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The Littlewood-Paley g-function associated with the Riemann-Liouville operator

Abstract. First, we study the Gauss and Poisson semigroups connected with the Riemann-Liouville operator. Next, we define and study the Littlewood-Paley g-function associated with the Riemann-Liouville operator for which we prove the L^p-boundedness for $p \in]1, 2]$.

1. Introduction

The usual Littlewood-Paley g-function is defined in the Euclidian space [21] by

$$\forall x \in \mathbb{R}^n;\ g(f)(x) = \left(\int_0^{+\infty} |\nabla P^t f(x)|^2 t\, dt \right)^{\frac{1}{2}},$$

where $(P^t)_{t>0}$ is the usual Poisson semigroup defined by

$$P^t f(x) = \frac{\Gamma(\frac{n+1}{2})}{\pi^{\frac{n+1}{2}}} \int_{\mathbb{R}^n} \frac{t f(y)}{(t^2 + |x-y|^2)^{\frac{n+1}{2}}}\, dy,$$

and ∇ is the gradient given by

$$\nabla = \Big(\frac{\partial}{\partial x_1}, \ldots, \frac{\partial}{\partial x_n}, \frac{\partial}{\partial t} \Big).$$

It is well known (see for example [21]) that the mapping

$$f \longmapsto g(f)$$

is bounded from the Lebesgue space $L^p(\mathbb{R}^n, dx)$, $p \in]1, +\infty[$ into itself. Moreover, the Littlewood-Paley theory plays an important role in the study of many function spaces as the Hardy space H^p. Many aspects of the Littlewood-Paley g-function connected with several hypergroups are studied [1, 2, 19, 23]. The authors have been especially interested by the boundedness of such operator when acting on the Lebesgue space L^p; $p \in]1, +\infty[$.

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In [3], the second author with the others define the so-called Riemann-Liouville operator $\mathscr{R}_\alpha$; $\alpha \geqslant 0$ by setting

$$\mathscr{R}_\alpha(f)(r,x) = \begin{cases} \dfrac{\alpha}{\pi} \displaystyle\int_{-1}^{1}\int_{-1}^{1} f(rs\sqrt{1-t^2}, x+rt)(1-t^2)^{\alpha-\frac{1}{2}} \\ \qquad\qquad \times (1-s^2)^{\alpha-1}\, dt\, ds, & \text{if } \alpha > 0, \\ \dfrac{1}{\pi} \displaystyle\int_{-1}^{1} f(r\sqrt{1-t^2}, x+rt) \dfrac{dt}{\sqrt{(1-t^2)}}, & \text{if } \alpha = 0, \end{cases}$$

where f is a continuous function on $\mathbb{R}^2$, even with respect to the first variable.

The Fourier transform associated with the operators $\mathscr{R}_\alpha$ is defined by;

$$\forall (\mu,\lambda) \in \Upsilon;\ \mathscr{F}_\alpha(f)(\mu,\lambda) = \int_0^{+\infty}\int_{\mathbb{R}} f(r,x) j_\alpha\big(r\sqrt{\mu^2+\lambda^2}\big) e^{-i\lambda x}\, d\nu_\alpha(r,x),$$

where

- $\Upsilon = \mathbb{R}^2 \cup \{(i\mu,\lambda);\ (\mu,\lambda) \in \mathbb{R}^2;\ |\mu| \leqslant |\lambda|\}$
- $d\nu_\alpha$ is the measure defined on $[0,+\infty[\times\mathbb{R}$ by

$$d\nu_\alpha(r,x) = \frac{r^{2\alpha+1} dr}{2^\alpha \Gamma(\alpha+1)} \otimes \frac{dx}{(2\pi)^{\frac{1}{2}}}.$$

- j_α is a modified Bessel function that will be defined in the second section.

Many harmonic analysis results related to the Fourier transform $\mathscr{F}_\alpha$ have been established [3, 4, 5, 18]. Also, the uncertainty principles play an important role in harmonic analysis [6, 7, 8, 12, 13, 15], for this reason, many of such principles are established for the Fourier transform $\mathscr{F}_\alpha$ [16, 17].

The aim of this work is to define and study the g-function associated with the Riemann-Liouville operator $\mathscr{R}_\alpha$. For this, we need first to define the Gauss and Poisson semigroups that will be denoted respectively by $(\mathscr{G}^t)_{t>0}$ and $(\mathscr{P}^t)_{t>0}$. The Poisson semigroup $(\mathscr{P}^t)_{t>0}$ allows us to define the Littlewood-Paley g-function by

$$\forall (r,x) \in [0,+\infty[\times\mathbb{R};\ g(f)(r,x) = \left(\int_0^{+\infty} |\nabla \mathscr{P}^t f(r,x)|^2 t\, dt \right)^{\frac{1}{2}},$$

where

$$\nabla = \Big(\frac{\partial}{\partial r}, \frac{\partial}{\partial x}, \frac{\partial}{\partial t}\Big).$$

Then, we have established the main result of this paper. Namely, for every $f \in L^p(d\nu_\alpha)$, $p \in]1,2]$, the function $g(f)$ belongs to the space $L^p(d\nu_\alpha)$ and we have

$$\|g(f)\|_{p,\nu_\alpha} \leqslant 2\frac{2^{\frac{2-p}{2}}}{p} \Big(\frac{p}{p-1}\Big)^{\frac{1}{p}} \|f\|_{p,\nu_\alpha},$$

where

$$\|f\|_{p,\nu_\alpha} = \left(\int_0^{+\infty} \int_{\mathbb{R}} |f(r,x)|^p \, d\nu_\alpha(r,x) \right)^{\frac{1}{p}}.$$

This paper is arranged as follows.

In the second section, we recall some harmonic analysis results related to the Fourier transform $\mathscr{F}_\alpha$. In the third section, we define and study the Gauss semigroup $(\mathscr{G}^t)_{t>0}$ and the Poisson semigroup $(\mathscr{P}^t)_{t>0}$ and we give their mutual connexion. The last section is devoted to establish the boundedness of the Littlewood-Paley g-function from $L^p(d\nu_\alpha)$; $p \in]1,2]$, into it self.

We want to add that in a coming paper; we will establish a principle of the maximum for the operator

$$\triangle_\alpha = \frac{\partial^2}{\partial r^2} + \frac{2\alpha+1}{r}\frac{\partial}{\partial r} + \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial t^2}.$$

We use this principle of the maximum to prove that for every $p \in [4,+\infty[$; there is $A_p > 0$ such that for every $f \in L^p(d\nu_\alpha)$; we have

$$\|g(f)\|_{p,\nu_\alpha} \leqslant A_p \|f\|_{p,\nu_\alpha}.$$

Using Marcinkiewisz interpolation theorem's; we deduce that for every $p \in]1,+\infty[$; there is $C_p > 0$ satisfying

$$\forall f \in L^p(d\nu_\alpha);\ \frac{1}{C_p}\|f\|_{p,\nu_\alpha} \leqslant \|g(f)\|_{p,\nu_\alpha} \leqslant C_p \|f\|_{p,\nu_\alpha}.$$

2. The Riemann-Liouville transform

In this section, we recall some harmonic analysis results related to the convolution product and the Fourier transform associated with Riemann-Liouville operator. For more details see [3, 4, 5, 18].

Let D and Ξ be the singular partial differential operators defined by

$$\begin{cases} D = \dfrac{\partial}{\partial x}; \\ \Xi = \dfrac{\partial^2}{\partial r^2} + \dfrac{2\alpha+1}{r}\dfrac{\partial}{\partial r} - \dfrac{\partial^2}{\partial x^2}; \qquad (r,x) \in]0,+\infty[\times\mathbb{R},\ \alpha \geqslant 0. \end{cases}$$

For all $(\mu,\lambda) \in \mathbb{C}^2$; the system

$$\begin{cases} Du(r,x) = -i\lambda u(r,x), \\ \Xi u(r,x) = -\mu^2 u(r,x), \\ u(0,0) = 1, \\ \dfrac{\partial u}{\partial r}(0,x) = 0;\ \forall x \in \mathbb{R}, \end{cases}$$

admits a unique solution $\varphi_{\mu,\lambda}$ given by

$$\forall (r,x) \in [0,+\infty[\times\mathbb{R};\ \varphi_{\mu,\lambda}(r,x) = j_\alpha\big(r\sqrt{\mu^2+\lambda^2}\big)e^{-i\lambda x}, \tag{2.1}$$

where j_α is the modified Bessel function defined by

$$j_\alpha(z) = 2^\alpha\Gamma(\alpha+1)\frac{J_\alpha(z)}{z^\alpha} = \Gamma(\alpha+1)\sum_{k=0}^{+\infty}\frac{(-1)^k}{k!\Gamma(\alpha+k+1)}\left(\frac{z}{2}\right)^{2k},$$

and J_α is the Bessel function of first kind and index α [10, 11, 14, 26]. The modified Bessel function j_α has the integral representation

$$j_\alpha(z) = \frac{\Gamma(\alpha+1)}{\sqrt{\pi}\Gamma(\alpha+\frac{1}{2})}\int_{-1}^{1}(1-t^2)^{\alpha-\frac{1}{2}}\exp(-izt)\,dt.$$

