(x + y)| \leq \delta, \\ &|2f(x + y - x_n)f(x_n) - f(x + y - 2x_n) - f(x + y)| \leq \delta, \\ &|2f(x - y + x_n)f(x_n) - f(x - y + 2x_n) - f(x - y)| \leq \delta, \\ &|2f(x - y - x_n)f(x_n) - f(x - y - 2x_n) - f(x - y)| \leq \delta \end{aligned}$$

which yield $|A_n - B_n| \le 4\delta$. Thus we obtain the inequality $\frac{1}{2}|\mu_n|^{-2}|A_n - B_n| \le |\mu_n|^{-2}2\delta$. The conditions (25) imply that $\varepsilon^2[f(x+y)+f(x-y)-2f(x)f(y)] = 0$. Since ε is not a zero divisor we have $\varepsilon^2 \neq 0$, thus f is a solution of (23).

Remark 2.4.2
Comments regarding Theorem 2.3.2 made in Remark 2.3.4 apply to Theorem 2.4.1 as well.

Remark 2.4.3
The assumption in Theorem 2.4.1 that A has no zero divisors is essential. Indeed, if $G = (\mathbb{C}, +)$, $\delta > 0$ and the function $f\colon G \to M$ is given by

$$f(x) = \begin{bmatrix} \dfrac{e^x + e^{-x}}{2} & 0 \\ 0 & \varepsilon \end{bmatrix} \qquad \text{for } x \in \mathbb{C},$$

where $|\varepsilon^2 - \varepsilon| = \delta$, then the function f is unbounded and satisfies (23) but it is not a solution of (23).

Remark 2.4.4
An inspection of the proofs of our Theorems 2.1.1, 2.3.2 and 2.4.1 shows that on can replace in these theorems the assumption that A has no zero divisors by the condition:

for every $\varepsilon \in A$ and $\varepsilon \neq 0$ there is $\varepsilon^2 \neq 0$ and ε^2 is not a zero divisor.

Remark 2.4.5
The first lines of the proofs of our Theorems 2.1.1, 2.3.2 and 2.4.1 lead to the conclusion that instead of the condition "A has a finite dimension" we might accept the condition "there exist: a sequence $x_n \in G$ such that $\lim_{n\to\infty} |f(x_n)| = \infty$ and a convergent subsequence of $|f(x_n)|^{-1}f(x_n)$". However the latter condition implies, in particular, that if $G = (\mathbb{Q}, +)$, then the dimension of A is finite. Indeed, if A were of infinite dimension, then there would exist a sequence $a_n \in A$ such that $|a_n| = 1$, having no convergent subsequence ([1], p.127–128). Let $\phi\colon \mathbb{Q} \to \mathbb{N}$ be a bijection and $f(x) := |x|a_{\phi(x)}$ for $x \in \mathbb{Q}$. Suppose that there exists a sequence $x_n \in \mathbb{Q}$ with a convergent subsequence of $|f(x_n)|^{-1}f(x_n) = a_{\phi(x_n)}$ ($|f(x)| = |x|$ for $x \in \mathbb{Q}$!) such that $\lim_{n\to\infty} |f(x_n)| = \lim_{n\to\infty} |x_n| = \infty$. Without loss of generality we may assume that sequence $a_{\phi(x_n)}$ itself is convergent and that $x_n \neq x_m$ for $n \neq m$. In this way we found a convergent subsequence of a_n, which is impossible.

Remark 2.4.6
In the following two cases the algebra A is isometrically isomorphic with $\mathbb{C}$, so that we have the situation already dealt with in the papers [2], [5] and [8]:

(a) A is a Banach algebra with unity and without topological zero divisors (see [1], p.467),

(b) A is a normed algebra of finite dimension, with unity and with multiplicative norm (since the condition (b) implies the condition (a)).

2.5. The Dhombres equation

B. Batko in [3] proved that the Dhombres equation

$$[f(x)+f(y)][f(x+y)-f(x)-f(y)]=0 \tag{27}$$

for f from an abelian group G to $\mathbb{C}$, is superstable. More precisely, if for some $\delta \geq 0$

$$|[f(x)+f(y)][f(x+y)-f(x)-f(y)]| \leq \delta \qquad \text{for } x,y \in G,$$

then f is either additive or $|f(x)| \leq \sqrt{\frac{\delta}{2}}$ for $x \in G$. It is possible to prove a more general result. To this end we need a lemma.

Lemma 2.5.1
Let A be a finite-dimensional normed algebra without the zero divisors. Then for all $a_n, b_n \in A$ the conditions: $a_n b_n \to 0$ and $a_n \to a \neq 0$ imply $b_n \to 0$.

