FOLIA 122

Annales Universitatis Paedagogicae Cracoviensis

Studia Mathematica XI (2012)

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Blowup and specialization methods for the study of linear systems

Abstract. The computation of the dimension of linear systems of curves with imposed base multiple points on surfaces is a difficult problem, with open conjectures that are being approached only with partial success. Among others, blowup-based techniques and degenerations show some promise of leading to satisfactory answers. We present an overview of such blowup-based techniques at an introductory level, with emphasis on clusters of infinitely near points and Ciliberto–Miranda's *blowup and twist*.

1. Introduction

1.1. An overwiew

Recent years have seen significant advances in the understanding of linear systems with imposed multiple points. The case of points in general position deserves special mention, with several relevant contributions to the open conjectures of Nagata–Biran–Szemberg and Segre–Harbourne–Gimigliano–Hirschowitz. Most of these rely to some extent on semicontinuity and degeneration methods, which often allow setting up induction arguments on the multiplicity or the number of points.

The formalism of blowups has become an essential tool in the study of linear systems with multiple points, especially when using degeneration methods: the geometry of the variety blown up at the imposed points is important; induction arguments often lead to consider points that are not in general position, but "infinitely near", i.e., on blowups; useful degenerations are often built by blowing up the total space of some family (sometimes trivial); etc.

These notes aim to overview the set of blowup-based tools that are being used for specializing and degenerating linear systems. We have taken a rather elementary approach, which should serve as a friendly introduction and guide to the original research articles. Sometimes full proofs are not given or the exposition restricts to particular cases for the sake of simplicity; in that case we include

AMS (2000) Subject Classification: 14C20, 14D06, (32C10, 14C05, 14E05).

references to the existing bibliography. In particular, we deal only with linear systems of curves on smooth surfaces defined over the field of complex numbers.

1.2. Two motivating conjectures

In 1959 M. Nagata, motivated by his solution to Hilbert's fourteenth problem, proposed the following conjecture.

Conjecture A.1 (Nagata, [36])
If d, m, n are positive integers with $n > 9$ and $d \leq m\sqrt{n}$, and $p_1, \dots, p_n \in \mathbb{P}^2$ are general points, there is no plane curve of degree d with multiplicity $\geq m$ at the n points.

If n is a square, Nagata proved the result to be true; as of today, there is no nonsquare n for which the result is known. Nagata's Conjecture turns out to be related to problems of sphere packings in symplectic geometry [33], and results known in the symplectic setting suggest the following generalization:

Conjecture A.2 (Nagata, Biran, Szemberg, Lazarsfeld, [31, 5.1])
Let S be a smooth projective surface and H an ample divisor on S. Suppose k is a positive integer such that there is a curve $C \in |kH|$ with $g(C) > 0$. Then, if d, m, n are positive integers with $n \geq k^2H^2$ and $d < m\sqrt{nH^2}$, and $p_1, \dots, p_n \in S$ are general points, there is no curve $D \subset S$ with $D \cdot H \leq d$ and multiplicity $\geq m$ at the n points.

Rather than just asking whether certain curves exist of a given degree and multiplicities, one is often interested in the dimension of such a system. To be precise, denote $\mathcal{L}_d(p_1^{m_1}, \dots, p_n^{m_n})$ the linear system of all plane curves of degree d with multiplicity at least m_i at n distinct points $p_1, \dots, p_n \in \mathbb{P}^2$. In principle, each point of multiplicity m imposes up to $\frac{m(m+1)}{2}$ linear conditions, which may or may not be independent, so the dimension of the linear system is at least

$$\operatorname{vdim}(d; m_1, \dots, m_n) = \max\Big\{-1, \frac{d(d+3)}{2} - \sum_{i=1}^{n} \frac{m_i(m_i+1)}{2}\Big\},$$

and one expects $\dim \mathcal{L}_d(p_1^{m_1}, \dots, p_n^{m_n}) = \operatorname{vdim}(d; m_1, \dots, m_n)$ in many cases:

Conjecture B.1 (Segre, [46])
Let $p_1, \dots, p_n \in \mathbb{P}^2$ be general points, and let $d, m_1, \dots, m_n$ be nonnegative integers. If $\dim \mathcal{L}_d(p_1^{m_1}, \dots, p_n^{m_n})$ is greater than $\operatorname{vdim}(d; m_1, \dots, m_n)$, then general members of $\mathcal{L}_d(p_1^{m_1}, \dots, p_n^{m_n})$ are nonreduced.

Of course, if general members of a linear system are nonreduced they must have a common multiple base curve; although Segre's Conjecture is still open, much more is known today about such multiple base curves and Segre's Conjecture can be equivalently formulated as follows (see [23], [28], [11]):

Conjecture B.2 (Segre–Harbourne–Gimigliano–Hirschowitz)
Let $p_1, \dots, p_n \in \mathbb{P}^2$ be general points, and let $d, m_1, \dots, m_n$ be nonnegative integers. If $\dim \mathcal{L}_d(p_1^{m_1}, \dots, p_n^{m_n})$ is greater than $\operatorname{vdim}(d; m_1, \dots, m_n)$, then there is an irreducible, reduced, rational curve $C \subset \mathbb{P}^2$ with $\sum(\operatorname{mult}_{p_i}(C))^2 = (\deg C)^2 + 1$ and $\sum(m_i \cdot \operatorname{mult}_{p_i}(C)) > d \cdot \deg C$ which is a multiple component of all curves in $\mathcal{L}_d(p_1^{m_1}, \dots, p_n^{m_n})$.

If the number of points is nine or less, then the conjecture is known to be true [22], [21], and this is in fact one of the original motivations for posing the conjecture. Blowing up the plane at the nine given points one gets a rational surface in which the opposite of the canonical divisor is effective, which allows, in fact, to compute the dimensions of the linear systems (if $r < 8$, it is possible to do the computation even for points in special positions [20]).

It is not hard to see that the Segre–Harbourne–Gimigliano–Hirschowitz Conjecture implies Nagata's Conjecture. Both conjectures turn out to be formidable challenges, and only particular cases have been solved so far. The methods reviewed in these notes have all had some relevance in the partial results that are known, and some of them show promise of leading to a general proof.

1.3. Preliminaries

We start by fixing notations and reviewing a few facts from the geometry of surfaces, which we shall assume are familiar to the reader, and will be used freely. Refer to [3], [5] or [26, V] for complete expositions and proofs.

1.3.1. Surfaces, curves and germs

A *suface* S denotes a connected two-dimensional complex analytic manifold. A curve in S is an effective Cartier divisor C, i.e., it is given by local equations $f_i = 0$ on suitable open sets U_i covering S, with f_i nonzero holomorphic functions agreeing up to units in the intersections $U_i \cap U_j$. Thus C determines (or is determined by) a line bundle $\mathcal{O}_S(C)$ and a nonzero section $f \in H^0(S, \mathcal{O}_S(C))$.

We use additive notation for curves, so if C and D are curves locally defined by equations $f = 0$, $g = 0$, then $C + D$ is the curve locally defined by $fg = 0$. C is irreducible if it can not be written as a nontrivial sum of curves $C + D$. In general, a Cartier divisor is an element in the free abelian group $\operatorname{Div} S$ generated by irreducible curves $C \subset S$. If ϕ is a nonzero rational (meromorphic) function, then the divisor defined by ϕ, $\operatorname{div}(\phi)$, is a Cartier divisor $Z - P$ (the "zeros" minus the "poles") so that, if ϕ is given locally by f/g, with f, g holomorphic, Z is given locally by $f = 0$, P by $g = 0$. Two divisors C, D are *linearly equivalent* if the difference $C - D$ is the divisor of a rational function. Similarly, a meromorphic 2-form ω defines a divisor K, called a *canonical divisor*, and two canonical divisors are always linearly equivalent.

If $\varphi\colon S' \to S$ is a holomorphic map, and C is a curve on S not containing $\varphi(S')$, then φ^*C is a curve on S' defined by lifting the local equations $f = 0$ to $f \circ \varphi = 0$. This definition extends immediately to divisors, and when φ is dominant (i.e., its image is not contained in a curve) φ^* is a group homomorphism $\operatorname{Div} S \to \operatorname{Div} S'$ respecting linear equivalence.

The Picard group of S, denoted $\operatorname{Pic} S$, is the group of isomorphism classes of line bundles on S. The definition of $\mathcal{O}_S(D)$ is extended to arbitrary divisors D by linearity; the map $D \mapsto \mathcal{O}_S(D)$ identifies $\operatorname{Pic} S$ with the group of linear equivalence classes in $\operatorname{Div} S$.

Given a point $O \in S$, the germs of functions holomorphic in a neighborhood of O describe a local ring, denoted $\mathcal{O}_O$. A germ of function $f \in \mathcal{O}_O$ determines a germ of curve $C : f = 0$. $\mathcal{O}_O$ is a unique factorization domain, so every germ f decomposes as a product of irreducible germs (uniquely up to an invertible factor). The germs of curve determined by the irreducible factors of f are called *branches* of the germ $C : f = 0$.

By fixing local coordinates x, y near O (so that O has coordinates $(0,0)$) one gets an isomorphism $\mathcal{O}_O \cong \mathbb{C}\{x,y\}$ with the ring of convergent power series; (x,y) is then the maximal ideal. The *multiplicity* of a germ of curve $C : f = 0$ at O, denoted $\operatorname{mult}_O C$, is the minimal order of a term in the power series f, or equivalently, the maximal n such that $f \in (x,y)^n$. It is independent of the choice of coordinates.

Theorem 1.3.1 (Newton–Puiseux, [6, Chapter 1])
Fix a point $O \in S$ and local coordinates x, y in a neighborhood of O. Let $f \in \mathcal{O}_O$ be an irreducible germ, and let $C : f = 0$ be the (unibranch) curve it determines. Then:

1. *There is a minimal (germ of) analytic parametrization of C of the form*
$$(x,y) = \eta(t) := (t^n, s(t)), \quad s \in \mathbb{C}\{t\}.$$
I.e. there are a positive integer n and a convergent power series s such that $f(t^n, s(t)) \equiv 0$ and every other parametrization η' factors uniquely through η.

2. *If C is not tangent to the y-axis, then n is the multiplicity of C at O, and s has order at least n. Otherwise, the order of s is the multiplicity of C and n is strictly greater.*

3. *n is uniquely determined, and s is uniquely determined up to conjugation $t \mapsto \zeta^k t$ in $\mathbb{C}\{t\}$, where $\zeta \in \mathbb{C}$ is a primitive n-th root of unity.*

1.3.2. Intersection numbers and linear systems

Among other uses, the Newton–Puiseux Theorem serves to define (and compute) intersection multiplicities of curves. Namely, if C, D are germs of curves at O, with C irreducible, η is a minimal parametrization of C as given by Theorem 1.3.1, and D is given by $g = 0$, then the intersection multiplicity of C and D at O is given by $[C,D]_O = \operatorname{ord}_t(g(\eta(t)))$. This definition can be extended by additivity to the case that C is reducible; it is symmetric and bilinear, and semicontinuous in linear families (see [48, IV.5], [6, 2.5]). It coincides with $\dim_{\mathbb{C}} \mathcal{O}/(f,g)$, where f and g are equations of the curves, [18, 3.3], [6, 3.11.10].

If C, D are curves on a surface S without common components, with C compact, then they meet in a finite set, and the sum of intersection multiplicities
$$C \cdot D := \sum_{O \in S} [C,D]_O \tag{1}$$

is finite. In particular, this is the case for every pair of curves without common components on a projective surface.

The set of all curves linearly equivalent to a given $C \subset S$ is denoted $|C|$, and when S is projective one has $|C| = \mathbb{P}(H^0(S, \mathcal{O}_S(C)))$, because two sections of $\mathcal{O}_S(C)$ determine the same curve if and only if they are scalar multiples of each other.

Two curves $C, D \subset S$ are *algebraically equivalent* if there exist a third curve $E \subset S$, a connected curve (not necessarily irreducible) T and a flat family of curves on S parametrized by T (i.e., a 2-dimensional complex subspace $X \subset T \times S$, flat over T) such that $C + E$ and $D + E$ are two fibers of $X \to T$. They are *homologically equivalent* if their classes in $H_2(S, \mathbb{Z})$ are the same. They are *numerically equivalent* if, for every other curve E on S, $C \cdot E = D \cdot E$. Since all curves linearly equivalent to C are parametrized by a projective space $|C|$, linear equivalence implies algebraic equivalence. On the other hand, algebraic equivalence implies homological equivalence and homological equivalence implies numerical equivalence [3], [19]. The group of divisors on S modulo linear equivalence is identified with the Picard group $\mathrm{Pic}(S)$; the group of divisors modulo algebraic equivalence is called the Néron–Severi group $\mathrm{NS}(S)$.

Theorem 1.3.2
On a projective surface S, the intersection number defines a symmetric bilinear form $\mathrm{Pic}\, S \times \mathrm{Pic}\, S \to \mathbb{Z}$*, given by* $(\mathcal{O}_S(C), \mathcal{O}_S(D)) \mapsto C \cdot D$ *on curves.*

Proof. See [5, I.4].

Theorem 1.3.2 allows to define intersection numbers $C \cdot D$ for arbitrary divisors, including $C^2 = C \cdot C$. It is worth noting that given two divisors C_1, C_2, there are always curves A_1, A_2, B_1, B_2, pairwise without common components, such that C_i is linearly equivalent to $A_i - B_i$. Hence the intersection number $C \cdot D$ can always be computed by means of (1), in principle.

A *linear system* is defined as the family of curves determined by a linear subspace $\mathcal{L} \subset |C|$ for some C. $|C|$ is called a *complete linear system.* The *base locus* of a linear system is the intersection of all curves in it; it is a Zariski closed subset of S, so it decomposes as a union of irreducible curves (*fixed components* of the system) and isolated points (*base points*). The components of the base locus may also appear with multiplicities in the system.

For two curves C, D, if $C \cdot D < 0$, then by (1) they share at least an irreducible component, and if C is irreducible, then it is a fixed component of the complete linear system $|D|$. A divisor D is said to be *nef* if $C \cdot D \geq 0$ for all curves C.