Consequently, for every $k\in\mathbb{N}$ and $z\in\mathbb{C}$; we have

$$|j_\alpha^{(k)}(z)| \leqslant e^{|\mathrm{Im}(z)|}. \tag{2.2}$$

The eigenfunction $\varphi_{\mu,\lambda}$ satisfies the following properties

- The function $\varphi_{\mu,\lambda}$ is bounded on $\mathbb{R}^2$ if, and only if $(\mu,\lambda)\in\Upsilon$, where Υ is the set defined by

$$\Upsilon = \mathbb{R}^2\cup\{(i\mu,\lambda);\ (\mu,\lambda)\in\mathbb{R}^2;\ |\mu|\leqslant|\lambda|\}$$

  and in this case

$$\sup_{(r,x)\in\mathbb{R}^2}|\varphi_{\mu,\lambda}(r,x)| = 1. \tag{2.3}$$

- The function $\varphi_{\mu,\lambda}$ has the following Mehler integral representation

$$\varphi_{\mu,\lambda}(r,x) = \begin{cases} \dfrac{\alpha}{\pi}\displaystyle\int_{-1}^{1}\int_{-1}^{1}\cos\left(\mu rs\sqrt{1-t^2}\right)\exp(-i\lambda(x+rt)) \\ \qquad\qquad \times(1-t^2)^{\alpha-\frac{1}{2}}(1-s^2)^{\alpha-1}\,dt\,ds, & \text{if } \alpha>0, \\ \dfrac{1}{\pi}\displaystyle\int_{-1}^{1}\cos\left(r\mu\sqrt{1-t^2}\right)\exp(-i\lambda(x+rt))\frac{dt}{\sqrt{1-t^2}}, & \text{if } \alpha=0. \end{cases}$$

  The precedent integral representation allows us to define the Riemann-Liouville transform $\mathscr{R}_\alpha$ associated with the operators D and Ξ by

$$\mathscr{R}_\alpha(f)(r,x) = \begin{cases} \dfrac{\alpha}{\pi}\displaystyle\int_{-1}^{1}\int_{-1}^{1} f\left(rs\sqrt{1-t^2},x+rt\right)(1-t^2)^{\alpha-\frac{1}{2}} \\ \qquad\qquad \times(1-s^2)^{\alpha-1}\,dt\,ds, & \text{if } \alpha>0, \\ \dfrac{1}{\pi}\displaystyle\int_{-1}^{1} f\left(r\sqrt{1-t^2},x+rt\right)\frac{dt}{\sqrt{1-t^2}}, & \text{if } \alpha=0, \end{cases}$$

  where f is any continuous function on $\mathbb{R}^2$, even with respect to the first variable.

- From the precedent integral representation of the eigenfunction $\varphi_{\mu,\lambda}$, we deduce that

$$\forall (r,x) \in [0,+\infty[\times\mathbb{R};\ \varphi_{\mu,\lambda}(r,x) = \mathscr{R}_\alpha(\cos(\mu.)e^{-i\lambda.})(r,x).$$

In the following, we will define the convolution product and the Fourier transform associated with the Riemann-Liouville operator. For this, we need the coming notation

- $L^p(d\nu_\alpha)$; $p \in [1,+\infty]$, is the Lebesgue space formed by the measurable functions f on $[0,+\infty[\times\mathbb{R}$ such that $\|f\|_{p,\nu_\alpha} < +\infty$, where

$$\|f\|_{p,\nu_\alpha} = \begin{cases} \left(\int_0^{+\infty}\int_{\mathbb{R}} |f(r,x)|^p \, d\nu_\alpha(r,x) \right)^{\frac{1}{p}}, & \text{if } p \in [1,+\infty[, \\ \operatorname*{ess\,sup}_{(r,x)\in[0,+\infty[\times\mathbb{R}} |f(r,x)|, & \text{if } p = +\infty. \end{cases}$$

Definition 2.1

i) For every $(r,x) \in [0,+\infty[\times\mathbb{R}$, the translation operator $\tau_{(r,x)}$ associated with Riemann-Liouville operator is defined on $L^p(d\nu_\alpha)$, $p \in [1,+\infty]$, by

$$\tau_{(r,x)}f(s,y) = \frac{\Gamma(\alpha+1)}{\sqrt{\pi}\Gamma(\alpha+\frac{1}{2})} \int_0^\pi f\left(\sqrt{r^2+s^2+2rs\cos\theta}, x+y\right) \sin^{2\alpha}(\theta)\, d\theta. \tag{2.4}$$

ii) The convolution product of $f, g \in L^1(d\nu_\alpha)$ is defined for every $(r,x) \in [0,+\infty[\times\mathbb{R}$, by

$$f * g(r,x) = \int_0^{+\infty}\int_{\mathbb{R}} \tau_{(r,-x)}(\check{f})(s,y) g(s,y)\, d\nu_\alpha(s,y), \tag{2.5}$$

where $\check{f}(s,y) = f(s,-y)$.

We have the following properties

- The eigenfunction $\varphi_{\mu,\lambda}$ satisfies the product formula

$$\tau_{(r,x)}(\varphi_{\mu,\lambda})(s,y) = \varphi_{\mu,\lambda}(r,x)\varphi_{\mu,\lambda}(s,y).$$

- For every $f \in L^p(d\nu_\alpha)$, $1 \leqslant p \leqslant +\infty$, and for every $(r,x) \in [0,+\infty[\times\mathbb{R}$, the function $\tau_{(r,x)}(f)$ belongs to $L^p(d\nu_\alpha)$ and we have

$$\|\tau_{(r,x)}(f)\|_{p,\nu_\alpha} \leqslant \|f\|_{p,\nu_\alpha}.$$

- For every $f \in L^p(d\nu_\alpha)$, $p \in [1,+\infty[$, we have

$$\lim_{(r,x)\to(0,0)} \|\tau_{(r,x)}(f) - f\|_{p,\nu_\alpha} = 0. \tag{2.6}$$

- For $f, g \in L^1(d\nu_\alpha)$, the function $f * g$ belongs to $L^1(d\nu_\alpha)$; the convolution product is commutative, associative and we have

$$\|f * g\|_{1,\nu_\alpha} \leqslant \|f\|_{1,\nu_\alpha} \|g\|_{1,\nu_\alpha}.$$

Moreover, if $1 \leqslant p, q, r \leqslant +\infty$ are such that $\frac{1}{r} = \frac{1}{p} + \frac{1}{q} - 1$ and if $f \in L^p(d\nu_\alpha)$, $g \in L^q(d\nu_\alpha)$, then the function $f * g$ belongs to $L^r(d\nu_\alpha)$, and we have the Young's inequality

$$\|f * g\|_{r,\nu_\alpha} \leqslant \|f\|_{p,\nu_\alpha} \|g\|_{q,\nu_\alpha}. \tag{2.7}$$

- Let φ be a nonnegative measurable function on $\mathbb{R} \times \mathbb{R}$, even with respect to the first variable, such that

$$\int_0^{+\infty} \int_{\mathbb{R}} \varphi(r, x)\, d\nu_\alpha(r, x) = 1.$$

Then by relation (2.6), the family $(\varphi_t)_{t>0}$ defined by

$$\forall (r, x) \in \mathbb{R} \times \mathbb{R};\ \varphi_t(r, x) = \frac{\varphi(\frac{r}{t}, \frac{x}{t})}{t^{2\alpha+3}},$$

is an approximation of the identity in $L^p(d\nu_\alpha)$; $p \in [1, +\infty[$, that is for every $f \in L^p(d\nu_\alpha)$, we have

$$\lim_{t \to 0^+} \|\varphi_t * f - f\|_{p,\nu_\alpha} = 0. \tag{2.8}$$

In the sequel, we use the following notations

- Υ_+ is the subset of Υ given by

$$\Upsilon_+ = \mathbb{R}_+ \times \mathbb{R} \cup \{(it, x);\ (t, x) \in \mathbb{R}^2;\ 0 \leqslant t \leqslant |x|\}.$$

- $\mathscr{B}_{\Upsilon_+}$ is the σ-algebra defined on Υ_+ by

$$\mathscr{B}_{\Upsilon_+} = \{\theta^{-1}(B),\ B \in \mathscr{B}_{\mathrm{or}}([0, +\infty[\times \mathbb{R})\},$$

where θ is the bijective function defined on the set Υ_+ by

$$\theta(\mu, \lambda) = \left(\sqrt{\mu^2 + \lambda^2}, \lambda\right). \tag{2.9}$$

- $d\gamma_\alpha$ is the measure defined on $\mathscr{B}_{\Upsilon_+}$ by

$$\forall A \in \mathscr{B}_{\Upsilon_+};\ \gamma_\alpha(A) = \nu_\alpha(\theta(A)).$$

- $L^p(d\gamma_\alpha)$; $p \in [1, +\infty]$, is the space of measurable functions f on Υ_+, such that

$$\|f\|_{p,\gamma_\alpha} < +\infty.$$

Proposition 2.2

i. *For all non negative measurable function g on Υ_+, we have*

$$\iint_{\Upsilon_+} g(\mu,\lambda)\,d\gamma_\alpha(\mu,\lambda)$$

$$= \frac{1}{2^\alpha\Gamma(\alpha+1)\sqrt{2\pi}}\Bigg(\int_0^{+\infty}\int_{\mathbb{R}} g(\mu,\lambda)(\mu^2+\lambda^2)^\alpha\mu\,d\mu\,d\lambda$$

$$+\int_{\mathbb{R}}\int_0^{|\lambda|} g(i\mu,\lambda)(\lambda^2-\mu^2)^\alpha\mu\,d\mu\,d\lambda\Bigg).$$

ii. *For all non negative measurable function f on $[0,+\infty[\times\mathbb{R}$ (respectively integrable on $[0,+\infty[\times\mathbb{R}$ with respect to the measure $d\nu_\alpha$) $f\circ\theta$ is a nonnegative measurable function on Υ_+ (respectively integrable on Υ_+ with respect to the measure $d\gamma_\alpha$) and we have*

$$\iint_{\Upsilon_+}(f\circ\theta)(\mu,\lambda)\,d\gamma_\alpha(\mu,\lambda) = \int_0^{+\infty}\int_{\mathbb{R}} f(r,x)\,d\nu_\alpha(r,x). \tag{2.10}$$

Definition 2.3

The Fourier transform associated with the Riemann-Liouville operator is defined on $L^1(d\nu_\alpha)$ by

$$\forall(\mu,\lambda)\in\Upsilon;\ \mathscr{F}_\alpha(f)(\mu,\lambda) = \int_0^{+\infty}\int_{\mathbb{R}} f(r,x)\varphi_{\mu,\lambda}(r,x)\,d\nu_\alpha(r,x),$$

where $\varphi_{\mu,\lambda}$ is the eigenfunction given by relation (2.1).