Proof. First of all, the sequence b_n is bounded. Really, in the contrary case there exists a subsequence b_{k_n} of b_n such that $|b_{k_n}| \to \infty$ and $b_{k_n} \neq 0$. Since the sequence $|b_{k_n}|^{-1} b_{k_n}$ is bounded, a certain its subsequence, which we denote by $|b_n|^{-1} b_n$, converges to a $b \neq 0$. We have $ab = \lim_{n\to\infty} |b_n|^{-1} a_n b_n = 0$. This is impossible, because A has no zero divisors.

Since the sequence $|b_n|$ is also bounded, there exists its convergent subsequence, say $|b_{k_n}|$. If it approached a nonzero limit, then, as the sequence b_{k_n} is also bounded, there would exist a subsequence of b_{k_n} convergent to a $b \neq 0$. This leads to the same contradiction as above.

We have proved that the limit of every convergent subsequence of $|b_n|$ equals zero. Thus $|b_n| \to 0$ and in consequence $b_n \to 0$, too.

Theorem 2.5.2
Let G be a groupoid and let A be a finite-dimensional normed algebra without the zero divisors. Then the Dhombres equation for $f\colon G \to A$ is superstable.

Proof. If the function f is unbounded, then there exists a sequence $x_n \in G$ such that $|f(x_n)| \to \infty$ and $f(x_n) \neq 0$. The sequence $|f(x_n)|^{-1} f(x_n)$ is bounded and A is finite-dimensional, thus we can assume that $|f(x_n)|^{-1} f(x_n) \to \varepsilon$ for some $\varepsilon \in A$. We have $|\varepsilon| = 1$, whence $\varepsilon \neq 0$. Assume that

$$|[f(x)+f(y)][f(x+y)-f(x)-f(y)]| \leq \delta \qquad \text{for some } \delta > 0 \text{ and } x,y \in G. \tag{28}$$

Taking here $y = x_n$ and dividing the resulting inequality by $|f(x_n)|$ we obtain

$$\lim_{n\to\infty} [|f(x_n)|^{-1} f(x) + |f(x_n)|^{-1} f(x_n)][f(x+x_n)-f(x)-f(x_n)] = 0.$$

Thus by Lemma 2.5.1

$$f(x) = \lim_{n\to\infty} [f(x+x_n) - f(x_n)], \qquad f(y+x) = \lim_{n\to\infty} [f(y+x+x_n) - f(x_n)].$$

Moreover, dividing by $|f(x_n)|^2$ the inequality obtained from (28) for $y = x_n$ and passing to the limit as $n \to \infty$, we get

$$\lim_{n\to\infty} |f(x_n)|^{-1} f(x + x_n) = \varepsilon.$$

Dividing by $|f(x_n)|$ the inequality

$$|[f(y) + f(x + x_n)][f(y + x + x_n) - f(y) - f(x + x_n)]| \le \delta$$

we have $f(y) = \lim_{n\to\infty}[f(y+x+x_n) - f(x+x_n)]$. Thus $f(y+x) = f(y)+f(x)$. Since the additive function are a solution of equation (27), the proof is finished.

Remark 2.5.3
The function $f\colon \mathbb{R} \to M$, given by

$$f(x) = \begin{bmatrix} x & 0 \\ 0 & \sqrt{\frac{1}{2}\delta} \end{bmatrix}$$

shows that the supposition in Theorem 2.5.2 that A has no zero divisors is essential.

2.6. The Mikusiński's equation.

Theorem 2.6.1
Let G be a group and let A be a finite-dimensional normed algebra without the zero divisors. Then the Mikusiński's equation

$$f(x + y)[f(x + y) - f(x) - f(y)] = 0$$

for $f\colon G \to A$ is superstable.

The proof is analogous to that of Theorem 2.5.2. We use the inequality

$$|f(x + y)[f(x + y) - f(x) - f(y)]| \le \delta$$

written in the form

$$|f(u)[f(u) - f(x) - f(-x + u)]| \le \delta,$$

taking $u = x_n$ such that $|f(x_n)|^{-1} f(x_n) \to \varepsilon$ for some $\varepsilon \in A$, provided the function f is unbounded.

Remark 2.6.2
Consult Remark 2.1.3 to see that the assumption "A has no zero divisors" is essential also in Theorem 2.6.1.

Remark 2.6.3
A normed algebra with multiplicative norm has no zero divisors. The real Banach algebra with multiplicative norm is isomorphic with $\mathbb{R}$ or $\mathbb{C}$ or with the field of quaternions (see [15] p.30).

Remark 2.6.4
B. Batko in [4] has proved the superstability of Mikusiński's equation after the first redaction of this paper and by the different method.

Problem
Let f be the function from an abelian semigroup to a finite-dimensional normed algebra without the zero divisors. Is the equation

$$[f(x+y)]^2 = [f(x)+f(y)]^2 \qquad ([f(x+y)+f(x)+f(y)][f(x+y)-f(x)-f(y)] = 0)$$

superstable?

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Institute of Mathematics
Pedagogical University
Podchorążych 2
PL-30-084 Kraków
Poland
E-mail: zmoszner@up.krakow.pl

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