The dimension of complete linear systems (equivalently, the dimension of H^0 groups) can not be determined from their numerical properties (intersection numbers) but on the other hand, if some vanishing theorem helps (for instance if $H^2 = 0$, as happens for effective divisors on rational surfaces), then the dimension of complete linear systems can at least be *bounded* using their Euler characteristics, which are determined by intersection numbers:

Theorem 1.3.3 (Riemann–Roch)
Let S be a projective surface. For every divisor D on X,

$$\chi(\mathcal{O}_S(D)) = \chi(\mathcal{O}_S) + \frac{D^2 - D\cdot K}{2},$$

where K is a canonical divisor.

Given a linear system $\mathcal{L} \subset |D|$ of finite dimension n (this is automatic if S is projective) and a point p not in the base locus of $\mathcal{L}$, the set of all divisors in $\mathcal{L}$ going through p is again a linear system $\mathcal{L} - p \subset \mathcal{L}$ of dimension $n-1$. Thus $\mathcal{L}$ determines a *rational map* $\varphi_{\mathcal{L}}\colon S \dashrightarrow \mathcal{L}^\vee \cong \mathbb{P}^n$ mapping each p to $\mathcal{L} - p$. See [5, II.6] for details and for the use of blowups to extend $\varphi_{\mathcal{L}}$ to the whole of S. The divisor D is called *very ample* if there exists a linear system $\mathcal{L} \subset |D|$ such that $\varphi_{\mathcal{L}}$ is an immersion, and *ample* if there exist a positive integer n and a linear system $\mathcal{L} \subset |nD|$ such that $\varphi_{\mathcal{L}}$ is an immersion.

It is quite remarkable that ampleness can be detected numerically:

Theorem 1.3.4 (Nakai–Moishezon criterion)
Let S be a projective surface. A divisor D on X is ample if and only if $D^2 > 0$ and $D \cdot C > 0$ for all irreducible curves C in X.

A similar result holds in higher dimensions, see [30], [31, 1.2.23]. The proof of the Nakai–Moishezon criterion relies on the Riemann–Roch Theorem and, more precisely, on one of its consequences: the fact that if D, H are divisors with H ample and $D^2 > 0$, $D \cdot H > 0$, then a multiple nD of D is linearly equivalent to an effective divisor. A less significant (but often useful) consequence is that a nef divisor always has nonnegative self intersection.

It is sometimes useful to consider "$\mathbb{Q}$-divisors", i.e., elements in $\mathbb{Q}\otimes \operatorname{Div} S$. The intersection form extends to $\mathbb{Q}\otimes \operatorname{Div} S$, and thus the notion of nef $\mathbb{Q}$-divisor makes sense; similarly, a $\mathbb{Q}$-divisor is called ample if it has an integer multiple which is very ample (or simply ample). See [31] for a complete exposition.

2. Complete ideals and unloading

The first two sections deal with *local* properties of complete ideals; the main reference is Casas-Alvero [6]. Subsequent sections use complete ideals and their properties in global settings.

2.1. Infinitely near points

Let S be a surface, and $O \in S$ a point. The blowup of S at O can be defined as follows (see [6, 3.1], [5, II.1], [26, I.4]). Let U be a neighborhood of O, where x, y are well defined analytic coordinate functions (and O has coordinates $(0,0)$). Fix projective coordinates $(u : v)$ in a projective line $\mathbb{P}^1$, and consider the subvariety $\tilde{U}$ of $U \times \mathbb{P}^1$ given by $xv - yu = 0$. It is a (smooth, connected) surface projecting onto U, and the restriction of the projection $\pi\colon \tilde{U} \to U$ to $\tilde{U} \setminus \pi^{-1}(O)$ is an isomorphism onto $U \setminus \{O\}$. Thus $\tilde{U}$ can be glued with the rest of S giving a new surface $\tilde{S}$ which is isomorphic to S except that O has been replaced by a curve $E \cong \mathbb{P}^1$ whose points correspond to the tangent directions at O. E is called the *exceptional divisor* or *exceptional curve* of the blowup. This construction (modulo isomorphism) does not depend on the coordinates chosen.

If $C : f = 0$ is a germ of curve with multiplicity n at O, then its pullback $\pi^* C \subset \tilde{S}$ consists of the exceptional curve E, with multiplicity n, and the *strict* transform $\tilde{C}$ of C. $\tilde{C}$ intersects E in at most n points; if f_n denotes the form of order n in the power series expansion of f, the intersection points of $\tilde{C}$ and E (with multiplicities!) are the zeroes (in $E \cong \mathbb{P}^1$) of $f_n(u, v)$. This set of points, with their multiplicities, is called the *tangent cone* of C at O. It is independent on the choice of coordinates as well.

Note that each branch of curve C at O has a unique tangent, the corresponding point of E appearing in the tangent cone with multiplicity equal to the multiplicity that the branch has at O.

Points on the exceptional curve $E = \pi^{-1}(O)$ of the blowup $\pi \colon \tilde{S} \to S$ of a point O are said to belong to the *first infinitesimal neigborhood* of the point O. Inductively, a point p belongs to the *k-th infinitesimal neigborhood* of O if it belongs to the first infinitesimal neighborhood of a point in the $(k-1)$-th infinitesimal neigborhood of O. Note that in this case, the point in the $(k-1)$-th neigborhood is uniquely determined; it is called the *immediate predecessor* of p. A point *infinitely near to O* is a point in some infinitesimal neigborhood of O.

Thus, a point p infinitely near O is a point on a surface S' with a birational morphism, composition of blowups, $\pi \colon S' \to S$ such that $\pi(p) = O$. It is not restrictive to assume that the restriction of π to $\pi^{-1}(S \setminus \{O\})$ is an isomorphism. If C is a curve through O, the strict transform of C at a point p infinitely near to O is defined inductively as the strict transform of the strict transform of C at the predecessor point of p. Set theoretically, it is also the closure of $\pi^{-1}(C \cap (S \setminus \{O\}))$.

If C is a curve through O and p is a point infinitely near to O, the multiplicity of C at p is defined to be the multiplicity at p of the strict transform of C, and we say p belongs to C if it belongs to its strict transform.

For every point p (possibly infinitely near), denote E_p its first infinitesimal neigborhood. Points infinitely near to p that belong to E_p (or to its strict transform after suitable blowups, see the preceding paragraph) are called points *proximate* to p. Sometimes $q \succ p$ is written to mean that q is proximate to p.

Each infinitely near point p is proximate to its immediate predecessor q. If it is not proximate to any other point, then it is called a *free* point. If it is proximate to some other point q', then it is the intersection point of the two corresponding exceptional components: $p = \tilde{E}_{q'} \cap E_q$, and thus it is proximate to exactly two points (it is easy to see that the blowup process never produces three exceptional components E_q, $\tilde{E}_{q'}$, $\tilde{E}_{q''}$ meeting at a point). Such points are called *satellite*.

A curve C is *singular* at an infinitely near point p if either it has multiplicity at least 2 at p, or it goes through a satellite point equal or infinitely near to p.

Theorem 2.1.1 (Noether's formula, [6, 3.3.1])
Let C and D be curves defined in a neighborhood of O. C and D have no common branch through O if and only if C and D share finitely many points infinitely near

to O. In such a case $[C,D]_O = \sum(\mathrm{mult}_p C)(\mathrm{mult}_p D)$, the summation running over all points p equal or infinitely near to O.

Proof. If C and D have a common branch through O, then they share infinitely many points infinitely near to O. So assume they have no common branch. In that case the intersection multiplicity is finite, and the finiteness of the set of common points will follow from the equality to be proved.

Both sides of the equality are additive in C, so it is not restrictive to assume C is irreducible. Let P be the point of C in the first neighborhood of O. Denote $\tilde{C}$, $\tilde{D}$ the strict transforms of C and D after blowing up O. We shall prove that $[C,D]_O = (\mathrm{mult}_O C)(\mathrm{mult}_O D) + [\tilde{C},\tilde{D}]_P$, and the result then follows by induction.

Assume that coordinates (x,y) have been chosen so that C is tangent to the x-axis. Let $\eta(t) = (t^n, s(t))$ be the minimal parametrization of C given by the Newton–Puiseux Theorem, and note that $n = \mathrm{mult}_O C$. C being tangent to the x-axis implies that $s(t) = a_m t^m + a_{m+1}t^{m+1} + \ldots$ with $m > n$.

Let $f = 0$ be an equation of D, and denote $e = \mathrm{mult}_O D$. We use x and $z = y/x$ as local coordinates in a neighborhood of P. The minimal parametrization of $\tilde{C}$ can be given as $\tilde{\eta}(t) = (x(t),(y/x)(t)) = (t^n, a_m t^{m-n} + a_{m+1}t^{m-n+1} + \ldots)$, and $\tilde{f} = x^{-e} f(x,xz)$ is an equation of $\tilde{D}$. The claimed equality now follows from the definition of the intersection multiplicity.

2.2. Weighted clusters

The goal of this section is to define the *complete ideals* that are associated to (multiple) infinitely near points. It is clear that the set of local equations of curves with multiplicity at least m at the point $O \in S$ is exactly the ideal $(x,y)^m \subset \mathcal{O}_O$, i.e., the m-th power of the maximal ideal at O. On the other hand, examples show that the naive definition for infinitely near points does not work:

Example 2.2.1
Let p be a point in the first neighborhood of O, and take coordinates so that p lies in the direction of $y = 0$. Then the set local equations of curves with multiplicity at least 1 at O and p, which is exactly the set of local equations of curves through O with $y = 0$ in the tangent cone, contains y and $y + x^2$ but does not contain x^2 (i.e., it is not an ideal!).

A *cluster* based at O is a finite set of points K infinitely near to O such that, for every $p \in K$, if q is a point such that p is infinitely near to q, then $q \in K$. (In forthcoming sections where a global setting is needed we drop the assumption that all points be infinitely near to a fixed point O). A *weighted cluster* is a cluster K with a map $m\colon K \to \mathbb{Z}$ (we usually denote $m_p \in \mathbb{Z}$ for the image of the point $p \in K$, and call it the multiplicity of p in the weighted cluster).

It is a cornerstone of the singularity theory of curves that for every singular (reduced) germ of curve C at O, the set of singular points of C equal and infinitely near to O form a cluster $K = \mathrm{Sing}\, C$, so that if $\pi_K\colon S_K \to S$ is the composition of the blowups of all points of K, then $\tilde{C}$ is nonsingular in S_K and $\pi_K^* C$ has only nonsingular components intersecting transversely. Moreover, two germs C, C' are topologically equivalent if and only if there is a bijection between their clusters of singular points, preserving multiplicities and proximities (in other words, if the Enriques diagrams – see Section 3 – of the weighted clusters $(\operatorname{Sing} C, \operatorname{mult} C)$, $(\operatorname{Sing} C', \operatorname{mult} C')$ coincide) [6].

Now, given a multiplicity m_p, and a germ of curve C with multiplicity at least m_p at some point p (equal or infinitely near to O), the *virtual transform* of C on the blowup of p with respect to the multiplicity m_p is defined as $\widehat{C} = \pi^* C - m_p E$. It is a curve (an *effective* divisor) and it coincides with the strict transform exactly when $\operatorname{mult}_p C = m_p$.

This allows to inductively define what it means for a curve to *go through* a weighted cluster (K, m). Let $p_1, \dots, p_r$ be the points of K in the first neighborhood of O. $K \setminus \{O\}$ is the disjoint union of $K_1, \dots, K_r$, where K_i is the cluster based at p_i which consists of the points in K equal or infinitely near to p_i. Then we say that a germ of curve C goes through (K, m) if C has multiplicity at least m_O at O and for $i = 1, \dots, r$, (the germ at p_i of) the virtual transform of C goes through (K_i, m). The set of all local equations of curves going through (K, m) will be denoted $\mathcal{H}_{K,m}$.

Proposition 2.2.2 ([6, 4.1.1])
For every weighted cluster (K, m), $\mathcal{H}_{K,m} \subset \mathcal{O}_O$ *is an ideal.*

Exercise 2.2.3
For every weighted cluster (K, m), the ideal $\mathcal{H}_{K,m} \subset \mathcal{O}_O$ is either the whole ring $\mathcal{O}_O$ or (x, y)-primary. If all multiplicities m_p are non-negative and at least one is positive, then the ideal $\mathcal{H}_{K,m}$ is (x, y)-primary.

Given a weighted cluster (K, m), let $\pi\colon S_K \to S$ be the composition of the blowups of all points of K, and denote E_p the total transform on S_K of the exceptional divisor above p. Using induction, it is not difficult to see that a germ of curve C goes through (K, m) if and only if the virtual transform divisor $\pi^* C - \sum m_p E_p$ is effective on S_K. Thus, the ideal associated to the cluster can be described as a direct image: $\mathcal{H}_{K,m} = \pi_*(\mathcal{O}_{S_K}(-\sum m_p E_p))$. An interesting consequence of this description is that $\mathcal{H}_{K,m}$ can be alternatively defined using valuations of the local ring $\mathcal{O}$, i.e., that it is a *complete*, or *integrally closed* ideal (see [50, Appendix 4]).

Exercise 2.2.4 (Noether's formula for clusters)
Let (K, m) be a weighted cluster, and $C : f = 0$ a germ of curve. For every germ $D : g = 0$ going through (K, m), the intersection multiplicity $[C, D]_O$ is at least $\sum m_p(\operatorname{mult}_p(C))$, the summation running over all points $p \in K$.

2.3. Local Bertini Theorem

Definition 2.3.1
A weighted cluster (K, m) is called *consistent* if all weights m_p are non-negative and, for every $p \in K$, the *proximity inequality* at p

$$m_p \geq \sum_{q \in K, q \succ p} m_q$$

is satisfied.

Theorem 2.3.2 ([6, 4.2.2])
Let (K, m) be a given weighted cluster. If there is a germ of curve C going through (K, m) with multiplicities equal to the weights, then (K, m) is consistent. Conversely, assume a finite set T of points infinitely near to O and not in K is fixed; if (K, m) is consistent, then there is a germ of curve going through it with multiplicities equal to the weights at all points of K and missing all points in T.