We have the following properties

- From relation (2.3), we deduce that for $f\in L^1(d\nu_\alpha)$ the function $\mathscr{F}_\alpha(f)$ belongs to the space $L^\infty(d\gamma_\alpha)$ and we have

$$\|\mathscr{F}_\alpha(f)\|_{\infty,\gamma_\alpha} \leqslant \|f\|_{1,\nu_\alpha}. \tag{2.11}$$

- For $f\in L^1(d\nu_\alpha)$, we have

$$\forall(\mu,\lambda)\in\Upsilon;\ \mathscr{F}_\alpha(f)(\mu,\lambda) = \widetilde{\mathscr{F}_\alpha}(f)\circ\theta(\mu,\lambda), \tag{2.12}$$

where for every $(\mu,\lambda)\in\mathbb{R}^2$,

$$\widetilde{\mathscr{F}_\alpha}(f)(\mu,\lambda) = \int_0^{+\infty}\int_{\mathbb{R}} f(r,x)j_\alpha(r\mu)\exp(-i\lambda x)\,d\nu_\alpha(r,x),$$

and θ is the function defined by relation (2.9).

- Let $f \in L^1(d\nu_\alpha)$ such that the function $\mathscr{F}_\alpha(f)$ belongs to the space $L^1(d\gamma_\alpha)$, then we have the following inversion formula for $\mathscr{F}_\alpha$, for almost every $(r,x) \in [0,+\infty[\times\mathbb{R}$,

$$f(r,x) = \iint_{\Upsilon_+} \mathscr{F}_\alpha(f)(\mu,\lambda)\overline{\varphi_{\mu,\lambda}(r,x)}\, d\gamma_\alpha(\mu,\lambda)$$

- Let $f \in L^1(d\nu_\alpha)$. For every $(r,x) \in [0,+\infty[\times\mathbb{R}$, we have

$$\forall(\mu,\lambda) \in \Upsilon;\ \mathscr{F}_\alpha(\tau_{(r,x)}(f))(\mu,\lambda) = \overline{\varphi_{\mu,\lambda}(r,x)}\mathscr{F}_\alpha(f)(\mu,\lambda).$$

- For $f, g \in L^1(d\nu_\alpha)$, we have

$$\forall(\mu,\lambda) \in \Upsilon;\ \mathscr{F}_\alpha(f * g)(\mu,\lambda) = \mathscr{F}_\alpha(f)(\mu,\lambda)\mathscr{F}_\alpha(g)(\mu,\lambda).$$

- Let $p \in [1,+\infty]$. From relation (2.10), the function f belongs to $L^p(d\nu_\alpha)$ if, and only if the function $f \circ \theta$ belongs to the space $L^p(d\gamma_\alpha)$ and we have

$$\|f \circ \theta\|_{p,\gamma_\alpha} = \|f\|_{p,\nu_\alpha}. \tag{2.13}$$

  Since the mapping $\widetilde{\mathscr{F}_\alpha}$ is an isometric isomorphism from $L^2(d\nu_\alpha)$ onto itself [24, 25], then the relations (2.12) and (2.13) show that the Fourier transform $\mathscr{F}_\alpha$ is an isometric isomorphism from $L^2(d\nu_\alpha)$ into $L^2(d\gamma_\alpha)$. Namely, for every $f \in L^2(d\nu_\alpha)$, the function $\mathscr{F}_\alpha(f)$ belongs to the space $L^2(d\gamma_\alpha)$ and we have

$$\|\mathscr{F}_\alpha(f)\|_{2,\gamma_\alpha} = \|f\|_{2,\nu_\alpha}. \tag{2.14}$$

Proposition 2.4
For every f in $L^p(d\nu_\alpha)$, $p \in [1,2]$; the function $\mathscr{F}_\alpha(f)$ lies in $L^{p'}(d\gamma_\alpha)$, $p' = \frac{p}{p-1}$, and we have

$$\|\mathscr{F}_\alpha(f)\|_{p',\gamma_\alpha} \leqslant \|f\|_{p,\nu_\alpha}.$$

Proof. The result follows from relations (2.11), (2.14) and the Riesz-Thorin theorem's [20, 22].

We denote by

- $\mathscr{S}_e(\mathbb{R}^2)$ the space of infinitely differentiable functions on $\mathbb{R}^2$, rapidly decreasing together with all their derivatives, even with respect to the first variable. The space $\mathscr{S}_e(\mathbb{R}^2)$ is endowed with the topology generated by the family of norms

$$\rho_m(\varphi) = \sup_{\substack{(r,x)\in[0,+\infty[\times\mathbb{R}\\ k+|\beta|\leqslant m}} (1+r^2+x^2)^k |D^\beta(\varphi)(r,x)|;\quad m \in \mathbb{N}. \tag{2.15}$$

- $\mathscr{D}_e(\mathbb{R}^2)$ the subspace of $\mathscr{S}_e(\mathbb{R}^2)$ of functions with compact support.

3. Gauss and Poisson semigroups associated with the Riemann-Liouville operator

In this section, we will define and study the Gauss and Poisson semigroups. Also, the maximal functions connected with these semigroups are checked.

Definition 3.1
The Gauss kernel g_t, $t > 0$, associated with the Riemann-Liouville operator is defined on $\mathbb{R}^2$ by

$$\begin{aligned} g_t(r,x) = \frac{e^{-\frac{(r^2+x^2)}{4t}}}{(2t)^{\alpha+\frac{3}{2}}} &= \iint\limits_{\Upsilon_+} e^{-t(\mu^2+2\lambda^2)}\overline{\varphi_{\mu,\lambda}(r,x)}\, d\gamma_\alpha(\mu,\lambda) \\ &= \widetilde{\mathscr{F}_\alpha}^{-1}(e^{-t(s^2+y^2)})(r,x). \end{aligned} \tag{3.16}$$

Lemma 3.2
The family $(g_t)_{t>0}$ is an approximation of the identity in the space $\mathscr{S}_e(\mathbb{R}^2)$; that is for every $f \in \mathscr{S}_e(\mathbb{R}^n)$; and every $t > 0$; the function $g_t * f$ belongs to $\mathscr{S}_e(\mathbb{R}^2)$ and for every $m \in \mathbb{N}$;

$$\lim_{t\to 0^+} \rho_m(g_t * f - f) = 0,$$

where ρ_m is the norm defined by relation (2.15)*.*

Proof. Since the Schwartz space $\mathscr{S}_e(\mathbb{R}^2)$ is stable under convolution product, we deduce that for every $f \in \mathscr{S}_e(\mathbb{R}^2)$; and every $t > 0$; the function $g_t * f$ belongs to the space $\mathscr{S}_e(\mathbb{R}^2)$. On the other hand, the transform $\widetilde{\mathscr{F}_\alpha}$ is a topological isomorphism from $\mathscr{S}_e(\mathbb{R}^2)$ onto itself which satisfies

$$\widetilde{\mathscr{F}_\alpha}(f * g) = \widetilde{\mathscr{F}_\alpha}(f)\widetilde{\mathscr{F}_\alpha}(g). \tag{3.17}$$

By relation (3.16), we get $\widetilde{\mathscr{F}_\alpha}(g_t)(r,x) = e^{-t(r^2+x^2)}$. So, we must show that for every $(k,\beta) \in \mathbb{N} \times \mathbb{N}^2$ and every $f \in \mathscr{S}_e(\mathbb{R}^2)$,

$$\lim_{t\to 0^+} \|(1+r^2+x^2)^k D^\beta(e^{-t(r^2+x^2)}f - f)\|_{\infty,\nu_\alpha} = 0.$$

Applying Leibniz formula, we get

$$\begin{aligned} &D^\beta(e^{-t(r^2+x^2)}f(r,x)) \\ &\quad = \sum_{\gamma\leqslant\beta} \frac{\beta!}{\gamma!(\beta-\gamma)!} D^\gamma(e^{-t(r^2+x^2)})D^{\beta-\gamma}(f)(r,x) \\ &\quad = \sum_{\gamma\leqslant\beta} \frac{\beta!}{\gamma!(\beta-\gamma)!}(-1)^{|\gamma|}\sqrt{t}^{|\gamma|} H_\gamma\big(r\sqrt{t}, x\sqrt{t}\big)e^{-t(r^2+x^2)}D^{\beta-\gamma}(f)(r,x), \end{aligned}$$

where H_γ is the Hermite polynomial on $\mathbb{R}^2$ with index γ.

Consequently,

$$\begin{aligned} &D^{\beta}(e^{-t(r^2+x^2)}f(r,x)-f(r,x)) \\ &\quad = \sum_{\substack{\gamma\leqslant\beta\\ \gamma\neq 0}} \frac{\beta!}{\gamma!(\beta-\gamma)!}(-1)^{|\gamma|}\sqrt{t}^{|\gamma|}H_{\gamma}\big(r\sqrt{t},x\sqrt{t}\big)e^{-t(r^2+x^2)}D^{\beta-\gamma}(f)(r,x) \\ &\qquad + (e^{-t(r^2+x^2)}-1)D^{\beta}(f)(r,x). \end{aligned}$$

Thus, for every t, $0\leqslant t<1$;

$$\begin{aligned} &\|(1+r^2+x^2)^k D^{\beta}(e^{-t(r^2+x^2)}f-f)\|_{\infty,\nu_\alpha} \\ &\quad \leqslant \sqrt{t}\Big[\sum_{\gamma\leqslant\beta}\frac{\beta!}{\gamma!(\beta-\gamma)!}\|H_{\gamma}e^{-(r^2+x^2)}\|_{\infty,\nu_\alpha}\|(1+r^2+x^2)^k D^{\beta-\gamma}(f)\|_{\infty,\nu_\alpha} \\ &\qquad + \|(1+r^2+x^2)^{k+1}D^{\beta}(f)\|_{\infty,\nu_\alpha}\Big]. \end{aligned}$$

The last inequality shows that for every $(k,\beta)\in\mathbb{N}\times\mathbb{N}^2$,

$$\lim_{t\to 0^+}\|(1+r^2+x^2)^k D^{\beta}(\widetilde{\mathscr{F}_\alpha}(g_t)f-f)\|_{\infty,\nu_\alpha}=0.$$

The proof is complete.

Proposition 3.3
For every $f\in\mathscr{S}_e(\mathbb{R}^2)$; the function $\mathscr{V}(f)$ defined by

$$\mathscr{V}(f)(r,x,t)=g_t*f(r,x),\quad \forall(r,x,t)\in\mathbb{R}^2\times]0,+\infty[,$$

is infinitely differentiable on $\mathbb{R}^2\times]0,+\infty[$ and satisfies the following equation

$$\begin{cases} \Lambda_\alpha(\mathscr{V}(f))=\dfrac{\partial}{\partial t}(\mathscr{V}(f)), \\ \lim\limits_{t\to 0^+}\mathscr{V}(f)(.,.,t)=f \quad \textit{uniformly.} \end{cases}$$

Where

$$\Lambda_\alpha=\frac{\partial^2}{\partial r^2}+\frac{2\alpha+1}{r}\frac{\partial}{\partial r}+\frac{\partial^2}{\partial x^2}. \tag{3.18}$$

Proof. For every $t>0$; the function g_t belongs to $\mathscr{S}_e(\mathbb{R}^2)$ and consequently, for every $f\in\mathscr{S}_e(\mathbb{R}^2)$, the function

$$(r,x)\longmapsto g_t*f(r,x)$$

belongs to the space $\mathscr{S}_e(\mathbb{R}^2)$ and for every $(\mu,\lambda)\in\mathbb{R}^2$;

$$\widetilde{\mathscr{F}_\alpha}(g_t*f)(\mu,\lambda)=\widetilde{\mathscr{F}_\alpha}(\mathscr{V}(f)(.,.,t))(\mu,\lambda)=e^{-t(\mu^2+\lambda^2)}\widetilde{\mathscr{F}_\alpha}(f)(\mu,\lambda).$$

This implies that for every $(r,x,t)\in\mathbb{R}^2\times]0,+\infty[$, we have

$$\mathscr{V}(f)(r,x,t)=\int_0^\infty\int_{\mathbb{R}} e^{-t(\mu^2+\lambda^2)}\widetilde{\mathscr{F}_\alpha}(f)(\mu,\lambda)j_\alpha(r\mu)e^{i\lambda x}\,d\nu_\alpha(\mu,\lambda).$$

From this equality; it follows that the function

$$(r,x,t)\longmapsto \mathscr{V}(f)(r,x,t)$$

is infinitely differentiable on $\mathbb{R}^2\times]0,+\infty[$ and we have

$$\begin{aligned}\frac{\partial}{\partial t}(\mathscr{V}(f))(r,x,t) &= -\int_0^\infty\int_{\mathbb{R}}(\mu^2+\lambda^2)e^{-t(\mu^2+\lambda^2)}\widetilde{\mathscr{F}_\alpha}(f)(\mu,\lambda)j_\alpha(r\mu)e^{i\lambda x}\,d\nu_\alpha(r,x)\\ &= \Lambda_\alpha(\mathscr{V}(f))(r,x,t),\end{aligned}$$

because $(\frac{\partial^2}{\partial r^2}+\frac{2\alpha+1}{\partial r})(j_\alpha(\mu r)) = -\mu^2 j_\alpha(r\mu)$ and $\frac{\partial^2}{\partial x^2}(e^{i\lambda x}) = -\lambda^2 e^{i\lambda x}$.

On the other hand; for $(r,x,t)\in\mathbb{R}^2\times]0,+\infty[$,

$$\begin{aligned}&f(r,x)-\mathscr{V}(f)(r,x,t)\\ &\quad=\int_0^\infty\int_{\mathbb{R}}(1-e^{-t(\mu^2+\lambda^2)})\widetilde{\mathscr{F}_\alpha}(f)(\mu,\lambda)j_\alpha(r\mu)e^{i\lambda x}\,d\nu_\alpha(r,x).\end{aligned}$$

So

$$\|f-\mathscr{V}(f)(.,.,t)\|_{\infty,\nu_\alpha}\leqslant t\int_0^\infty\int_{\mathbb{R}}(\mu^2+\lambda^2)|\widetilde{\mathscr{F}_\alpha}(f)(\mu,\lambda)|\,d\nu_\alpha(\mu,\lambda),$$

which means that

$$\lim_{t\to 0^+}\|\mathscr{V}(f)(.,.,t)-f\|_{\infty,\nu_\alpha}=0.$$

Proposition 3.4

i. *For every $p\in[1,+\infty]$; the operator $\mathscr{G}^t$, $t>0$, defined by*

$$\mathscr{G}^t(f)=g_t*f \tag{3.19}$$

is a bounded positive operator from $L^p(d\nu_\alpha)$ into itself and for every $f\in L^p(d\nu_\alpha)$, we have

$$\|\mathscr{G}^t(f)\|_{p,\nu_\alpha}\leqslant\|f\|_{p,\nu_\alpha}.$$

ii. *For every $p\in[1,+\infty[$, the family $(\mathscr{G}^t)_{t>0}$ is a strongly continuous semigroup of operators on $L^p(d\nu_\alpha)$, that is*

- *For $s,t>0$; $\mathscr{G}^s\circ\mathscr{G}^t=\mathscr{G}^{s+t}$,*
- *For every $f\in L^p(d\nu_\alpha)$, $\lim_{t\to 0^+}\|\mathscr{G}^t(f)-f\|_{p,\nu_\alpha}=0$.*

The family $(\mathscr{G}^t)_{t>0}$ is called Gauss semigroup associated with the Riemann-Liouville operator $\mathscr{R}_\alpha$.

Proof. i. Let $g(r,x)=e^{-\frac{r^2+x^2}{2}}$, g is a measurable positive function and we have

$$g_t(r,x)=\frac{g(\frac{r}{\sqrt{2t}},\frac{x}{\sqrt{2t}})}{(\sqrt{2t})^{2\alpha+3}}.$$

So

$$\int_0^\infty\int_{\mathbb{R}} g_t(r,x)\,d\nu_\alpha(r,x) = \int_0^\infty\int_{\mathbb{R}} g(r,x)\,d\nu_\alpha(r,x) = 1.$$

From relation (2.7), for every $f \in L^p(d\nu_\alpha)$; and every $t > 0$, the function $\mathscr{G}^t(f) = g_t * f$ belongs to $L^p(d\nu_\alpha)$ and we have

$$\|\mathscr{G}^t(f)\|_{p,\nu_\alpha} \leqslant \|g_t\|_{1,\nu_\alpha}\|f\|_{p,\nu_\alpha} = \|f\|_{p,\nu_\alpha}.$$

ii. From relation (3.16), we have

$$\forall(\mu,\lambda) \in \mathbb{R}^2;\ \widetilde{\mathscr{F}_\alpha}(g_t)(\mu,\lambda) = e^{-t(\mu^2+\lambda^2)}.$$

So, from relation (3.17); for s, $t > 0$; we get

$$\widetilde{\mathscr{F}_\alpha}(g_t * g_s)(\mu,\lambda) = e^{-(t+s)(\mu^2+\lambda^2)} = \widetilde{\mathscr{F}_\alpha}(g_{t+s})(\mu,\lambda),$$

and consequently; $g_s * g_t = g_{s+t}$ which involves that for every $f \in L^p(d\nu_\alpha)$;

$$\mathscr{G}^s(\mathscr{G}^t(f)) = \mathscr{G}^{s+t}(f).$$

Moreover, from relation (2.8),

$$\lim_{t\to 0^+} \|\mathscr{G}^t(f) - f\|_{\infty,\nu_\alpha} = 0.$$

The proof is complete.