Proof. The first part of the statement is clear from the fact that the sum of the multiplicities of a germ of curve at all points (on the germ) proximate to p is exactly the intersection multiplicity of the strict transform of the germ with the exceptional divisor of blowing up p, and this is exactly the multiplicity of the germ at p.

For the converse statement, one uses induction on the number of points of K. If $K = \{O\}$ the statement is obvious, so assume K has more points than just O. Blow up O, and let as before $p_1, \ldots, p_r$ be the points of K in the first neighborhood of O, and K_i, $i = 1, \ldots, r$ the cluster which consists of the points in K equal or infinitely near to p_i. Define also

$$T_i = (T \cap K_i) \cup \{\text{first point proximate to } O \text{ inf. near } p_i \text{ not in } K\}.$$

By induction, there is a germ of curve $\tilde{C}_i$ at p_i going through $(K_i, m|_{K_i})$ and missing all points in T_i. Using Noether's formula, this in particular implies

$$[\tilde{C}_i, E]_{p_i} = m_i \stackrel{\text{def}}{=} \sum_{p \in K_i, p \succ O} m_p,$$

and thus $E \not\subset \tilde{C}_i$. Therefore there is a germ of curve C_i at O of multiplicity m_i whose strict transform at p_i is $\tilde{C}_i$. Pick moreover $t = m_O - \sum_{i=1}^{r} m_i \geq 0$ smooth branches $C'_1, \ldots, C'_t$ through O and missing all points $p_1, \ldots, p_r$ and all points in T (observe that $t \geq 0$ because of the proximity inequality at O). It is clear that $C = C_1 + \ldots + C_r + C'_1 + \ldots + C'_t$ goes through (K, m) with multiplicities equal to the weights.

Definition 2.3.3
A germ of curve C is said to go *sharply* through a weigted cluster (K, m) if it has multiplicities equal to the weights and no singular point outside of K.

Corollary 2.3.4 (local Bertini Theorem)
If (K, m) is consistent, then general curves through (K, m) (i.e., defined by an equation general in $\mathcal{H}_{K,m}$) go sharply through (K, m).

This form of Bertini's Theorem is not found in most modern texts, even though it has been known for a long time, see Zariski's remark in [49, Chapter 2]. The proof follows easily from 2.3.2; the interested reader will find details in [6, 4.2.7].

2.4. Unloading

If a weighted cluster (K, m) does not satisfy the proximity inequality at a point $p \in K$, that is, $m_p < \sum_{q \in K, q \succ p} m_q$, then for every germ of curve C through

(K,m), its virtual transform $\widehat{C}$ in the blowup of p contains E_p as a component (the intersection multiplicities $[\widehat{C},E_p]_q$ at points proximate to p add up to more than m by Noether's formula 2.2.4, which means the virtual transform of C at p has multiplicity strictly bigger than m_p). Consider the weights m' obtained from m by the change $m'_p = m_p + 1$. At each point q proximate to p, the virtual transform of every germ of curve through (K,m), with respect to the new weights m', has multiplicity at least $m_q - 1$ (because the sum of this virtual transform plus E_p is the virtual transform with respect to the weights m, which has multiplicity at least m_q). So consider m'' obtained from m by the change $m''_q = m_q - 1$ at all points q proximate to p. Then $\mathcal{H}_{K,m} = \mathcal{H}_{K,m''}$, and we have *unloaded* a unit of multiplicity from the points proximate to p onto p itself.

Example 2.4.1
Given a point p in the first neighborhood of O, what is the set I of all germs whose total transform at p has multiplicity at least $a \geq 0$? Since the total transform coincides with the virtual transform with respect to the multiplicity 0, $I = \mathcal{H}_{K,m}$, where $m_O = 0$, $m_p = a \geq 1$. By unloading a unit of multiplicity, $\mathcal{H}_{K,m} = \mathcal{H}_{K,m''}$, where $m''_O = 1$, $m''_p = a - 1$. If $a > 2$, this is still not consistent, and another unit can be unloaded. Write $a = 2k + r$, with $r \in \{0,1\}$. The reader may check, after unloading k units of multiplicity, that $I = \mathcal{H}_{K,m} = \mathcal{H}_{K,n}$, where $n_O = k + r$, $n_p = k$.

Exercise 2.4.2
For each weighted cluster (K,m), denote $D_{K,m}$ the divisor $-\sum_{p\in K} m_p E_p$ on S_K, and let $\tilde{E}_p$ denote the *strict* transform of E_p in S_K. Show that:

1. The proximity inequality $m_p \geq \sum_{q\in K, q\succ p} m_q$ is equivalent to $D_{K,m}\cdot \tilde{E}_p \geq 0$.
2. Assume $D_{K,m}\cdot \tilde{E}_p < 0$ and let m'' be the weights defined above. Then $D_{K,m''} = D_{K,m} - \tilde{E}_p$.

Proposition 2.4.3
Given a non-consistent weighted cluster (K,m), a finite number of unloading steps lead to a consistent weighted cluster (K,n) such that $\mathcal{H}_{K,m} = \mathcal{H}_{K,n}$.

Corollary 2.4.4
General members of $\mathcal{H}_{K,m}$ have multiplicity n_p at each $p \in K$.

Proof of 2.4.3. Let $C : f = 0$ be a germ of curve through (K,m). It exists because of 2.2.3. In S_K, we have

$$\pi_K^* C = \tilde{C} + \sum_{p\in K} v_p \tilde{E}_p$$

for some nonnegative integers v_p. Let $D_{K,m} = -\sum_{p\in K} m_p E_p = \sum_{p\in K} r_p \tilde{E}_p$. By 2.4.2, each unloading step consists in adding 1 to a coefficient r_p; so for every weighted cluster (K,n) obtained from (K,m) by unloading, one has $D_{K,n} = -\sum_{p\in K} n_p E_p = -\sum_{p\in K} s_p \tilde{E}_p$ with $s_p \geq r_p$ for all p. On the other hand, since

$\mathcal{H}_{K,m} = \mathcal{H}_{K,n}$, the germ C goes through (K,n), which means that $\pi_K^* C + D_{K,n}$ is effective; thus $s_p \leq v_p$. Therefore each coefficient s_p is bounded, there is a finite number of possible weights n and one of them must be consistent.

3. Enriques diagrams

3.1. Number of conditions

PROPOSITION 3.1.1

Let (K,m) be a consistent weighted cluster. Then

$$\dim_{\mathbb{C}} \frac{\mathcal{O}_O}{\mathcal{H}_{K,m}} = \sum_{p \in K} \frac{m_p(m_p+1)}{2}.$$

This formula is sometimes attributed to Hoskin and Deligne, although it was already known to Enriques [15]. Our proof is taken from Casas-Alvero [6].

Proof. We argue by induction on the number $c = \sum_{p\in K} \frac{m_p(m_p+1)}{2}$. Clearly, if $c = 0$ the statement is correct (since on a consistent cluster $c = 0$ implies $m_p = 0$ at all points).

So assume $c > 0$. Choose a point $q \in K$ with no points proximate to it with positive multiplicity, and define a new weighted cluster (Q,n) as follows:

- $Q = K \cup \{q_1, \ldots, q_{m_q-1}\}$, where the $m_q - 1$ points q_i are arbitrarily chosen free points on the first neighborhood of q,
- $n_p = m_p$ for all $p \in K$, $p \neq q$,
- $n_q = m_q - 1$,
- $n_{q_i} = 1, i = 1, \ldots, m_p - 1$.

Clearly, (Q,n) is consistent, and since $\sum_{p\in Q} \frac{n_p(n_p+1)}{2} = c - 1$, by induction we get $\dim_{\mathbb{C}} \frac{\mathcal{O}_O}{\mathcal{H}_{Q,n}} = c - 1$.

Now let (Q',n') be obtained from (Q,n) by adding a further simple point q_{m_q} on the first neigborhood of q. It is clear that $\mathcal{H}_{Q',n'} \subset \mathcal{H}_{Q,n}$ and, since only a point of multiplicity one was added, $\dim_{\mathbb{C}} \frac{\mathcal{O}_O}{\mathcal{H}_{Q',n'}} \leq c$. On the other hand, by 2.3.2, $\mathcal{H}_{Q',n'} \neq \mathcal{H}_{Q,n}$, so it follows that $\dim_{\mathbb{C}} \frac{\mathcal{O}_O}{\mathcal{H}_{Q',n'}} = c$. But (Q',n') is not consistent; after unloading, we see that $\mathcal{H}_{Q',n'} = \mathcal{H}_{K,m}$.

3.2. The Enriques diagram of a cluster

A convenient way to graphically represent clusters is in the form of directed trees; points of the cluster are represented by vertices, and an edge is drawn to each point from its immediate predecessor. Enriques introduced a convention for encoding proximities between points, and their free/satellite nature, which is most easily described by giving the *rules* to draw the *Enriques diagram* of any cluster:

- If q is in the first neighborhood of p, and free, draw a curved edge from p to q, whose tangent at p coincides with the preceding one (unless $p = O$ in which case there is no preceding edge).
- Suppose there is an edge from p to q (which can be curved or straight). Then all points proximate to p in successive neighborhoods of q are joined with edges on a half-line orthogonal (at q) to the edge pq).

In order to avoid selfintersections, the half-lines in a succesion of satellite points are chosen alternatively to the right and to the left.

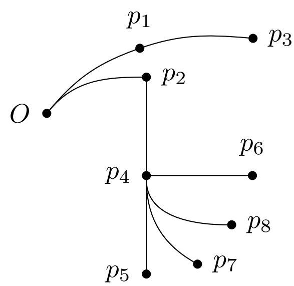


Figure 1. Enriques diagram of a nine-point cluster.

Example 3.2.1
Figure 1 depicts a cluster $K = (O, p_1, \ldots, p_8)$ in which p_4, p_5 are satellites proximate to the origin O (in addition to each beigh proximate to its immediate predecessor), p_6 is a satellite proximate to p_2 and p_4, and the remaining vertices are free. It will be helpful, for this first diagram, to include pictures corresponding to the surfaces on which lie the points of K. This is done in Figures 2–4, where the notation S_Q denotes the surface obtained by blowing up at the cluster Q.

It is apparent here that the Enriques diagram provides a much more compact representation of the cluster.

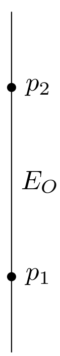


Figure 2. $S_{\{O\}}$

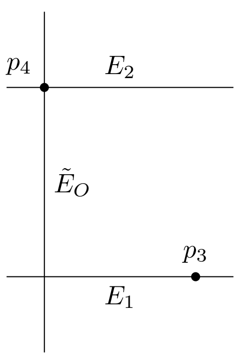


Figure 3. $S_{\{O,p_1,p_2\}}$

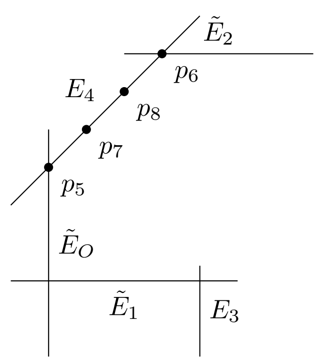


Figure 4. $S_{\{O,p_1,p_2,p_3,p_4\}}$

Exercise 3.2.2
Draw a schematic picture of the surface S_K obtaied after blowing up all points of K in the preceding example. Denoting as customary E_O (resp. E_i) the pullback to S_K of the exceptional divisor above O (resp. p_i), write them as linear combinations of the irreducible divisors $\tilde{E}_O$, $\tilde{E}_i$. Check that in the example $\tilde{E}_i = E_i - \sum_{p_j \succ p_i} E_j$, and prove that this equality holds in general.

3.3. Operations with complete ideals

Given two clusters K and Q, the union $K \cup Q$ is obviously a cluster. Given two weighted clusters (K, m) and (Q, n), their sum is defined to be $(K, m) + (Q, n) = (K \cup Q, m + n)$, where m_q is taken to be zero for $q \in Q \setminus K$ and symmetrically, n_p is taken to be zero for $p \in K \setminus Q$. Note that if (K, m) and (Q, n) are consistent, then so is $(K \cup Q, m + n)$.

Proposition 3.3.1
Given two consistent weighted clusters (K, m) and (Q, n), the product $\mathcal{H}_{K,m} \cdot \mathcal{H}_{Q,n}$ coincides with the complete ideal $\mathcal{H}_{K \cup Q, m+n}$ of the sum of the clusters.

Proof. The inclusion $\mathcal{H}_{K,m} \cdot \mathcal{H}_{Q,n} \subset \mathcal{H}_{K \cup Q, m+n}$ is clear, so let us prove the converse. It is not restrictive – and it simplifies notations – to assume $K = Q = K \cup Q$. We argue by induction, as the case $K = \{O\}$ is clear. So let, as before, $p_1, \ldots, p_r$ be the points of K in the first neighborhood of O, and K_i the cluster based at p_i which consists of the points in K equal or infinitely near to p_i.

It will be enough to show that a general member h of $\mathcal{H}_{K,m+n}$ can be written as a product fg, where $f \in \mathcal{H}_{K,m}$ and $g \in \mathcal{H}_{K,n}$. Since $(K, m + n)$ is a consistent cluster, by Bertini, a general member h of $\mathcal{H}_{K,m+n}$ has multiplicity exactly $m + n$ at O, its virtual transform h_i at each p_i coincides with its strict transform, and the germ $\tilde{C}_i : h_i = 0$ has no point on the exceptional divisor E outside of K. By the induction hypothesis, $h_i = f_i g_i$, where $f_i \in \mathcal{H}_{K,m}$ and $g_i \in \mathcal{H}_{K,n}$. Since $\tilde{C}_i$ does not contain E, and $\tilde{C}_i = \tilde{F}_i + \tilde{G}_i$ with $\tilde{F}_i : f_i = 0$ and $\tilde{G}_i : g_i = 0$, it follows that $\tilde{F}_i$ is the strict transform of a germ $F_i : f_i^\circ = 0$ of curve at O, and $\tilde{G}_i$ is the strict transform of a germ $G_i : g_i^\circ = 0$ of curve at O. Then it is not hard to check that $f = \prod f_i^\circ \in \mathcal{H}_{K,m}$, $g = \prod g_i^\circ \in \mathcal{H}_{K,n}$, and h differs from fg by a unit.

Definition 3.3.2
A complete ideal is called *irreducible* if it can not be written as the product of two nontrivial complete ideals.

Definition 3.3.3
The *excess multiplicity* (or *excess*) at a given point p of a weighted cluster (K, m) is $\rho_p = m_p - \sum_{q \in K, q \succ p} m_q$. So the cluster is consistent if and only if the excess is nonnegative at every point. Note that multiplicities determine and are in turn determined by excesses in every weigthed cluster.

Theorem 3.3.4 (Zariski)
Let (K, m) be a consistent weighted cluster.

1. *$\mathcal{H}_{K,m}$ is an irreducible complete ideal if and only if (K, m) has exactly one point p with positive excess, and $\rho_p = 1$.*

2. *Every complete ideal decomposes uniquely as a product of irreducible complete ideals.*

Proof. Since every complete ideal can be defined by an essentially unique consistent weighted cluster, we restrict in this proof to considering consistent clusters only.

Note that the excess multiplicity at a point p of the weighted cluster $\mathcal{H}_{K\cup Q,m+n}$ is the sum of the excesses at that point of $\mathcal{H}_{K,m}$ and $\mathcal{H}_{Q,n}$. The irreducibility of a weighted cluster with only one point with positive excess, which is equal to 1, follows immediately from this remark.

On the other hand, for each point p in a cluster K, there is a unique set of weights $m^{(p)} = \{m_q^{(p)}\}_{q\in K}$ such that the excesses $\rho_q^{(p)} = m_q^{(p)} - \sum_{q'\in K, q'\succ q} m_{q'}^{(p)}$ are $\rho_q^{(p)} = \delta_{pq}$ (Kronecker's delta); namely the multiplicities are determined by $m_q^{(p)} = \rho_q^{(p)} + \sum_{q'\in K, q'\succ q} m_{q'}^{(p)}$. The weighted cluster $(K, m^{(p)})$ is consistent by definition, and every consistent weighted cluster can be written uniquely as a sum of such clusters: $(K, m) = \sum_{p\in K} \rho_p(K, m^{(p)})$. By the previous proposition it follows that every complete ideal decomposes uniquely as a product of complete ideals with only one point with positive excess equal to 1, which we already proved are irreducible. It also follows that this decomposition is trivial exactly when (K, m) has only one point with positive excess equal to 1.

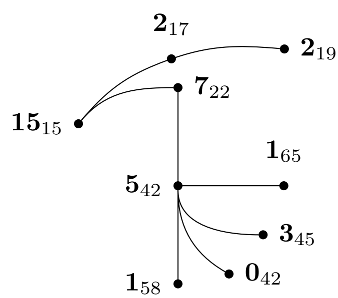


Figure 5. Weighted Enriques diagram of Exercises 3.3.5 and 3.3.9. Each point p is labeled twice, its multiplicity m_p in boldface, its value $v(m)_p$ as a subscript.

Exercise 3.3.5
Let (K, m) be the cluster of Example 3.2.1, weighted with the multiplicities indicated in Figure 5. Compute (the Enriques diagrams of) the Zariski decomposition of $\mathcal{H}_{K,m}$.

Quotient ideals $(I : J)$ of complete ideals can also be dealt with by cluster computations:

Proposition 3.3.6
Given two consistent weighted clusters (K, m) and (Q, n), the quotient $(\mathcal{H}_{K,m} : \mathcal{H}_{Q,n})$ coincides with the complete ideal $\mathcal{H}_{K\cup Q,m+(-n)}$. Given a consistent weighted cluster (K, m) and a curve germ $C : f = 0$, the quotient $(\mathcal{H}_{K,m} : f)$ coincides with the complete ideal $\mathcal{H}_{K\cup Q,m-\operatorname{mult} C}$, where we take $(m - \operatorname{mult} f)_p = m_p - \operatorname{mult}_p(C)$.

Note that even though (K, m) and (Q, n) are assumed to be consistent, $(Q, -n)$ and hence $\mathcal{H}_{K\cup Q,m+(-n)}$ will in general not be consistent. Thus to know the effective behavior of elements in the quotient it will be necessary to perform an unloading computation. The proof of 3.3.6 is similar to that of 3.3.1.

Intersections of complete ideals are also complete, but their description is better handled by expressing the divisor $D_{K,m}$ in terms of irreducible exceptional divisors rather than multiplicities. Given a weighted cluster (K, m), its associated divisor can be written as $D_{K,m} = \sum_{p\in K\cup Q} -v(m)_p \tilde{E}_p$, where $v(m)_p = m_p + \sum_{q\in K, p\succ q} v(m)_q$ is the *value* at p associated to the multiplicities m. It is not hard to compute the $v(m)_p$ from the m_p and conversely.

Definition 3.3.7
Given two weighted clusters (K, m) and (Q, n), consider their associated divisors on $S_{K\cup Q}$, written as combinations of the irreducible exceptional divisors: $D_{K,m} = \sum_{p\in K\cup Q} -v(m)_p \tilde{E}_p$ and $D_{Q,n} = \sum_{p\in K\cup Q} -v(n)_p \tilde{E}_p$. Their meet is defined to be $(K \cup Q, m \wedge n)$, where $m \wedge n$ is the unique system of multiplicities such that $v(m \wedge n)_p = \max(v(m)_p, v(n)_p)$ for all $p \in K \cup Q$.

Proposition 3.3.8
Given two weighted clusters (K, m) and (Q, n), $\mathcal{H}_{K,m} \cap \mathcal{H}_{Q,n} = \mathcal{H}_{K\cup Q,m\wedge n}$.

We leave these proofs to the interested reader. Note that $(K \cup Q, m \wedge n)$ need not be consistent even when (K, m) and (Q, n) are, and thus to know the effective behavior of members of the intersection one often needs to do an unloading computation.

Exercise 3.3.9
Check that the subscript values in Figure 5 correspond to the divisor $D_{K,m}$ associated to the cluster of Exercise 3.3.5. Compute the intersection $\mathcal{H}_{K,m} \cap \mathfrak{m}_O^{16}$ and the quotient $(\mathcal{H}_{K,m} : \mathfrak{m}_O^{16})$.

3.4. Clusters with many roots

So far we worked in the neighborhood of a point $O \in S$, but in the sequel we shall work globally, i.e., it will become relevant to consider sets of points of S and points infinitely near to points of S; with a slight abuse of language, we still call them clusters: a *cluster* on S is a finite set of points K proper or infinitely near in S, such that, for each point $p \in K$, K contains all points to which p is infinitely near. We denote S_K the surface obtained by blowing up S at all points in K, and $\pi_K\colon S_K \to S$ the composition of the blowups. The proper points of K (those which are infinitely near to no other point) are the *roots* of K. Thus a set of n distinct points is also considered to be a cluster, consisting of roots only.

The combinatorics of proximities between points of a cluster are still conveniently encoded in their Enriques diagrams, which now are finite unions of trees (also called "forests"), a tree for each root, each drawn according to the rules above. The "simplest" Enriques diagram consisting of n vertices with no edges (corresponding to clusters with n roots only) will be denoted $\mathbf{D}_0(n)$, or simply $\mathbf{D}_0$ if n is clear from the context.

A weighted cluster determines a divisor $D_{K,m}$ in S_K and a sheaf of ideals $\mathcal{H}_{K,m} = (\pi_K)_*(D_{K,m})$, supported at the roots of K, all whose stalks are complete ideals in their local rings. Consistence and unloading of the weights works locally, so all the preceding results apply to the global setting. Now however, linear equivalence of curves and divisors comes into play, and we become interested in finite dimensional linear systems of curves through clusters.

Given a divisor class H and a weighted cluster (K, m), we are interested in the linear system of all curves in $|H|$ going through (K, m). This will be denoted $\mathcal{L}_H(K, m)$, and can be identified with $\mathbb{P}(H^0(S, \mathcal{H}_{K,m}(H)))$. Via the blowup π_K, it can also be identified with $\mathbb{P}(H^0(S_K, \mathcal{O}_{S_K}(\pi_K^*(H) + D_{K,m})))$. The linear systems with which Conjecture B.2 is concerned are obtained as particular cases, when K is a cluster of n general points (in particular, with Enriques diagram $\mathbf{D}_0(n)$) in $S = \mathbb{P}^2$.

4. Varieties of clusters

4.1. Blowup of a section

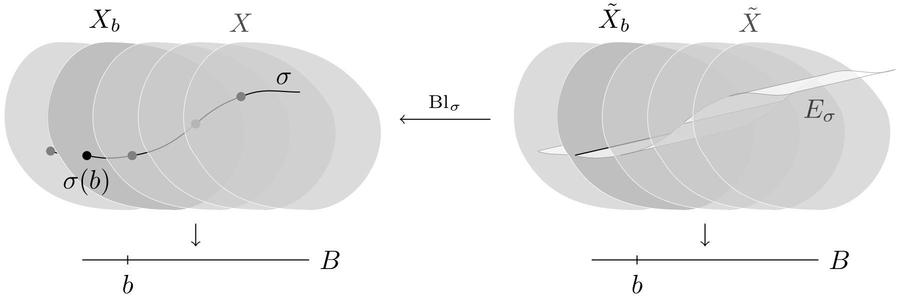
The blowup construction introduced in the first section can be generalized to higher dimensions as follows. Assume Y is a d-dimensional complex variety and $X \subset Y$ is a smooth closed subvariety of dimension r. For each $p \in X \subset Y$ take coordinates $(y_1, \ldots, y_d)$ in a neighborhood U of p such that X is defined by $y_1 = \ldots = y_{d-r} = 0$. The blowup of U along $X \cap U$ is the subset of $U \times \mathbb{P}^{d-r-1}$ defined by the equations $y_i u_j = y_j u_i$ for all $1 \le i < j \le d-r$, and the map $\mathrm{Bl}_{X\cap U}$ is the projection onto the first factor. Since Y can be covered by such open sets U, the blowup of Y along X can then be defined by glueing the local blowups. Modulo isomorphism, it is independent of all choices made. $\mathrm{Bl}_X^{-1}(X)$ is a divisor, called the *exceptional divisor* as before, which is a $\mathbb{P}^{d-r-1}$-bundle over X, i.e., it is locally a product $\mathrm{Bl}_{X\cap U}^{-1}(X \cap U) \cong (X \cap U) \times \mathbb{P}^{d-r-1}$.

Further generalizations are possible, namely one can blow up an arbitrary scheme (or complex space) Y along a closed subscheme X. The result is in general a new scheme $\tilde{Y}$ with a proper morphism $\tilde{Y} \xrightarrow{\mathrm{Bl}_X} Y$ which is an isomorphism on $\mathrm{Bl}_X^{-1}(Y \setminus X)$ and such that $\mathrm{Bl}_X^{-1}(X)$ is a divisor, satisfying a universal property (see [26, II.8]). We don't need this extra generality here.

The case of interest to us comes from a family $f\colon X \to B$ of smooth projective surfaces with a section $\sigma\colon B \to X$. The blowup of X centered at the image of σ

$$\begin{tikzcd} \tilde{X} \arrow[r, "\mathrm{Bl}_\sigma"] \arrow[dr, "\tilde{f}"'] & X \arrow[d, "f"'] \\ & B \arrow[u, bend right, "\sigma"'] \end{tikzcd}$$

is naturally a new family $\tilde{f}\colon \tilde{X} \to B$, whose fiber $\tilde{X}_b$ is for each $b \in B$ the blowup of the fiber X_b of f at the point $\sigma(b) \in X_b$ of the section. The exceptional divisor E_σ of the blowup is the union of the exceptional $\mathbb{P}^1$'s at each fiber.



To see it, again take analytic local coordinates $(x, y, b_1, \dots, b_r)$ near a point of the section so that f is the projection to the b-coordinates (B is assumed to have dimension r) and σ is the zero section. Then the blowup of S along $\sigma(B)$ has equation $xv - yu = 0$ in $S \times \mathbb{P}^1$, so that fixing the values of $b_1, \dots, b_r$ we recover the definition of the blowup of the point $(0, 0)$ on a surface given in Section 2.1.

4.2. Kleiman's iterated blowups

Given a smooth projective surface S, the set of all clusters with exactly one point can be identified with the surface itself; we denote $X_1 = S$ this variety which parametrizes the 1-point clusters. Consider the second projection $p_2\colon X_1 \times X_1 \to X_1$ as a (trivial) family of surfaces, and let $\Delta\colon X_1 \to X_1 \times X_1$ be the diagonal section. The construction of Section 4.1

$$\begin{tikzcd} X_2 \arrow[r, "\mathrm{Bl}_\Delta"] \arrow[dr, "f_1"'] & X_1 \times X_1 \arrow[d, "p_2"'] \\ & X_1 \arrow[u, bend right, "\Delta"'] \end{tikzcd}$$

yields a new family of smooth projective surfaces, $f_1\colon X_2 \to X_1$ whose fiber over the point $K = \{p\}$, $p \in S$, is the surface $\tilde{S}_p$ obtained by blowing up p. (The blowup map is the restriction to $\tilde{S}_p$ of $g_1 := p_1 \circ \mathrm{Bl}_\Delta$, where $p_1\colon X_1 \times X_1$ is the projection on the first factor). Thus points in X_2 can be naturally identified with *ordered* clusters of two points. Outside of the exceptional divisor, X_2 is isomorphic to the complement of the diagonal in $S \times S$, and corresponding points parametrize pairs of distinct points in S. Points on the exceptional divisor parametrize clusters $K = \{q_1, q_2\}$, where q_2 lies on the first neigborhood of q_1.