PROPOSITION 3.5
For every $f \in \mathscr{D}_e(\mathbb{R}^2)$, the maximal function $\mathscr{M}(f)$ defined on $\mathbb{R}^2$ by

$$\mathscr{M}(f)(r,x) = \sup_{s>0} \frac{1}{s}\left|\int_0^s \mathscr{G}^t(f)(r,x)\,dt\right|, \tag{3.20}$$

belongs to the space $L^p(d\nu_\alpha)$, $p \in]1,+\infty[$, moreover

$$\|\mathscr{M}(f)\|_{p,\nu_\alpha} \leqslant 2\Big(\frac{p}{p-1}\Big)^{\frac{1}{p}}\|f\|_{p,\nu_\alpha}.$$

Proof. The result follows immediately from [9, theorem 7, pp 693].

DEFINITION 3.6
For every $t > 0$, the Poisson kernel p_t associated with the Riemann-Liouville operator is defined on $\mathbb{R}^2$ by

$$\begin{aligned} p_t(r,x) &= \iint_{\Upsilon_+} e^{-t\sqrt{s^2+2y^2}}\,\overline{\varphi_{s,y}(r,x)}\,d\gamma_\alpha(s,y) = \mathscr{F}_\alpha^{-1}\big(e^{-t\sqrt{s^2+2y^2}}\big)(r,x) \\ &= \widetilde{\mathscr{F}_\alpha}^{-1}\big(e^{-t\sqrt{s^2+y^2}}\big)(r,x). \end{aligned} \tag{3.21}$$

Lemma 3.7
For every $(r,x,t)\in\mathbb{R}^2\times]0,+\infty[$, we have

$$p_t(r,x)=\frac{2^{\alpha+\frac{3}{2}}\Gamma(\alpha+2)}{\sqrt{\pi}}\frac{t}{(t^2+r^2+x^2)^{\alpha+2}}.$$

Proof. We know that for every $x\in\mathbb{R}$; we have

$$\frac{1}{\sqrt{\pi}}\int_0^{\infty}\frac{e^{-u}}{\sqrt{u}}e^{-\frac{x^2}{4u}}\,du=e^{-|x|}.$$

From Definition 3.6, and applying Fubini's theorem, we get

$$\begin{aligned}
p_t(r,x)&=\frac{1}{\sqrt{\pi}}\int_0^{\infty}\frac{e^{-u}}{\sqrt{u}}\left(\iint_{\Upsilon_+}e^{-\frac{t^2}{4u}(s^2+2y^2)}\overline{\varphi_{s,y}(r,x)}\,d\gamma_\alpha(s,y)\right)\\
&=\frac{1}{\sqrt{\pi}}\int_0^{\infty}\frac{e^{-u}}{\sqrt{u}}g_{\frac{t^2}{4u}}(r,x)\,du\\
&=\frac{2^{\alpha+\frac{3}{2}}}{\sqrt{\pi}}t^{-2\alpha-3}\int_0^{\infty}e^{-\frac{u}{t^2}(r^2+x^2+t^2)}u^{\alpha+1}\,du\\
&=\frac{2^{\alpha+\frac{3}{2}}\Gamma(\alpha+2)}{\sqrt{\pi}}\frac{t}{(t^2+r^2+x^2)^{\alpha+2}}.
\end{aligned}\tag{3.22}$$

Proposition 3.8
Let $f\in\mathscr{S}_e(\mathbb{R}^2)$, the function $\mathscr{U}(f)$ defined on $\mathbb{R}^2\times]0,+\infty[$ by

$$\mathscr{U}(f)(r,x)=p_t*f(r,x)$$

is infinitely differentiable and satisfies the equation

$$\begin{cases}\Lambda_\alpha\big(\mathscr{U}(f)\big)+\dfrac{\partial^2}{\partial t^2}\big(\mathscr{U}(f)\big)=0,\\ \displaystyle\lim_{t\to 0^+}\mathscr{U}(f)(.,.,t)=f \quad \textit{uniformly}.\end{cases}$$

Proof. From relation (3.21), for every $(\mu,\lambda)\in\mathbb{R}^2$, we have

$$\widetilde{\mathscr{F}_\alpha}(\mathscr{U}(f)(.,.,t))=\widetilde{\mathscr{F}_\alpha}(p_t)(\mu,\lambda)\widetilde{\mathscr{F}_\alpha}(f)(\mu,\lambda)=e^{-t\sqrt{\mu^2+\lambda^2}}\widetilde{\mathscr{F}_\alpha}(f)(\mu,\lambda).$$

So, for every $(r,x,t)\in\mathbb{R}^2\times]0,+\infty[$;

$$\begin{aligned}
\mathscr{U}(f)(r,x,t)&=\widetilde{\mathscr{F}_\alpha}^{-1}\big(e^{-t\sqrt{\mu^2+\lambda^2}}\widetilde{\mathscr{F}_\alpha}(f)\big)(r,x)\\
&=\int_0^{\infty}\int_{\mathbb{R}}e^{-t\sqrt{\mu^2+\lambda^2}}\widetilde{\mathscr{F}_\alpha}(f)(\mu,\lambda)j_\alpha(r\mu)e^{i\lambda x}\,d\nu_\alpha(\mu,\lambda).
\end{aligned}$$

From relation (2.2) and the fact that the function $\widetilde{\mathscr{F}_\alpha}(f)$ belongs to the space $\mathscr{S}_e(\mathbb{R}^2)$; we deduce that the function $\mathscr{U}(f)$ is infinitely differentiable on

$\mathbb{R}^2\times]0,+\infty[$. Moreover,

$$\begin{aligned}\Lambda_\alpha(\mathscr{U}(f))(r,x,t)&\\ &=-\int_0^\infty\int_{\mathbb{R}}(\mu^2+\lambda^2)e^{-t\sqrt{\mu^2+\lambda^2}}\widetilde{\mathscr{F}_\alpha}(f)(\mu,\lambda)j_\alpha(r\mu)e^{i\lambda x}\,d\nu_\alpha(\mu,\lambda)\\ &=-\frac{\partial^2}{\partial t^2}(\mathscr{U}(f))(r,x,t).\end{aligned}$$

On the other hand; for every $(r,x,t)\in\mathbb{R}^2\times]0,+\infty[$; we get

$$\begin{aligned}|f(r,x)-\mathscr{U}(f)(r,x,t)|&\leqslant\int_0^\infty\int_{\mathbb{R}}\left|1-e^{-t\sqrt{\mu^2+\lambda^2}}\right||\widetilde{\mathscr{F}_\alpha}(f)(\mu,\lambda)|\,d\nu_\alpha(\mu,\lambda)\\ &\leqslant t\int_0^\infty\int_{\mathbb{R}}\sqrt{\mu^2+\lambda^2}|\widetilde{\mathscr{F}_\alpha}(f)(\mu,\lambda)|\,d\nu_\alpha(\mu,\lambda),\end{aligned}$$

which means that

$$\|\mathscr{U}(f)(.,.,t)-f\|_{\infty,\nu_\alpha}\leqslant t\int_0^\infty\int_{\mathbb{R}}\sqrt{\mu^2+\lambda^2}|\widetilde{\mathscr{F}_\alpha}(f)(\mu,\lambda)|\,d\nu_\alpha(\mu,\lambda),$$

and proves that

$$\lim_{t\to0^+}\|\mathscr{U}(f)(.,.,t)-f\|_{\infty,\nu_\alpha}=0.$$

Proposition 3.9

i. *For every $p\in[1,+\infty]$; the operator $\mathscr{P}^t$, $t>0$, defined by*

$$\mathscr{P}^t(f)=p_t*f$$

is a bounded positive operator from $L^p(d\nu_\alpha)$ into itself and for every $f\in L^p(d\nu_\alpha)$, we have

$$\|\mathscr{P}^t(f)\|_{p,\nu_\alpha}\leqslant\|f\|_{p,\nu_\alpha}.$$

ii. *For every $p\in[1,+\infty[$, the family $(\mathscr{P}^t)_{t>0}$ is a strongly continuous semigroup of operators on $L^p(d\nu_\alpha)$, that is*

- *For $s,t>0$; $\mathscr{P}^s\circ\mathscr{P}^t=\mathscr{P}^{s+t}$,*
- *For every $f\in L^p(d\nu_\alpha)$, $\lim_{t\to0^+}\|\mathscr{P}^t(f)-f\|_{p,\nu_\alpha}=0$.*

The family $(\mathscr{P}^t)_{t>0}$ is called Poisson semigroup associated with the Riemann-Liouville operator $\mathscr{R}_\alpha$.

Proof. i. Let $p(r,x)=\frac{2^{\alpha+\frac{3}{2}}\Gamma(\alpha+2)}{\sqrt{\pi}}\frac{1}{(1+r^2+x^2)^{\alpha+2}}$, p is a measurable positive function and we have

$$p_t(r,x)=\frac{1}{t^{2\alpha+3}}p\Big(\frac{r}{t},\frac{x}{t}\Big).$$

So,

$$\int_0^\infty\int_{\mathbb{R}} p_t(r,x)\,d\nu_\alpha(r,x) = \int_0^\infty\int_{\mathbb{R}} p(r,x)\,d\nu_\alpha(r,x) = 1. \tag{3.23}$$

From relation (2.7), for every $f \in L^p(d\nu_\alpha)$; and every $t > 0$, the function $\mathscr{P}^t(f) = p_t * f$ belongs to $L^p(d\nu_\alpha)$ and we have

$$\|\mathscr{P}^t(f)\|_{p,\nu_\alpha} \leqslant \|p_t\|_{1,\nu_\alpha}\|f\|_{p,\nu_\alpha} = \|f\|_{p,\nu_\alpha}.$$

ii. From relation (3.21), we have

$$\forall(\mu,\lambda) \in \mathbb{R}^2;\ \widetilde{\mathscr{F}_\alpha}(p_t)(\mu,\lambda) = e^{-t\sqrt{\mu^2+\lambda^2}}.$$

So, from relation (3.17); for $s,t > 0$; we get

$$\widetilde{\mathscr{F}_\alpha}(p_t * p_s)(\mu,\lambda) = e^{-(t+s)\sqrt{\mu^2+\lambda^2}} = \widetilde{\mathscr{F}_\alpha}(p_{t+s})(\mu,\lambda),$$

and consequently; $p_s * p_t = p_{s+t}$ which involves that for every $f \in L^p(d\nu_\alpha)$;

$$\mathscr{P}^s(\mathscr{P}^t(f)) = \mathscr{P}^{s+t}(f).$$

Moreover, from relations (2.8) and (3.23),

$$\lim_{t\to 0^+} \|\mathscr{P}^t(f) - f\|_{p,\nu_\alpha} = 0.$$

This finishes the proof.