Kleiman introduced in [29] varieties which naturally parametrize ordered n-point clusters, for each n. The construction (which can be used in greater generality) works iteratively, as follows. Assume the varieties X_{n-1} and X_{n-2} have been defined, with maps $f_{n-2}, g_{n-2} \colon X_{n-1} \to X_{n-2}$ such that X_{n-2} parametrizes ordered clusters of $n-2$ points of S, and the fiber $S_K = f_{n-2}(K)$ over $K \in X_{n-2}$ is the surface obtained by blowing up all points in K, the blowup map being the restriction of $g_1 \circ \ldots \circ g_{n-2}$ to S_K. Thus X_{n-1} parametrizes ordered clusters of $n-1$ points, by identifying a cluster $K' \in X_{n-1}$ with its last point; this point lies on the surface obtained by blowing up the first $n-2$ points of K', which constitute the cluster $K = f_{n-2}(K') \in X_{n-2}$. Consider the fibered product

$$X_{n-1} \times_{X_{n-2}} X_{n-1} = \{(K_1, K_2) \in X_{n-1} \times X_{n-1} \mid f_{n-2}(K_1) = f_{n-2}(K_2)\},$$

which is a projective smooth variety, and the projection p_2 onto the second factor, as a family of surfaces. Again, the construction of Section 4.1 applied to the diagonal section Δ_{n-1}, yields a new family $f_{n-1} \colon X_n \to X_{n-1}$ and X_n parametrizes ordered clusters of n points, the blowup map being the restriction of $g_1 \circ \ldots \circ g_{n-1}$, where $g_{n-1} := p_1 \circ \mathrm{Bl}_{\Delta_{n-1}}$.

We call X_n the variety of the n-point clusters and, for each $K \in X_n$ we identify the surface S_K obtained by blowing up all points of K with $f_{n+1}^{-1}(K) \subset X_{n+1}$.

It is important to note that the exceptional divisors of the iterated blowups in this construction are *relative divisors*. We denote by $\mathcal{E}_i \subset X_{i+1}$ the exceptional divisor of the i-th blowup and, by abuse of notation, also its pullback by $g_{i+1} \circ \ldots \circ g_{n+1}$ to X_{n+1}. Then, for each ordered cluster of n points $K = \{p_1, \ldots, p_n\} \in X_n$, the (total transform, or pull-back, of) the exceptional divisor E_i above p_i is precisely $\mathcal{E}_i|_{S_K}$. Thus, it is possible to apply semicontinuity to families of clusters, and in particular, any cluster of n points can be considered as a specialization of a set of n distinct points:

Theorem 4.2.1
For each divisor class H on S, given multiplicities $m_1, \ldots, m_n$, the function $K \mapsto \dim \mathcal{L}_H(K, m)$ is upper-semicontinuous for the Zariski topology of X_n (where $K = \{p_1, \ldots, p_n\} \in X_n$ are ordered n-point clusters, all weighted with the same multiplicities $m_{p_i} = m_i$).

Proof. Apply the semicontinuity theorem [26, III.12.8] to $\mathcal{O}_{X_{n+1}}((g_1 \circ \ldots \circ g_n)^* H - \sum m_i \mathcal{E}_i)$, and note that $\dim H^0(S, \mathcal{H}_{K,m}(H)) = \dim H^0(S_K, \mathcal{O}_{S_K}(\pi_K^*(H) - \sum m_i E_i))$.

4.3. Proximity strata

Fix now an Enriques diagram $\mathbf{D}$ of n points, with an ordering. How to describe the set $\mathrm{Cl}(\mathbf{D}) = \{K \in X_n \mid K \text{ has diagram } \mathbf{D}\}$? The description of X_2 given above shows the way to go: the exceptional divisors $\mathcal{E}_i$ contain clusters with special diagrams.

Consider the proximity matrix $P_{\mathbf{D}} = (m_{ij})_{i,j=1,\dots,n}$, where

$$m_{ij} = \begin{cases} 1 & \text{if } i = j, \\ -1 & \text{if } p_i \text{ is proximate to } p_j, \\ 0 & \text{otherwise.} \end{cases}$$

This is an invertible matrix, and in fact, given a cluster K, on S_K one has

$$\begin{pmatrix} \tilde{E}_1 \\ \vdots \\ \tilde{E}_n \end{pmatrix} = P_{\mathbf{D}}^T \begin{pmatrix} E_1 \\ \vdots \\ E_n \end{pmatrix} \iff K \text{ has Enriques diagram } \mathbf{D}.$$

Based on this observation, we define *virtual exceptional divisors* $E_i^{\mathbf{D}}$ on any S_K by the equality

$$\begin{pmatrix} E_1^{\mathbf{D}} \\ \vdots \\ E_n^{\mathbf{D}} \end{pmatrix} = P_{\mathbf{D}}^T \begin{pmatrix} E_1 \\ \vdots \\ E_n \end{pmatrix}$$

(so that they coincide with the irreducible exceptional divisors if and only if K has diagram $\mathbf{D}$). Now we can describe $\mathrm{Cl}(\mathbf{D})$ as a locally closed subset of X_n. For this we consider

$$\begin{aligned} \mathrm{Eff}(\mathbf{D}) &:= \{K \in X_n \mid E_i^{\mathbf{D}} \text{ effective for all } i\} \\ &= \{K \in X_n \mid P_{\mathbf{D}} P_K^{-1} \geq 0\} \end{aligned}$$

which is Zariski-closed in X_n by semicontinuity applied to $\mathcal{O}_{X_{r+1}}(\mathcal{E}_i^{\mathbf{D}})$.

Proposition 4.3.1
Given an ordered Enriques diagram $\mathbf{D}$ of n points, let Σ be the set of all diagrams $\mathbf{D}'$ such that $P_{\mathbf{D}} P_{\mathbf{D}'}^{-1} \geq 0$.

1. $\mathrm{Cl}(\mathbf{D}) = \mathrm{Eff}(\mathbf{D}) \setminus \bigcup_{\mathbf{D}' \in \Sigma} \mathrm{Eff}(\mathbf{D}')$ (*in particular it is locally closed in X_n for the Zariski topology*).

2. $\mathrm{Cl}(\mathbf{D})$ *is irreducible and its dimension equals twice the number of roots* (*proper points*) *plus the number of free infinitely near points in the diagram.*

The proof of the first claim is essentially contained in the previous discussion. The second is proved by induction on the number of points, giving a construction of $\mathrm{Cl}(\mathbf{D})$ as an iterated blowup, in [44].

4.4. Applications

Assume now S is irreducible, and let H be a divisor class on S. Since $\mathrm{Cl}(\mathbf{D})$ is irreducible, it makes sense, given multiplicities $m_1, \dots, m_n$, to ask about

$$h^0(H, \mathbf{D}, m) = h^0(S_K, \mathcal{O}_{S_K}(H - D_{K,m})), \quad \text{where } K \text{ is general in } \mathrm{Cl}(\mathbf{D}),$$

and $\dim \mathcal{L}_H(\mathbf{D}, m) = h^0(H, \mathbf{D}, m) - 1$, which is the dimension of the linear system of curves in $|H|$ going through a general cluster in $\mathrm{Cl}(\mathbf{D})$ weighted with multiplicities m. In this context, the following conjecture (which is a generalization of Conjecture B.2) was posed by Greuel, Lossen and Shustin:

CONJECTURE 4.4.1
Let $S = \mathbb{P}^2$ and $H = \mathcal{O}_{\mathbb{P}^2}(d)$. If the multiplicities $m_1, \ldots, m_n$ are consistent for some diagram $\mathbf{D}$ of n vertices, and $d \geq m_i + m_j + m_k$ for every set of distinct indices $i, j, k \in \{1, \ldots, n\}$, then

$$h^0(H, \mathbf{D}, m) = \max\left\{0, \binom{d+2}{2} - \sum \binom{m_i+1}{2}\right\}.$$

DEFINITION 4.4.2
Given two Enriques diagrams with the same number of vertices r, we say that $\mathbf{D}$ specializes to $\mathbf{D}'$ and write $\mathbf{D} \rightsquigarrow \mathbf{D}'$ if $\overline{\mathrm{Cl}(\mathbf{D})} \supset \mathrm{Cl}(\mathbf{D}')$.

By using semicontinuity again, it is clear that $\mathbf{D} \rightsquigarrow \mathbf{D}'$ implies $h^0(H, \mathbf{D}, m) \leq h^0(H, \mathbf{D}', m)$. This is of interest even in order to obtain upper bounds for the dimension of linear systems of curves with points in general position, as the diagram consisting of n points in general position specializes to every other diagram of n points.

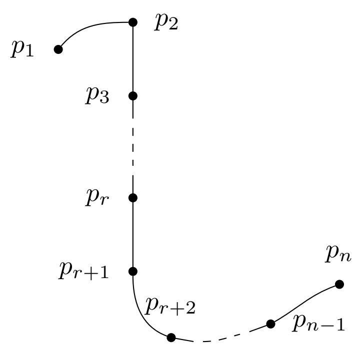


Figure 6. The enriques diagram $\mathbf{D}_r$ of Example 4.4.3.

EXAMPLE 4.4.3
Consider the Enriques diagram $\mathbf{D}_r$ of Figure 6. It has each point infinitely near to the previous one, and r points $(p_2, \ldots, p_{r+1})$ proximate to the root p_1; p_2 and the points after p_{r+1} are free. It is not hard to see that $\mathbf{D}_r \rightsquigarrow \mathbf{D}_{r+1}$: each cluster K with diagram $\mathbf{D}_{r+1}$ is the limit for $t \mapsto 0$ of a family of clusters K_t with diagram $\mathbf{D}_r$, whose $(r+2)$-th point (lying on the exceptional divisor E_{r+1}) varies with t, so that for $t = 0$ it becomes the satellite point $q_{r+2} = E_{r+1} \cap \tilde{E}_1$. A complete ad-hoc proof can be found in [43].

As an example of the usefulness of these diagrams, let us mention the main result of [42], whose proof relies on unloading computations on them. Other applications can be found in [43], [41], [40], etc.

Theorem 4.4.4
Let (K, m) be a consistent weighted cluster of n points in the plane, and let $\mathbf{D}$ be the diagram of K. Assume all points have multiplicity 2 and K is general in $\mathrm{Cl}(\mathbf{D})$. Then $\dim \mathcal{L}_d(K, m) = \max\{\frac{d(d+3)}{2} - 3n, -1\}$ for all degrees $d \geq 1$.

There is no criterion known that allows to decide, given two arbitrary diagrams $\mathbf{D}$ and $\mathbf{D}'$, whether $\mathbf{D} \rightsquigarrow \mathbf{D}'$ or not. It follows from Proposition 4.3.1 that a necessary condition is $P_{\mathbf{D}} P_{\mathbf{D}'}^{-1} \geq 0$, and this condition is sufficient if and only if

$$\forall\, \mathbf{D}' \text{ s.t. } P_{\mathbf{D}} P_{\mathbf{D}'}^{-1} \geq 0,\ \dim \mathrm{Cl}(\mathbf{D}) > \dim \mathrm{Cl}(\mathbf{D}'). \tag{2}$$

Diagrams $\mathbf{D}$ for which (2) is satisfied have $\mathrm{Eff}(\mathbf{D})$ irreducible, and are called *prime*. At present there is no known geometric criterion characterizing prime Enriques diagrams, although (2) can be checked for any given diagram (because there are only finitely many Enriques diagrams with n points). A general exposition of these matters, including proofs that a large class of diagrams are prime, can be found in [44]. In particular, the diagrams $\mathbf{D}_r$ of Example 4.4.3 are prime, and thus $\mathbf{D}_r \rightsquigarrow \mathbf{D}_{r+1}$ can also be proved by means of a matrix computation. For non-prime diagrams, we have the following conjecture:

Conjecture 4.4.5
Let $\mathbf{D}$, $\mathbf{D}'$ be diagrams with $P_{\mathbf{D}} P_{\mathbf{D}'}^{-1}$. Then $\overline{\mathrm{Cl}(\mathbf{D})} \cap \mathrm{Cl}\, \mathbf{D}' \neq \emptyset$.

Other interesting irreducible subsets of X_n have been exploited in the literature in order to bound dimensions of linear systems of points in general position. The interested reader can find details in [25], [34].

5. Degeneration to the normal cone

In the two last sections we continue to assume that S is a *projective* surface, and we are interested in linear systems of curves on S going through given weighted clusters, with special interest in the case of general points, motivated by Conjectures A.1 and B.2. We start by reviewing the definition and main properties of the Hilbert scheme for projective surfaces; for a beautiful exposition of the subject we refer the reader to [35]. The general theory of Hilbert schemes can be found in [37] (for quasiprojective schemes) or [13] (for complex varieties).

5.1. The Hilbert scheme

Given a projective surface S there exists a scheme $\mathrm{Hilb}\, S$, called the Hilbert scheme of curves in S, and a flat family $\eta\colon \mathcal{C} \to \mathrm{Hilb}\, S$ of curves,

$$\begin{tikzcd} \mathcal{C} \arrow[r, hook] \arrow[dr, "\eta"'] & S \times \mathrm{Hilb}\, S \arrow[d, "p_2"] \\ & \mathrm{Hilb}\, S \end{tikzcd}$$

such that every flat family of curves in S uniquely factors through η. In particular, each closed fiber $\mathcal{C}_\xi \subset S \times \{\xi\} \cong S$ of η is a curve of S and every curve appears as one of these. Moreover, $\operatorname{Hilb} S$ has countably many irreducible components, each of them a projective variety, and for each divisor class D,

1. the intersection multiplicity $\mathcal{C}_\xi \cdot D$ is constant on every connected component $\mathbf{F}$ of $\operatorname{Hilb} S$ (it is independent of $\xi \in \mathbf{F}$): we denote it by $\mathbf{F} \cdot D$;

2. assuming D ample, for each $n \geq 0$, there are finitely many components $\mathbf{F}$ with $\mathbf{F} \cdot D \leq n$;

3. for every irreducible component $\mathbf{F}$ of $\operatorname{Hilb} S$, and every integer $m \geq 0$, the *m-th incidence locus* $I_m(\mathbf{F}) = \{(p, \xi) \in S \times \mathbf{F} \mid \operatorname{mult}_p(\mathcal{C}_\xi) \geq m\}$ is closed in $S \times \mathbf{F}$. Moreover, for each $\mathbf{F}$ there is an integer $m_{\mathbf{F}}$ such that the m-th incidence locus is empty for $m > m_{\mathbf{F}}$.

The existence of the Hilbert scheme and its properties are deep facts in the theory of projective varieties.