Lemma 3.10
We have the following connexion between the Gauss and Poisson semigroups, that is

$$\mathscr{P}^t(f)(r,x) = \frac{1}{\sqrt{\pi}}\int_0^\infty \frac{e^{-u}}{\sqrt{u}}\mathscr{G}^{\frac{t^2}{4u}}(f)(r,x)\,du.$$

Proof. Let $f \in L^p(d\nu_\alpha)$, $p \in [1,+\infty]$; for every $(r,x,t) \in \mathbb{R}^2\times]0,+\infty[$, the integral

$$\frac{1}{\sqrt{\pi}}\int_0^\infty \frac{e^{-u}}{\sqrt{u}}\mathscr{G}^{\frac{t^2}{4u}}(f)(r,x)\,du$$

is well defined.

Moreover, from relations (2.5), (3.19) and applying Fubini's theorem, we get

$$\begin{aligned}
&\frac{1}{\sqrt{\pi}}\int_0^\infty \frac{e^{-u}}{\sqrt{u}}\mathscr{G}^{\frac{t^2}{4u}}(f)(r,x)\,du \\
&\quad= \frac{1}{\sqrt{\pi}}\int_0^\infty \frac{e^{-u}}{\sqrt{u}}\left(\int_0^\infty\int_{\mathbb{R}} \tau_{(r,-x)}(\check{f})(s,y)g_{\frac{t^2}{4u}}(s,y)\,d\nu_\alpha(s,y)\right)du \\
&\quad= \int_0^\infty\int_{\mathbb{R}} \tau_{(r,-x)}(\check{f})(s,y)\left(\frac{1}{\sqrt{\pi}}\int_0^\infty \frac{e^{-u}}{\sqrt{u}}g_{\frac{t^2}{4u}}(s,y)\,du\right)d\nu_\alpha(s,y).
\end{aligned}$$

By relation (3.22), we obtain

$$\begin{aligned}\frac{1}{\sqrt{\pi}}\int_0^{\infty}\frac{e^{-u}}{\sqrt{u}}\mathscr{G}^{\frac{t^2}{4u}}(f)(r,x)\,du &= \int_0^{\infty}\int_{\mathbb{R}}\tau_{(r,-x)}(\breve{f})(s,y)p_t(s,y)\,d\nu_\alpha(s,y)\\ &= \mathscr{P}^t(f)(r,x).\end{aligned}$$

Proposition 3.11
For every $f\in\mathscr{D}_e(\mathbb{R}^2)$, the maximal function f^* defined on $\mathbb{R}^2$ by

$$f^*(r,x)=\sup_{t>0}|\mathscr{P}^t(f)(r,x)| \tag{3.24}$$

belongs to the space $L^p(d\nu_\alpha)$; $p\in]1,+\infty[$, and we have

$$\|f^*\|_{p,\nu_\alpha}\leqslant 2\Big(\frac{p}{p-1}\Big)^{\frac{1}{p}}\|f\|_{p,\nu_\alpha}. \tag{3.25}$$

Proof. Let $f\in\mathscr{D}_e(\mathbb{R}^2)$. From Lemma 3.10, for every $(r,x,t)\in\mathbb{R}^2\times]0,+\infty[$, we have

$$\mathscr{P}^t(f)(r,x)=\frac{1}{\sqrt{\pi}}\int_0^{\infty}\frac{e^{-u}}{\sqrt{u}}\mathscr{G}^{\frac{t^2}{4u}}(f)(r,x)\,du=\frac{t}{2\sqrt{\pi}}\int_0^{\infty}\frac{e^{\frac{-t^2}{4s}}}{s^{\frac{3}{2}}}\mathscr{G}^s(f)(r,x)\,ds.$$

Integrating by parts and using the fact that for every $s>0$, $|\int_0^s\mathscr{G}^u(f)(r,x)\,du|\leqslant s\|f\|_{\infty,\nu_\alpha}$, we get

$$\mathscr{P}^t(f)(r,x)=-\frac{t}{2\sqrt{\pi}}\int_0^{\infty}s\frac{d}{ds}\Big[\frac{e^{\frac{-t^2}{4s}}}{s^{\frac{3}{2}}}\Big]\Big[\frac{1}{s}\int_0^s\mathscr{G}^u(f)(r,x)\,du\Big]\,ds.$$

Thus, for every $(r,x,t)\in\mathbb{R}^2\times]0,+\infty[$; we have

$$|\mathscr{P}^t(f)(r,x)|\leqslant\mathscr{M}(f)(r,x)\Big|\frac{t}{2\sqrt{\pi}}\int_0^{\infty}s\frac{d}{ds}\Big(\frac{e^{\frac{-t^2}{4s}}}{s^{\frac{3}{2}}}\Big)\,ds\Big|=\mathscr{M}(f)(r,x).$$

So; for every $(r,x)\in\mathbb{R}^2$; $f^*(r,x)\leqslant\mathscr{M}(f)(r,x)$; where $\mathscr{M}(f)$ is the maximal function defined by relation (3.20). Using Proposition 3.5, we deduce that

$$\|f^*\|_{p,\nu_\alpha}\leqslant 2\Big(\frac{p}{p-1}\Big)^{\frac{1}{p}}\|f\|_{p,\nu_\alpha}.$$

4. The Littlewood-Paley g-function associated with the Riemann-Liouville operator

This section is devoted to study the boundedness of the g-function. We start this section by some intermediate results.

Lemma 4.1
Let f be a function of $\mathscr{S}_e(\mathbb{R}^2)$; and let $\mathscr{U}(f)$ be the function defined on $\mathbb{R}^2\times]0,+\infty[$ by

$$\mathscr{U}(f)(r,x,t)=\mathscr{P}^t(f)(r,x)=p_t*f(r,x).$$

Then for every $k\in\mathbb{N}$, and $(r,x,t)\in\mathbb{R}^2\times]0,+\infty[$, we have

$$\Big|\Big(\frac{\partial}{\partial t}\Big)^k(\mathscr{U}(f))(r,x,t)\Big|\leqslant\frac{\Gamma(2\alpha+k+3)}{2^{\alpha+\frac{1}{2}}\Gamma(\alpha+\frac{3}{2})}\frac{\|f\|_{1,\nu_\alpha}}{t^{2\alpha+k+3}}. \tag{4.26}$$

Proof. From the proof of Proposition 3.8 and for every $(r,x,t)\in\mathbb{R}^2\times]0,+\infty[$, we have

$$\mathscr{U}(f)(r,x,t)=\int_0^\infty\int_{\mathbb{R}}e^{-t\sqrt{\mu^2+\lambda^2}}\widetilde{\mathscr{F}_\alpha}(f)(\mu,\lambda)j_\alpha(r\mu)e^{i\lambda x}\,d\nu_\alpha(\mu,\lambda). \tag{4.27}$$

So, for every $k\in\mathbb{N}$,

$$\begin{aligned}&\Big(\frac{\partial}{\partial t}\Big)^k(\mathscr{U}(f))(r,x,t)\\&\quad=(-1)^k\int_0^\infty\int_{\mathbb{R}}(\mu^2+\lambda^2)^{\frac{k}{2}}e^{-t\sqrt{\mu^2+\lambda^2}}\widetilde{\mathscr{F}_\alpha}(f)(\mu,\lambda)j_\alpha(r\mu)e^{i\lambda x}\,d\nu_\alpha(\mu,\lambda).\end{aligned}$$

Consequently, for every $(r,x,t)\in\mathbb{R}^2\times]0,+\infty[$;

$$\begin{aligned}\Big|\Big(\frac{\partial}{\partial t}\Big)^k(\mathscr{U}(f))(r,x,t)\Big|&\leqslant\|\widetilde{\mathscr{F}_\alpha}(f)\|_{\infty,\nu_\alpha}\int_0^\infty\int_{\mathbb{R}}(\mu^2+\lambda^2)^{\frac{k}{2}}e^{-t\sqrt{\mu^2+\lambda^2}}d\nu_\alpha(\mu,\lambda)\\&\leqslant\|f\|_{1,\nu_\alpha}\int_0^\infty\int_{\mathbb{R}}(\mu^2+\lambda^2)^{\frac{k}{2}}e^{-t\sqrt{\mu^2+\lambda^2}}\,d\nu_\alpha(\mu,\lambda)\\&=\frac{\Gamma(2\alpha+k+3)\|f\|_{1,\nu_\alpha}}{2^{\alpha+\frac{1}{2}}\Gamma(\alpha+\frac{3}{2})}\frac{1}{t^{2\alpha+k+3}}.\end{aligned}$$