Two curves are algebraically equivalent if and only if they are parametrized by points on the same connected component $\mathbf{F}$ of the Hilbert scheme, so the first claim above is actually a restatement of the fact that algebraic equivalence implies numerical equivalence. The third is a consequence of semicontinuity of multiplicities in flat families, as proved in [32], and explains how singularities behave in families:

Corollary 5.1.1
For each weighted Enriques diagram $(\mathbf{D}, m)$, and each component $\mathbf{F}$ of the Hilbert scheme of curves on S, the weighted incidence locus

$$I_{\mathbf{D},m}(\mathbf{F}) = \{(K, \xi) \in \operatorname{Cl}(\mathbf{D}) \times \mathbf{F} \mid \mathcal{C}_\xi \textit{ goes through } (K, m)\}$$

is closed in $\operatorname{Cl}(\mathbf{D}) \times \mathbf{F}$.

Corollary 5.1.2
For each weighted Enriques diagram $(\mathbf{D}, m)$, and each component $\mathbf{F}$ of the Hilbert scheme of curves on S, the equisingular stratum

$$ES_{\mathbf{D},m}(\mathbf{F}) = \{\xi \in \mathbf{F} \mid \operatorname{Sing}(\mathcal{C}_\xi) \textit{ has Enriques diagram } (\mathbf{D}, m)\}$$

is locally closed in $\mathbf{F}$.

5.2. Seshadri constants

Definition 5.2.1 ([12])
If S is a smooth projective surface and L is an ample divisor class (or line bundle) on S, the *Seshadri constant* of L at n points $p_1, \ldots, p_n \in S$ is the real number

$$\epsilon(L; p_1, \ldots, p_n) = \inf \left\{ \frac{L \cdot C}{\sum \operatorname{mult}_{p_i} C} \right\},$$

where the infimum is taken over all curves C on S through at least one of the points.

Seshadri constants are a very active area of research; a nice exposition of our state of knowledge on this subject, together with many relevant references, can be found in [4]. The Seshadri constant depends semicontinously on the position of the points and attains its maximum for very general points (i.e., for sets of points – or clusters – in a countable intersection of Zariski-open subsets of the parameter space); we denote this maximum $\epsilon(L, n)$:

Proposition 5.2.2 ([38],[31, 5.1.11])
Given a smooth projective surface S and an ample divisor class L, for every $\varepsilon > 0$, the locus

$$\mathrm{Cl}(\mathbf{D}_0(n), L, \varepsilon) = \{K = (p_1, \dots, p_n) \in \mathrm{Cl}(\mathbf{D}_0(n)) \mid \epsilon(L; p_1, \dots, p_n) \leq \varepsilon\}$$

is Zariski-closed in $\mathrm{Cl}(\mathbf{D}_0(n))$.

Note that an n-tuple of distinct points is the same as a cluster of n points with the Enriques diagram $\mathbf{D}_0(n)$ consisting of n roots.

Proof. Given a rational number t, consider the $\mathbb{Q}$-divisor $L_t = \pi^*L - t(E_1 + \dots + E_n)$ on the blowup $\tilde{S}$ of S along $p_1, \dots, p_n$. The definition of Seshadri constants can be reformulated saying that L_t is nef if and only if $t \leq \epsilon(L; p_1, \dots, p_n)$. Since every nef divisor has non-negative selfintersection, $\epsilon(L; p_1, \dots, p_n) \leq \sqrt{L^2/n}$ for every set of n distinct points, and hence $\mathrm{Cl}(\mathbf{D}_0(n), L, \varepsilon) = \mathrm{Cl}(\mathbf{D}_0(n))$ for all $\varepsilon \geq \sqrt{L^2/n}$. Thus we assume in the rest of the proof that $\varepsilon < \sqrt{L^2/n}$.

Remark 5.2.3
The upper bound $\epsilon(L; p_1, \dots, p_n) \leq \sqrt{L^2/n}$ has a well-known generalization to higher dimensions [31, 5.1.9]. The search for lower bounds, on the other hand, turns out to be much more difficult. The techniques reviewed in this section have been motivated (at least in part) by the challenge of obtaining such lower bounds, and we will stress these applications.

For the n-tuples of points with $\epsilon(L; p_1, \dots, p_n) \leq \varepsilon < \sqrt{L^2/n}$, there are rational values of t, $\varepsilon \leq t < \sqrt{L^2/n}$, such that L_t is not nef. Since $L_t^2 > 0$, there is a multiple aL_t which is linearly equivalent on $\tilde{S}$ to an effective divisor D, with $a \in \mathbb{N}$ *independent of the position of the points.* L_t not being nef means that there exist irreducible curves C with $L \cdot C < t \sum \mathrm{mult}_{p_i} C$. The strict transform $\tilde{C}$ of such a curve C satisfies $\tilde{C} \cdot D < 0$, and hence must be a component of D. In particular, $C \cdot L \leq D \cdot \pi^*L = aL^2$.

Consider now the components $\mathbf{F}_1, \dots, \mathbf{F}_k$ with $\mathbf{F}_i \cdot L \leq aL^2$ (which we recall are finite in number), and for each i consider the (finite) set of multiplicities $M_i = \{m = (m_1, \dots, m_n) \mid 0 \leq m_j \leq m_{\mathbf{F}_i},\ \mathbf{F}_i \cdot L < \varepsilon \sum m_j\}$. Then

$$\mathrm{Cl}(\mathbf{D}_0, L, \varepsilon) = \bigcup_{i=1}^{k} \Bigg(\bigcup_{m \in M_i} p_{\mathrm{Cl}(\mathbf{D}_0)}(I_{\mathbf{D}_0, m}(\mathbf{F}_i)) \Bigg), \tag{3}$$

where $p_{\mathrm{Cl}(\mathbf{D}_0)}$ denotes in each case the projection of $\mathrm{Cl}(\mathbf{D}_0) \times \mathbf{F}_i$ onto the first factor. Since the right hand side is a finite union of closed subsets, the result follows.

Further results along these lines can be found in [47], [38], [24], [45]. It turns out that, except for (possibly) the upper bound $\sqrt{L^2/n}$, the set of Seshadri constants of L at varying n-tuples of points has no accumulation points. In addition, every Seshadri constant strictly less than $\sqrt{L^2/n}$ is obtained as $\frac{L\cdot C}{\sum \operatorname{mult}_{p_i} C}$ for an adequate curve C. In particular it is rational. It is not known at present whether irrational Seshadri constants exist; if they do, they must therefore be of the form $\sqrt{L^2/n}$, and Nagata's Conjecture predicts that this does happen.

5.3. Blowup of a vertical center

Fix a smooth projective surface S and let $Y \subset S$ be an irreducible smooth curve or a point. Take an open disk $\Delta \subset \mathbb{C}$ around $0 \in \Delta$ and blow up the trivial family $X = S \times \Delta$ along $Y \times \{0\}$.

$$
\begin{tikzcd}
\tilde{X} \arrow[r, "\mathrm{Bl}_{Y\times\{0\}}"] \arrow[dr, "f"'] & X = S \times \Delta \arrow[d, "p_2"] & Y \times \{0\} \arrow[l, hook'] \\
 & \Delta &
\end{tikzcd}
$$

Since the blowup is an isomorphism outside of its center, all fibers of the new family $f\colon \tilde{X} \to \Delta$ except the central one coincide with the corresponding fiber of the initial family X, and are thus isomorphic to S. On the other hand, the central fiber $\tilde{X}_0$ consists of two components: the exceptional divisor Z (which is now a surface in the three-dimensional variety $\tilde{X}$) and the *strict transform* $\tilde{S}$ of the original fiber $S \times \{0\}$.

If $Y \subset S$ is a curve, then the exceptional divisor is locally a product of Y and $\mathbb{P}^1$, i.e., Z is a surface ruled over Y, the lines of the ruling being the fibers of $\mathrm{Bl}_{Y\times\{0\}}|_Z\colon Z \to Y$. Moreover, since Y is already a divisor on S in this case, the blowup process does not affect S, which means the map $\mathrm{Bl}_{Y\times\{0\}}|_{\tilde{S}}\colon \tilde{S} \to S$ is an isomorphism (and one often writes S instead of $\tilde{S}$). The intersection $Z \cap S$ is a curve, which on S can be identified with the original curve Y and on Z is the zero section of the ruling.

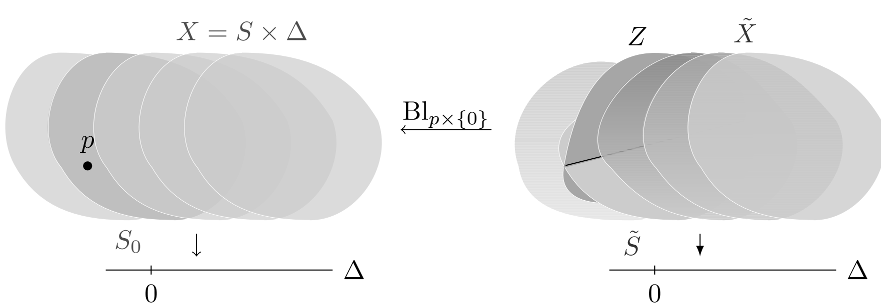


Figure 7. When blowing up a point in a family of surfaces, only the fiber containing the point ($p \in S_0 = S \times \{0\}$ in this case) is affected, as a new component $Z \cong \mathbb{P}^2$ appears.

If $Y = \{p\} \subset S$ is a point, then the exceptional divisor is $Z \cong \mathbb{P}^2$, whereas the strict transform $\tilde{S}$ is (isomorphic to) the blowup of S centered at p. The intersection $Z \cap \tilde{S}$ is again a curve; a line on $Z \cong \mathbb{P}^2$, and the exceptional curve of blowing up p on $\tilde{S}$.

It is often useful to blow up X along more than one point in the central fiber; the degeneration obtained this way has $\tilde{X}_0$ formed by as many components as blown up points plus one (the exceptional divisors and S blown up). It is also possible to modify an existing degeneration by blowing up its total space along sections, or along a curve C contained in a component of the central fiber (in which case a new central component is added – the exceptional divisor – and the central components met by C are also blown up). All these possibilities have been used to bound or compute dimensions of linear systems, and they will be illustrated here and in the last section.

5.4. Applications to Seshadri constants

The following result, which should be understood as a lower bound on the left hand side of the inequality, is an illustrating example of what can be achieved by applying semicontinuity to the families just described.

THEOREM 5.4.1 ([39])
For every projective smooth surface S with an ample divisor class L, and every partition $n = n_1 + n_2 + \ldots + n_r$, the following inequality holds:

$$\epsilon(L, n) \geq \epsilon(L, r)\epsilon(\mathcal{O}_{\mathbb{P}^2}(1), \max\{n_i\}).$$

Our proof is taken from [45], where this result was extended to consider Seshadri constants on higher dimensional varieties (Hilbert schemes and degenerations can of course be considered in the more general setting, with only some technicalities needed to deal with cycles of arbitrary codimension in the variety).

Proof. By the upper bound 5.2.3 applied to the two constants in the right hand side, if $\epsilon(L; n) = \sqrt{L^2/n}$, we are done; so assume $\varepsilon = \epsilon(L; n) < \sqrt{L^2/n}$. Since $\mathrm{Cl}(\mathbf{D}_0(n), L, \varepsilon) = \mathrm{Cl}(\mathbf{D}_0(n))$ by hypothesis, one of the closed sets on the right in the decomposition (3) must be $\mathrm{Cl}(\mathbf{D}_0(n)) = p_{\mathrm{Cl}(\mathbf{D}_0(n))}(I_{\mathbf{D}_0(n),m}(\mathbf{F}))$, (because $\mathrm{Cl}(\mathbf{D}_0(n))$ is irreducible, and so can not be the union of finitely many proper closed subsets).

Choose a set $Y = \{p_1, \ldots, p_r\}$ of r very general points of S. Take an open disk $\Delta \subset \mathbb{C}$ around $0 \in \Delta$ and blow up the trivial family $X = S \times \Delta$ along $Y \times \{0\}$. As in the one-point blowup described in the preceding section, all fibers of the family $f\colon \tilde{X} \to \Delta$ except $\tilde{X}_0$ are isomorphic to S, and $\tilde{X}_0$ consists of $r + 1$ components: the exceptional divisor is made of r components $Z = Z_1 \cup \ldots \cup Z_r$, each isomorphic to $\mathbb{P}^2$, and the strict transform $\tilde{S}$ of $S \times \{0\}$ is the blowup of S at the r general points.

Now choose n_i very general points $p_{i1}, \ldots, p_{in_i}$ of the plane Z_i for each $i = 1, \ldots, r$, and let σ_{ij} be a section of $f\colon \tilde{X} \to \Delta$ through the point p_{ij}. For $t \in \Delta$ near enough to 0, the n points $\sigma_{ij}(t)$ are distinct, so after possibly shrinking the disk we get a map $\sigma\colon \Delta^0 \to \mathrm{Cl}(\mathbf{D}_0(n))$, $\sigma(t) = (\sigma_{11}(t), \ldots, \sigma_{rn_r}(t))$, where $\Delta^0 = \Delta \setminus \{0\}$.