Lemma 4.2
Let f be a function of $\mathscr{D}_e(\mathbb{R}^2)$ and let a be a positive real number such that $\mathrm{supp}(f)\subset B_a=\{(r;x)\in\mathbb{R}^2,\ r^2+x^2\leqslant a^2\}$. *Then for every $(r,x,t)\in\mathbb{R}^2\times]0,+\infty[$ such that $r^2+x^2\geqslant 4a^2$, we have*

$$|\mathscr{U}(f)(r,x,t)|\leqslant\frac{a^{2\alpha+3}2^{2\alpha+4}\Gamma(\alpha+2)}{(2\alpha+3)\Gamma(\alpha+\frac{3}{2})\sqrt{\pi}}\frac{\|f\|_{\infty,\nu_\alpha}}{(t^2+r^2+x^2)^{\alpha+\frac{3}{2}}} \tag{4.28}$$

$$\Big|\frac{\partial}{\partial r}(\mathscr{U}(f))(r,x,t)\Big|\leqslant\frac{a^{2\alpha+3}2^{2\alpha+8}\Gamma(\alpha+3)}{(2\alpha+3)\Gamma(\alpha+\frac{3}{2})\sqrt{\pi}}\frac{\|f\|_{\infty,\nu_\alpha}}{(t^2+r^2+x^2)^{\alpha+2}}, \tag{4.29}$$

$$\Big|\frac{\partial}{\partial x}(\mathscr{U}(f))(r,x,t)\Big|\leqslant\frac{a^{2\alpha+3}2^{2\alpha+8}\Gamma(\alpha+3)}{(2\alpha+3)\Gamma(\alpha+\frac{3}{2})\sqrt{\pi}}\frac{\|f\|_{\infty,\nu_\alpha}}{(t^2+r^2+x^2)^{\alpha+2}}. \tag{4.30}$$

Proof. From relation (2.4) and Lemma 3.7, we have

$$\begin{aligned}\tau_{(r,-x)}(p_t)(s,y) \qquad\qquad (4.31)\\ &= \frac{2^{\alpha+\frac{3}{2}}\Gamma(\alpha+2)}{\sqrt{\pi}}\frac{\Gamma(\alpha+1)}{\sqrt{\pi}\Gamma(\alpha+\frac{1}{2})}\int_0^\pi \frac{t\sin^{2\alpha}\theta d\theta}{(t^2+(r^2+s^2+2rs\cos\theta)+(x-y)^2)^{\alpha+2}}\\ &\leqslant \frac{2^{\alpha+\frac{3}{2}}\Gamma(\alpha+2)}{\sqrt{\pi}}\frac{t}{(t^2+(r-s)^2+(x-y)^2)^{\alpha+2}}\frac{\Gamma(\alpha+1)}{\sqrt{\pi}\,\Gamma(\alpha+\frac{1}{2})}\int_0^\pi \sin^{2\alpha}\theta\,d\theta\end{aligned}$$

and

$$\tau_{(r,-x)}(p_t)(s,y) \leqslant \frac{2^{\alpha+\frac{3}{2}}\Gamma(\alpha+2)}{\sqrt{\pi}}\frac{t}{(t^2+(r-s)^2+(x-y)^2)^{\alpha+2}}. \qquad (4.32)$$

Let $f \in \mathscr{D}_e(\mathbb{R}^2)$; $\mathrm{supp}(f) \subset B_a$. We have

$$\begin{aligned}\mathscr{U}(f)(r,x,t) = p_t * f(r,x) &= \int_0^\infty\int_{\mathbb{R}} \tau_{(r,-x)}(p_t)(s,y)f(s,y)\,d\nu_\alpha(s,y)\\ &= \iint_{B_a^+} \tau_{(r,-x)}(p_t)(s,y)f(s,y)\,d\nu_\alpha(s,y),\end{aligned}$$

where

$$B_a^+ = \{(r,x);\ r^2+x^2 \leqslant a^2,\ r \geqslant 0\}.$$

From relation (4.32), for every $(r,x,t) \in \mathbb{R}^2\times]0,+\infty[$;

$$\begin{aligned}|\mathscr{U}(f)(r,x,t)| &\leqslant \frac{2^{\alpha+\frac{3}{2}}\Gamma(\alpha+2)\|f\|_{\infty,\nu_\alpha}}{\sqrt{\pi}}\iint_{B_a^+}\frac{t\,d\nu_\alpha(s,y)}{(t^2+(r-s)^2+(x-y)^2)^{\alpha+2}}\\ &\leqslant \frac{2^{\alpha+\frac{3}{2}}\Gamma(\alpha+2)\|f\|_{\infty,\nu_\alpha}}{\sqrt{\pi}}\iint_{B_a^+}\frac{d\nu_\alpha(s,y)}{(t^2+(r-s)^2+(x-y)^2)^{\alpha+\frac{3}{2}}}\\ &= \frac{2^{\alpha+\frac{3}{2}}\Gamma(\alpha+2)\|f\|_{\infty,\nu_\alpha}}{\sqrt{\pi}}\iint_{B_a^+}\frac{d\nu_\alpha(s,y)}{(t^2+\|(r,x)-(s,y)\|^2)^{\alpha+\frac{3}{2}}}.\end{aligned}$$

For every $(r,x) \in \mathbb{R}^2$ such that $r^2+x^2 \geqslant 4a^2$ and for every $(s,y) \in B_a^+$; we have

$$\|(r,x)-(s,y)\| \geqslant \|(r,x)\| - \|(s,y)\| = \sqrt{r^2+x^2} - \sqrt{s^2+y^2} \geqslant \frac{1}{2}\|(r,x)\|.$$

This implies that for every $(r,x,t) \in \mathbb{R}^2\times]0,+\infty[$; $r^2+x^2 \geqslant 4a^2$,

$$\begin{aligned}|\mathscr{U}(f)(r,x,t)| &\leqslant \frac{2^{3\alpha+\frac{9}{2}}\Gamma(\alpha+2)}{\sqrt{\pi}}\frac{\|f\|_{\infty,\nu_\alpha}}{(t^2+r^2+x^2)^{\alpha+\frac{3}{2}}}\nu_\alpha(B_a^+)\\ &= \frac{a^{2\alpha+3}2^{2\alpha+4}\Gamma(\alpha+2)}{(2\alpha+3)\Gamma(\alpha+\frac{3}{2})\sqrt{\pi}}\|f\|_{\infty,\nu_\alpha}\frac{1}{(t^2+r^2+x^2)^{\alpha+\frac{3}{2}}}.\end{aligned}$$

From relation (4.31); we have

$$\frac{\partial}{\partial r}(\tau_{(r,-x)}(p_t)(s,y)) = \frac{2^{\alpha+\frac{3}{2}}\Gamma(\alpha+2)}{\sqrt{\pi}}\frac{\Gamma(\alpha+1)}{\sqrt{\pi}\Gamma(\alpha+\frac{1}{2})}(-2(\alpha+2)) \times \int_0^{\pi}\frac{t(r+s\cos\theta)\sin^{2\alpha}\theta\,d\theta}{(t^2+(r^2+s^2+2rs\cos\theta)+(x-y)^2)^{\alpha+3}},$$

and consequently,

$$\left|\frac{\partial}{\partial r}(\tau_{(r,-x)}(p_t)(s,y))\right| \leqslant \frac{2^{\alpha+\frac{5}{2}}\Gamma(\alpha+3)}{\sqrt{\pi}}\frac{t(r+s)}{(t^2+(r-s)^2+(x-y)^2)^{\alpha+3}} \leqslant \frac{2^{\alpha+\frac{5}{2}}\Gamma(\alpha+3)}{\sqrt{\pi}}\frac{r+s}{(t^2+(r-s)^2+(x-y)^2)^{\alpha+\frac{5}{2}}}.$$

Hence,

$$\left|\frac{\partial}{\partial r}(\mathscr{U}(f))(r,s,t)\right| \leqslant \iint_{B_a^+}\left|\frac{\partial}{\partial r}(\tau_{(r,-x)}(p_t)(s,y))\right||f(s,y)|\,d\nu_\alpha(s,y) \leqslant \frac{2^{\alpha+\frac{5}{2}}\Gamma(\alpha+3)}{\sqrt{\pi}}\|f\|_{\infty,\nu_\alpha}\iint_{B_a^+}\frac{(r+s)d\nu_\alpha(s,y)}{(t^2+(r-s)^2+(x-y)^2)^{\alpha+\frac{5}{2}}}.$$

But, for every (r,x); $r^2+x^2 \geqslant 4a^2$ and every $(s,y)\in B_a^+$; we have

$$\frac{r+s}{(t^2+(r-s)^2+(x-y)^2)^{\alpha+\frac{5}{2}}} \leqslant \frac{2\sqrt{r^2+x^2}}{(t^2+\frac{1}{4}(r^2+x^2))^{\alpha+\frac{5}{2}}} \leqslant \frac{2^{2\alpha+6}}{(t^2+r^2+x^2)^{\alpha+2}}.$$

This implies that

$$\left|\frac{\partial}{\partial r}(\mathscr{U}(f))(r,s,t)\right| \leqslant \frac{2^{3\alpha+\frac{17}{2}}\Gamma(\alpha+3)}{\sqrt{\pi}}\|f\|_{\infty,\nu_\alpha}\frac{1}{(t^2+r^2+x^2)^{\alpha+2}}\nu_\alpha(B_a^+).$$

Then, the result follows from the fact that

$$\nu_\alpha(B_a^+) = \frac{a^{2\alpha+3}}{(2\alpha+3)2^{\alpha+\frac{1}{2}}\Gamma(\alpha+\frac{3}{2})}.$$

We get the result (4.30) as the same way as the precedent inequality.