Form the fibered product $I_{\sigma,m}(\mathbf{F}) = \Delta^0 \times_{\mathrm{Cl}(\mathbf{D}_0(n))} I_{\mathbf{D}_0(n),m}(\mathbf{F})$, or pullback, with the incidence variety:

$$\begin{tikzcd}
I_{\sigma,m}(\mathbf{F}) \arrow[r, "\sigma_{\mathbf{F}}"] \arrow[d, "p_{\Delta}"] & I_{\mathbf{D}_0(n),m}(\mathbf{F}) \arrow[d, "p_{\mathrm{Cl}(\mathbf{D}_0(n))}"] \\
\Delta^0 \arrow[r, "\sigma"] \arrow[u, dashed, bend left, "\Xi"] & \mathrm{Cl}(\mathbf{D}_0(n))
\end{tikzcd}$$

and pick a section Ξ of p_{Δ}. Composed with the projection to $\mathbf{F}$, Ξ determines a 1-dimensional flat family of curves parametrized by $t \in \Delta^0$ such that the fiber $\mathcal{C}_t$ has multiplicity m_{ij} at $\sigma_{pij}(t)$:

$$\begin{tikzcd}
 & S \times \Delta^0 \arrow[rrr, "id_S \,\times\, (p_{\mathbf{F}} \circ \sigma_{\mathbf{F}} \circ \Xi)"] \arrow[ddl, bend left] & & & S \times \mathrm{Hilb}\, S \arrow[ddl, bend left] \\
\mathcal{C}_t \subset \mathcal{C}_{\Delta^0} \arrow[ur, hook] \arrow[rr] \arrow[d] & & \mathcal{C}_{\mathbf{F}} \arrow[r, hook] \arrow[d, "\eta_{\mathbf{F}}"] & \mathcal{C} \arrow[ur, hook] \arrow[d, "\eta"] & \\
t \in \Delta^0 \arrow[rr, "p_{\mathbf{F}} \circ \sigma_{\mathbf{F}} \circ \Xi"] & & \mathbf{F} \arrow[r, hook] & \mathrm{Hilb}\, S &
\end{tikzcd}$$

Let $\mathcal{C}_{\Delta}$ be the closure of $\mathcal{C}_{\Delta^0}$ in $\tilde{X}$. $\mathcal{C}_{\Delta} \cap \tilde{X}_0$ is called the *flat limit* of the family of curves; it is an effective divisor on the central fiber of the degeneration, consisting of an effective divisor on each of its components which agree on the intersections. Denote $\mathcal{C}_i = \mathcal{C}_{\Delta} \cap Z_i$ and $\mathcal{C}_{\tilde{S}} = \mathcal{C}_{\Delta} \cap \tilde{S}$ the different pieces of the flat limit. By semicontinuity of multiplicities [32], $\mathrm{mult}_{p_{ij}} \mathcal{C}_i \geq m_{ij}$, and by the genericity of the points on Z_i, we have a bound on the degree:

$$\mathcal{C}_i \cdot \mathcal{O}_{\mathbb{P}^2}(1) \geq \epsilon(\mathcal{O}_{\mathbb{P}^2}(1), n_i)(m_{i1} + \ldots + m_{in_i}).$$

This degree is the number of points (counted with multiplicities) in which the central fiber meets the line $Z_i \cap \tilde{S}$, which on the other hand is the exceptional divisor of blowing up p_i on S. Therefore, $\mathcal{C}_{\tilde{S}}$ is the strict transform of a curve C on S with multiplicity $\mathcal{C}_i \cdot \mathcal{O}_{\mathbb{P}^2}(1)$ at p_i, which again by genericity of the p_i, must have degree

$$C \cdot L \geq \epsilon(L, r) \sum_{i=1}^{r} \mathcal{C}_i \cdot \mathcal{O}_{\mathbb{P}^2}(1) \geq \epsilon(L, r) \sum_{i=1}^{r} \epsilon(\mathcal{O}_{\mathbb{P}^2}(1), n_i)(m_{i1} + \ldots + m_{in_i}).$$

It follows that $\mathbf{F} \cdot L = \mathcal{C}_t \cdot L = C \cdot L \geq \epsilon(L, r)\epsilon(\mathcal{O}_{\mathbb{P}^2}(1), \max\{n_i\}) \sum m_{ij}$.

To illustrate the applicability of Theorem 5.4.1, let us mention one connection with M. Nagata's Conjecture A.1, which can be reformulated as follows:

Conjecture 5.4.2 (Nagata)
For every integer $n \geq 9$, $\epsilon(\mathcal{O}_{\mathbb{P}^2}(1), n) = \sqrt{1/n}$.

Corollary 5.4.3 (of 5.4.1)
$\epsilon(\mathcal{O}_{\mathbb{P}^2}(1), nm) \geq \epsilon(\mathcal{O}_{\mathbb{P}^2}(1), n)\epsilon(\mathcal{O}_{\mathbb{P}^2}(1), m)$; therefore, if Nagata's Conjecture is true for n and for m points, then it is true for nm points.

More applications to Nagata's Conjecture and its generalization A.2 can be found in [39], [45].

5.5. Limit linear systems

Specializing points to different components of a degeneration can be used not only to study the *existence* of curves of given degree with multiple points in general position, as in the proof of 5.4.1, but also to study the *dimension* of the corresponding linear system.

Consider the situation at the end of Section 4: H is an effective, ample, divisor class on S, and $m = (m_1, \dots, m_n)$ are positive multiplicities. One may fix in addition an Enriques diagram $\mathbf{D}$, and assume that the weighted diagram $(\mathbf{D}, m)$ is consistent; in this section we consider general distinct points, so $\mathbf{D} = \mathbf{D}_0(n)$ will be the diagram consisting of n roots without proximities. It is interesting and often difficult to give upper bounds for the dimension $\dim_H(\mathbf{D}, m)$ of the linear system of curves in $|H|$ with multiplicities m_i at the points of a general cluster $K \in \mathrm{Cl}(\mathbf{D})$.

Ciliberto–Miranda in [7] established the framework for applying to this problem the degenerations of Section 5.3. The basic idea is as follows (further developments will be explained in the last section). Consider the family $\tilde{X}$ obtained by blowing up $S \times \Delta$ (Δ a disk) at a general point p of the central fiber, and let $Z \cong \mathbb{P}^2$, $\tilde{S}$, be the two components of the new central fiber. Recall that $E = Z \cap \tilde{S}$ is a line of Z and the special curve of blowing up S at p. The Picard group of this central fiber is the fibered product of $\operatorname{Pic} Z$ and $\operatorname{Pic} \tilde{S}$ over $\operatorname{Pic} E$, that is, a line bundle on the central fiber is a pair of line bundles H_Z on Z and $H_{\tilde{S}}$ on $\tilde{S}$ whose restrictions to E agree. The Picard group of $Z \cong \mathbb{P}^2$ is generated by $\mathcal{O}_{\mathbb{P}^2}(1)$, while the Picard group of $\tilde{S}$ is $\operatorname{Pic} S \oplus \mathbb{Z}E$. Hence in order that the restrictions to E agree, one must have $H_Z = \mathcal{O}_{\mathbb{P}^2}(a)$ and $H_{\tilde{S}} = H_S - aE$ for some integer a and some line bundle H_S on S.

We are interested in line bundles that are possible limits of the given H on S. Denote by $\pi\colon \tilde{X} \to S$ the composition of the blowup with the projection $S \times \Delta \to S$. Then $\tilde{H} = \pi^*(H)$ is a line bundle on $\tilde{X}$ whose restriction to general fibers is H. Its restriction to $\tilde{S}$ is also the pullback of H, whereas its restriction to Z is trivial. Consider, for each integer a, the line bundle $\tilde{H}_a = \tilde{H} - aZ$. Twisting a line bundle on $\tilde{X}$ like H by a divisor supported on a fiber does not change its restriction to general fibers. On the other hand, the restriction of $\tilde{H}_a$ to $\tilde{S}$ is $H - aE$, and the restriction of $\tilde{H}_a$ to Z is $\mathcal{O}_{\mathbb{P}^2}(a)$. So every such pair, $H - aE$, $\mathcal{O}_{\mathbb{P}^2}(a)$ is a possible limit of H for this degeneration, and it is not hard to see that these are all.

Now pick $r < n$ general points on $\tilde{S}$ plus $n - r$ general points on Z, and create n sections $\sigma_1, \dots, \sigma_n$ of the family $f\colon \tilde{X} \to \Delta$ going through the n points, as in the proof of 5.4.1. Thus for $t \neq 0$ near 0, $\mathcal{L}_H(\sigma_1(t)^{m_1}, \dots, \sigma_n(t)^{m_n})$ is a linear system of curves in $|H|$ with n general points of multiplicities m in general position. Now, for *every* a, there is a possible limit system in the central fiber, and the dimension of each of these is (by semicontinuity) an upper bound on $\dim \mathcal{L}_H(m)$. The limit linear system for a given value of a is the projectivization of the space of global sections

$$H^0(\tilde{S}, H - aE - p_1^{m_1} - \ldots - p_r^{m_r}) \bigoplus_{H^0(E,\mathcal{O}_{\mathbb{P}^1}(a))} H^0(Z, \mathcal{O}_{P^2}(a) - p_{r+1}^{m_{r+1}} - \ldots - p_n^{m_n}),$$

(i.e., pairs of global sections whose restrictions to E coincide) so its dimension essentially depends on the dimensions of two linear systems $\mathcal{L}_{(H-aE)}(m_1, \ldots, m_r)$ and $\mathcal{L}_a(m_{r+1}, \ldots, m_n)$ with less points. Using this degeneration it is then possible to argue by induction on the number of points, and to prove, for instance:

Theorem 5.5.1 ([9])
Conjecture B.2 holds for $m_1 = \ldots = m_n \leq 12$.

This result has been improved by [14] to cover multiplicity up to 42.

6. When everything fails

6.1. Three fat points come together

Blowing down Z to the point p in the degeneration just described turns the sections $\sigma_1, \ldots, \sigma_r$ into a family of r points approaching p. Thus, the computations in the Ciliberto–Miranda method explained at the end of the previous section can be understood as the computation of the *limit* of $\mathcal{L}_H(m_1, \ldots, m_n)$, when r of the points come together (this limit exists, because the Grassmannian of linear subspaces of $|H|$ of dimension $\dim \mathcal{L}_H(m_1, \ldots, m_n)$ is complete). Let us look at a concrete example in detail.

What is the limit of three colliding double points? A first attempt at this computation is to pick a double point with two double points infinitely near to it. This is a nonconsistent cluster, which after unloading is equivalent to a triple point with two simple points infinitely near to it. But the number of conditions (Section 3.1) imposed by three double points is $3 \cdot 3 = 9$ whereas the number of conditions imposed by a triple and two simple points is $6 + 2 = 8$, so in general (i.e., whenever H is positive enough) not all curves in $|H|$ going through the cluster with multiplicities $3, 1, 1$ are limits of curves with three double points.

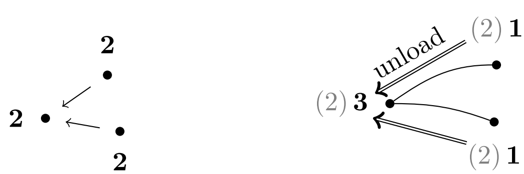


Figure 8. When two double points become infinitely near to a third, unloading gives multiplicities $3, 1, 1$.

The degeneration of the previous section does let us compute this limit. Consider the family $\tilde{X}$ obtained by blowing up $S \times \Delta$ (Δ a disk) at a general point p

of the central fiber, and let $Z \cong \mathbb{P}^2$, $\tilde{S}$, be the two components of the new central fiber. Pick three points on Z, and sections through them. The limit linear system we are looking for lives in $\tilde{S}$.

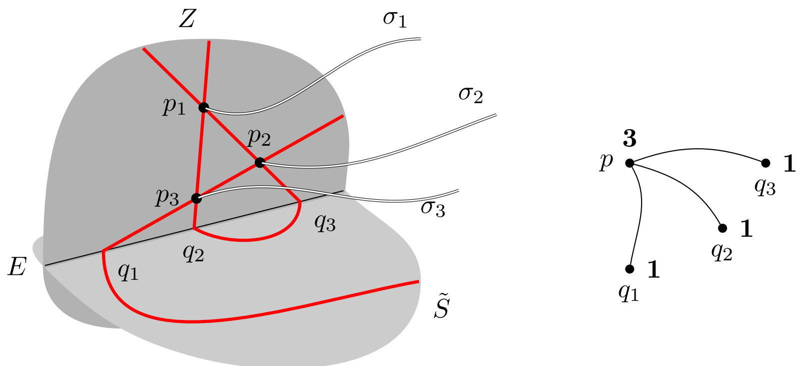


Figure 9. The central fiber of the degeneration computing the limit of three double points. $\mathcal{L}_Z = \mathcal{L}_3(p_1^2, p_2^2, p_3^2)$ is a triangle, so the three points q_1, q_2, q_3 must belong to every limit curve in $\tilde{S}$, which means the (blown down) curve on S goes through the cluster with depicted Enriques diagram.

We know that limit curves must have multiplicity at least 3; the degeneration tells us that again: since there are three double points on $Z \cong \mathbb{P}^2$, any effective divisor containing them must have degree 3 at least. So take the limit linear system whose components are $\mathcal{L}_{\tilde{S}} = \mathcal{L}_{H-3E}$ on $\tilde{S}$, $\mathcal{L}_Z = \mathcal{L}_3(p_1^2, p_2^2, p_3^2)$ on Z. The only curve of degree 3 in the plane Z with 3 double points is the triangle with vertices p_1, p_2, p_3. Since a divisor on the central fiber is made up of divisors on Z and $\tilde{S}$ which agree on E, and the three intersection points q_1, q_2, q_3, of the sides of the triangle with E belong to all (i.e., the only) effective divisors $\mathcal{L}_Z$, it follows that the limit curves in $\tilde{S}$ of multiplicity 3 also go through these three fixed points. Thus the limit of three approaching double points is a triple point with three infinitely near points of multiplicity 1 (which impose the correct number of conditions).

Exercise 6.1.1
Compute the limit of three triple points coming together using the same idea.

6.2. Everything fails if they are too fat

The method just explained is quite powerful, but it is not flexible enough to cover all cases. Let us look at a concrete example again.

What is the limit of three colliding 4-ple points? The first attempt, picking a 4-ple point with two 4-ple points infinitely near to it, gives a nonconsistent cluster, which after unloading is equivalent to a 6-ple point with two double points infinitely near to it. Again the number of conditions doesn't match: three 4-ple points impose $3 \cdot 10 = 30$ conditions, whereas the number of conditions imposed by a 6-ple and two double points is $21 + 2 \cdot 3 = 27$, so in general not all curves going through the cluster with multiplicities $6, 2, 2$ are limits of curves with three 4-ple points. We already expected that from the computation with double points.

Let us now try the degeneration, putting again the three 4-ple points on the plane Z. Any effective divisor on the plane containing them must have degree 6 at least . So take the limit linear system whose components are $\mathcal{L}_{\tilde{S}} = \mathcal{L}_{H-6E}$ on $\tilde{S}$, $\mathcal{L}_Z = \mathcal{L}_6(p_1^4, p_2^4, p_3^4)$ on Z. The only section of degree 6 in Z with 3 4-ple points is the triangle with vertices p_1, p_2, p_3, with each line counted twice. This intersects E in the same three points as before, but with multiplicity 2. Therefore, the limit curves of multiplicity 6 have to meet E at these three points, and have intersection multiplicity 2 with E there (they have to be tangent to the exceptional divisor). In terms of clusters, they must go through the three points and also through their three satellites. This imposes $21 + 3 + 3 = 27$ conditions again, so we still don't know the limit!

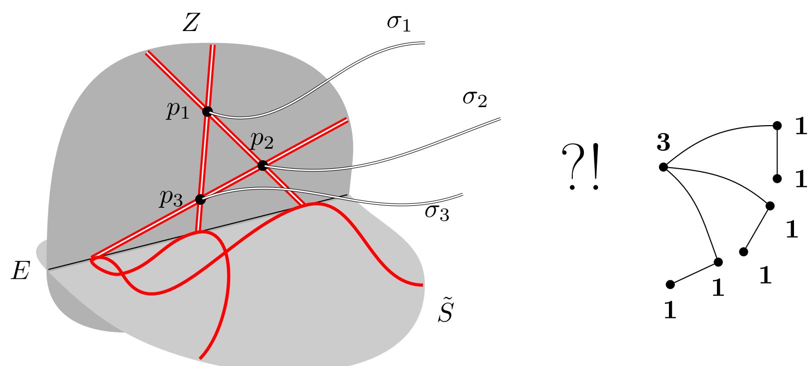


Figure 10. The same degeneration fails to compute the limit of three quadruple points: $\mathcal{L}_Z = \mathcal{L}_3(p_1^4, p_2^4, p_3^4)$ is a triangle of double lines, so the three points q_1, q_2, q_3 must be tangency points with E for every limit curve in $\tilde{S}$, which means the (blown down) curve on S goes through the cluster with depicted Enriques diagram; but this is not enough.

In fact, the linear system $\mathcal{L}_Z$ is special: its virtual dimension is -1 and nevertheless it is effective. Thus the chosen degeneration is not optimal for this problem, because the system in at least one component of the central fiber is not of the expected dimension.

This kind of problem arises in general, whenever a given degeneration is chosen and used for systems of increasing multiplicity. Although it would be possible to solve this particular limit by different means (e.g., using a different degeneration, or computing algebraically [16]), to find a degeneration for each problem is in general not feasible (usually cases with few points of high multiplicity are the most difficult).

6.3. Blow-up and twist

The lines appearing as fixed components in the linear system $\mathcal{L}_Z$ of the previous example are instances of the curves appearing in the satement of the SHGH Conjecture B.2, and actually $\mathcal{L}_Z$ has dimension larger than expected because of them. Blowing up Z at the three points turns the lines into (-1)-curves, i.e., rational curves with selfintersection -1.

The last bit of technique that has been introduced to the study of degenerations as applied to linear systems is designed to overcome this situation: a degeneration has been found which "does not work" because the linear system on one of the central components is special *due to* (-1)*-curves*. In that case, Ciliberto–Miranda [10] propose to modify the degeneration by blowing up the total space along each (-1) curve whose intersection with the linear system is negative, and modify the linear system by twisting with some appropriate multiple of the new exceptional divisor. Note that if the SHGH conjecture is true for the number of points on that component (for instance, if there are 9 or less points there, as happens in the previous example) then such (-1)-curves are the only possible source of special central systems.

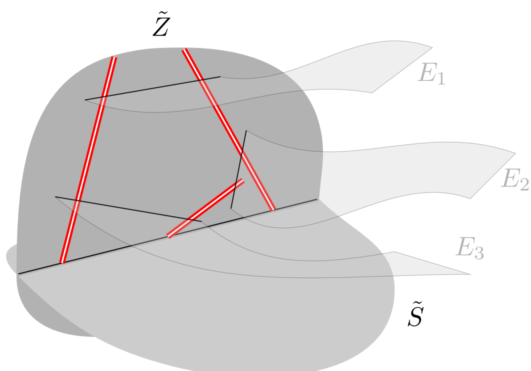


Figure 11. The previous degeneration after blowing up the sections; at this point we still see the double lines (which now don't intersect) but nothing new on $\tilde{S}$.

Let us apply this idea to the case of three 4-ple points coming together. First we blow up the family already constructed along the three sections σ_1, σ_2, σ_3; in this way general members of the family are blowups of S at three general points. Call E_i the exceptional divisor of the blowup along σ_i. The restriction of the line bundle $H - 4E_1 - 4E_2 - 4E_3$ to general surfaces determines the complete linear system of (virtual transforms of) curves on S with three general 4-tuple points. The central fiber surface consists of $\tilde{Z}$ (the plane blown up at three points) and $\tilde{S}$, unchanged from the previous degeneration. In this way, the three offending lines have become three disjoint (-1)-curves C_1, C_2, C_3 in $\tilde{Z}$. Now blow up the family again, this time along C_1, C_2, C_3. The resulting degeneration $g\colon Y \to \Delta$ has a central fiber Y_0 consisting of:

- $\widehat{S}$, the blowup of $\tilde{S}$ (which was the blowup of S at the point p) at the three additional points $q_1 = C_1 \cap \tilde{S}$, $q_2 = C_2 \cap \tilde{S}$, $q_3 = C_3 \cap \tilde{S}$, which are points of $\tilde{Z} \cap \tilde{S} = E$, i.e., infinitely near to p.
- $\tilde{Z}$, the blowup of a plane at three general points, (not affected by blowing up curves on it).

- W_1, W_2, W_3, the three new exceptional divisors, ruled surfaces over $\mathbb{P}^1$. It is not hard to show that they are isomorphic to $\mathbb{F}_0 = \mathbb{P}^1 \times \mathbb{P}^1$.

In general, the exceptional divisor obtained by blowing up a (-1) curve meeting the singular locus of the central fiber in τ points is isomorphic to $\mathbb{F}_{\tau-1}$. The intersections between components are as follows:

× $\tilde{E} = \widehat{S} \cap \tilde{Z}$ is the strict transform of E after blowing up q_1, q_2, q_3 on $\widehat{S}$ and a line on $\tilde{Z}$.

× $F_i := \widehat{S} \cap W_i$ is the exceptional divisor of blowing up q_i on $\widehat{S}$ and a line of the ruling in the ruled surface W_i.

× $\tilde{C}_i = \tilde{Z} \cap W_i$ is the line C_i on $\tilde{Z}$ and the zero-section of the ruling in W_i. (which via the identification of W_i with $\mathbb{P}^1 \times \mathbb{P}^1$ is a line of the other ruling).

For each i, $\widehat{S} \cap \tilde{Z} \cap W_i$ is a point.

It is also useful to note that the preimage of $\tilde{Z}$ in Y (its total transform) is $\bar{Z} = \tilde{Z} \cup W_1 \cup W_2 \cup W_3$. Now consider line bundles on the degeneration of the form $A_a = H - 4E_1 - 4E_2 - 4E_3 - a_0\bar{Z} - a_1W_1 - a_2W_2 - a_3W_3$. We already know from the previous computation that a_0 must be 6 at least in order to have an effective system on $\tilde{Z}$. Now the restrictions to W_i of H, E_i, E_j and E_k (where $\{i, j, k\} = \{1, 2, 3\}$) are linearly equivalent to 0, 0, F_i and F_i respectively, whereas the restriction of $\tilde{Z}$, W_i, W_j and W_k are linearly equivalent to $\tilde{C}_i$, $-\tilde{C}_i - F_i$, 0 and 0 respectively (so that the restriction of $\bar{Z}$ is $-F_i$). We conclude that the restriction of A_a to W_i is $(a_0 + a_i - 8)F_i + a_i\tilde{C}_i$ which, in order to be effective, needs $a_0 + a_i \geq 8$. Taking the minimal a_i that makes the limit line bundle effective on all central components gives $a_0 = 6$, $a_1 = a_2 = a_3 = 2$, and this restricts to $\widehat{S}$ as a point of multiplicity 6 with tree double points infinitely near to it. Such a cluster imposes $21 + 3 \cdot 3 = 30$ conditions, and we have proved that every curve on S which arises as a limit of curves with three general points when the three points collide goes through such a cluster. So that is the limit we were looking for.

The preceding computation is a particular case of a general result, which can be applied whenever a limit line bundle intersects negatively a (-1)-curve:

Proposition 6.3.1 ([10])
Let $f\colon X \to \Delta$ be a degeneration of surfaces such that the central fiber is $V \cup W$ with V an irreducible surface, W a union of irreducible (possibly nonreduced) surfaces, and let A be a line bundle on X. Assume there is a (-1)-curve C in V with $\sigma = A \cdot L < 0$ meeting W transversely in τ points (counted with multiplicity if there are nonreduced components). Then global sections of A vanish along C with multiplicity at least $-\sigma/\tau > 0$.

Corollary 6.3.2
In the setting of Proposition 6.3.1, for every limit divisor of global sections of A_t (the restriction of A to a general fiber X_t of the family) to the central fiber, components of W have τ points of multiplicity $-\sigma/\tau$ (at the intersection points $C \cap W$).

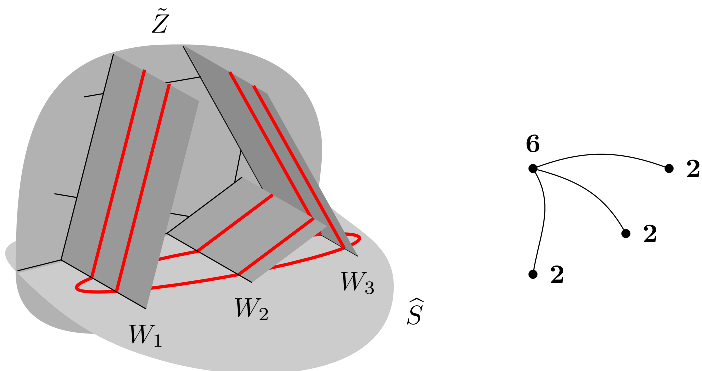


Figure 12. The central fiber after blowing up the three lines appearing doubled on Z. The minimal twist to have an effective system on the central fiber gives a trivial system on $\tilde{Z}$, a system composed of 2 fibers $\equiv C_i$ on each of the three new components $W_i \cong \mathbb{P}^1 \times \mathbb{P}^1$, and a system meeting each of the new exceptionals F_i in two points on $\widehat{S}$; for each curve in the system these two points must agree with the fibers on W_i.

Thus, we have a general, algorithmic process to modify an existing degeneration (blow up the (-1)-curve C; if E_C is the new exceptional divisor, change the line bundle to $A' = A+(\sigma/\tau)E_C$) to avoid limit linear systems of dimension bigger than expected.

Remark 6.3.3
If $\tau > 1$, the new line bundle A' still intersects $C' = E_c \cap V$ negatively.

6.4. Perspectives

In the preceding sections we saw how to apply degenerations to bound the dimensions of linear systems. It should be mentioned that there exists another method to compute limits of linear systems to bound dimensions, algebraically rather than by geometric degenerations, developed after the 'Horace method' of [27], in [2], [1], [17], [40]. The limit systems computed in this way, just like the linear systems in the components of the degenerations presented above, often involve infinitely near points.

Both methods show the potential of eventually leading to a solution of the conjectures stated in the introduction, and it is often the case that a partial result proved with one of them is soon replicated with the other. The remaining paragraphs contain some speculations on possible future developments using the degeneration + blowup + twist method.

When the components on the central fiber of the degeneration have *few* imposed multiple points (more generally, when the blown up components are *anticanonical*) the linear systems on them behave as expected [22]; so either their virtual dimension is the actual dimension (in which case the bound on the dimension of the general linear system is also the virtual dimension, and hence the actual value is computed) or they have a multiple fixed component which is a rational curve of negative selfintersection. In this second case, the strategy of the last section tells us to blow up the offending curve and to obtain a new degeneration, which after a suitable twist with the new component, must decrease strictly the discrepancy between the virtual and the actual dimensions. One expects that iteratively applying such blowups and twists, a degeneration would be found that computes the actual dimension, for an arbitrary initial problem.

This approach has been able to tackle every particular problem so far, including for instance the emptiness of $\mathcal{L}_{174}(55^{10})$ in the plane ("le cas inviolé" in [28]) proved in [8], but two caveats are in order. First, some of the multiple fixed components might be double curves of the central fiber (intersection of two components) which have negative selfintersection "on both sides". Blowing up *these* is technically somewhat more complicated, as the resulting central fiber is no longer reduced. However, this kind of difficulty does not seem to be insurmountable.

The second difficulty that we can see at this point is more serious in nature. The process of iteratively blowing up curves converts the components of the central fiber in blowups of $\mathbb{P}^2$ at *ever more* points, so that eventually non-anticanonical components appear. Thus one can no longer invoke [22] to show that any possible discrepancy from the virtual dimension comes from negative curves. Moreover, the points blown up on these components are no longer completely general. So it is not clear that an induction argument can be applied.

Thus the challenge posed by Conjectures A.1 and B.2 remains, as the path toward a proof outlined in the previous paragraphs needs to break through the non-anticanonical components that appear, with some argument unknown at this point.

Acknowledgements

These notes correspond to lectures given at the summer school of the SFB/TR 45 Bonn-Essen-Mainz, financed by the Deutsche Forschungsgemeinschaft, that took place March 23-27, 2009 at the Pedagogical University of Cracow, with the goal of improving the training of PhD students and postdocs in the area, in particular of the members of the SFB/TR 45. I thank the organizers of the school, Stefan Müller-Stach and Tomasz Szemberg, for their invitation to give these lectures, and I thank Brian Harbourne, fellow speaker at the Summer School in Krakow, for his encouragement and support. I also thank Tomasz Szemberg, Eduardo Casas-Alvero and Ciro Ciliberto for their careful reading of these notes and useful suggestions. Partially supported by the Spanish Ministerio de Educación y Ciencia, grant MTM2009-10359.

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Received: 2 December 2011; final version: 19 March 2012;
available online: 27 April 2012.