Theorem 4.3
Let Δ_α be the partial differential operator defined by

$$\Delta_\alpha = \Lambda_\alpha + \frac{\partial^2}{\partial t^2},$$

where Λ_α is given by relation (3.18). Then, for every non negative function $f \in \mathscr{D}_e(\mathbb{R}^2)$ and every $p \in]1,2]$, we have

$$\int_0^\infty\int_0^\infty\int_{\mathbb{R}} \Delta_\alpha((\mathscr{U}(f))^p)(r,x,t)\,d\nu_\alpha(r,x)t\,dt = \|f\|_{p,\nu_\alpha}^p. \tag{4.33}$$

Proof. Let f be a non negative function, $f \in \mathscr{D}_e(\mathbb{R}^2)$. Then $\mathscr{U}(f)$ is a positive function and from Proposition 3.8,

$$\Delta_\alpha(\mathscr{U}(f)) = 0.$$

Moreover; we have

$$\Delta_\alpha((\mathscr{U}(f))^p) = p(p-1)(\mathscr{U}(f))^{p-2}|\nabla(\mathscr{U}(f))|^2 \geqslant 0, \tag{4.34}$$

where

$$\nabla(\mathscr{U}(f)) = \Big(\frac{\partial}{\partial r}(\mathscr{U}(f)), \frac{\partial}{\partial x}(\mathscr{U}(f)), \frac{\partial}{\partial t}(\mathscr{U}(f))\Big).$$

Then, we have

$$\begin{aligned}&\int_0^\infty\int_0^\infty\int_{\mathbb{R}} \Delta_\alpha((\mathscr{U}(f))^p)(r,x,t)\,d\nu_\alpha(r,x)t\,dt\\ &= \lim_{A\to+\infty}\int_0^A\int_0^A\int_{-A}^A \Big(\Lambda_\alpha((\mathscr{U}(f))^p)(r,x,t) + \frac{\partial^2}{\partial t^2}((\mathscr{U}(f))^p)(r,x,t)\Big)\,d\nu_\alpha(r,x)t\,dt.\end{aligned}$$

From relation (4.27); we deduce that for every $(r,x,t) \in \mathbb{R}^2\times]0,+\infty[$ and for every $k \in \mathbb{N}$; we have

$$\Big|\frac{\partial^k}{\partial t^k}(\mathscr{U}(f))(r,x,t)\Big| \leqslant \int_0^\infty\int_{\mathbb{R}} (\mu^2+\lambda^2)^{\frac{k}{2}}|\widetilde{\mathscr{F}_\alpha}(f)(\mu,\lambda)|\,d\nu_\alpha(\mu,\lambda) < +\infty.$$

It follows that, the function

$$\begin{aligned}&\frac{\partial^2}{\partial t^2}((\mathscr{U}(f))^p)\\ &\qquad = p(p-1)(\mathscr{U}(f))^{p-2}\Big(\frac{\partial}{\partial t}(\mathscr{U}(f))\Big)^2 + p(\mathscr{U}(f))^{p-1}\frac{\partial^2(\mathscr{U}(f))}{\partial t^2}\end{aligned}$$

is bounded on $[0,A]\times[-A,A]\times[0,A]$.

As the same way; the function

$$\Lambda_\alpha((\mathscr{U}(f))^p) = \frac{\partial^2}{\partial r^2}((\mathscr{U}(f))^p) + \frac{2\alpha+1}{r}\frac{\partial}{\partial r}((\mathscr{U}(f))^p) + \frac{\partial^2}{\partial x^2}((\mathscr{U}(f))^p)$$

is bounded on $[0,A]\times[-A,A]\times[0,A]$.

Then, by Fubini's theorem; we get

$$\int_0^A\int_0^A\int_{-A}^A \Delta_\alpha((\mathscr{U}(f))^p)(r,x,t)\,d\nu_\alpha(r,x)t\,dt = I_1(A)+I_2(A)+I_3(A), \tag{4.35}$$

where

$$I_1(A) = C_\alpha \int_0^A\int_{-A}^A \left(\int_0^A \frac{\partial}{\partial r}\Big[r^{2\alpha+1}\frac{\partial}{\partial r}((\mathscr{U}(f))^p)\Big](r,x,t)dr\right)dx\,t\,dt,$$

$$I_2(A) = C_\alpha \int_0^A\int_0^A \left(\int_{-A}^A \frac{\partial^2}{\partial x^2}[(\mathscr{U}(f))^p](r,x,t)\,dx\right)r^{2\alpha+1}\,dr\,t\,dt,$$

$$I_3(A) = \int_0^A\int_{-A}^A \left(\int_0^A \Big(\frac{\partial}{\partial t}\Big)^2[(\mathscr{U}(f))^p](r,x,t)t\,dt\right)d\nu_\alpha(r,x),$$

with $C_\alpha = \frac{1}{2^\alpha\Gamma(\alpha+1)\sqrt{2\pi}}$.

Now,

$$I_1(A) = pC_\alpha \int_0^A\int_{-A}^A A^{2\alpha+1}\frac{\partial}{\partial r}(\mathscr{U}(f))(A,x,t)(\mathscr{U}(f))^{p-1}(A,x,t)\,dx\,t\,dt.$$

Let $a>0$ such that $\operatorname{supp}(f)\subset B_a$ and let $A\geqslant 2a$. By relations (4.28) and (4.29), we have

$$|I_1(A)| \leqslant \frac{C_1A^{2\alpha+4}}{A^{(2\alpha+3)(p-1)}A^{2\alpha+4}} = \frac{C_1}{A^{(2\alpha+3)(p-1)}};$$

which involves that

$$\lim_{A\to+\infty} I_1(A) = 0. \tag{4.36}$$

As the same way;

$$\begin{aligned}I_2(A) = pC_\alpha \int_0^A\int_0^A \Big[&\frac{\partial}{\partial x}(\mathscr{U}(f))(r,A,t)(\mathscr{U}(f))^{p-1}(r,A,t)\\ &-\frac{\partial}{\partial x}(\mathscr{U}(f))(r,-A,t)(\mathscr{U}(f))^{p-1}(r,-A,t)\Big]r^{2\alpha+1}\,dr\,t\,dt,\end{aligned}$$

and by relations (4.28) and (4.30); we obtain

$$|I_2(A)| \leqslant \frac{C_2\ A^{2\alpha+4}}{A^{(2\alpha+3)(p-1)}A^{2\alpha+4}} = \frac{C_2}{A^{(2\alpha+3)(p-1)}};$$

so,

$$\lim_{A\to+\infty} I_2(A) = 0. \tag{4.37}$$

Let us checking the integral $I_3(A)$. We have

$$\int_0^A \Big(\frac{\partial}{\partial t}\Big)[(\mathscr{U}(f))^p](r,x,t)t\,dt$$
$$= pA\frac{\partial}{\partial t}(\mathscr{U}(f))(r,x,A)(\mathscr{U}(f))^{p-1}(r,x,A) - (\mathscr{U}(f))^p(r,x,A) + f^p(r,x).$$

However,

$$\int_0^A\int_{-A}^A \mathscr{U}^p(f)(r,x,A)\,d\nu_\alpha(r,x) \leqslant \int_0^\infty\int_{\mathbb{R}} \mathscr{U}^p(f)(r,x,A)\,d\nu_\alpha(r,x)$$
$$= \|p_A * f\|^p_{p,\nu_\alpha} \leqslant \|p_A\|^p_{p,\nu_\alpha}\|f\|^p_{1,\nu_\alpha}.$$

By a simple computation and using Lemma 3.7, we deduce that

$$\lim_{A\to+\infty}\|p_A\|^p_{p,\nu_\alpha} = 0,$$

and then

$$\lim_{A\to+\infty}\int_0^A\int_{-A}^A \mathscr{U}^p(f)(r,x,A)\,d\nu_\alpha(r,x) = 0.$$

On the other hand, by relation (4.26), we have

$$pA\int_0^A\int_{-A}^A \Big|\frac{\partial}{\partial t}(\mathscr{U}(f))(r,x,A)\Big|(\mathscr{U}(f))^{p-1}(r,x,A)\,d\nu_\alpha(r,x) \leqslant \frac{C_3}{A^{(2\alpha+3)(p-1)}},$$

which implies that

$$\lim_{A\to+\infty} pA\int_0^A\int_{-A}^A \frac{\partial}{\partial t}(\mathscr{U}(f))(r,x,A)(\mathscr{U}(f))^{p-1}(r,x,A)\,d\nu_\alpha(r,x) = 0.$$

hence,

$$\lim_{A\to+\infty} I_3(A) = \int_0^\infty\int_{\mathbb{R}} (f(r,x))^p\,d\nu_\alpha(r,x) = \|f\|^p_{p,\nu_\alpha}. \tag{4.38}$$

Then, the desired result follows from relations (4.35), (4.36), (4.37) and (4.38) .

Definition 4.4

The Littlewood-Paley g-function associated with the Riemann-Liouville operator is defined for $f \in \mathscr{D}_e(\mathbb{R}^2)$ by

$$g(f)(r,x) = \Big(\int_0^\infty |\nabla(\mathscr{U}(f))(r,x,t)|^2 t\,dt\Big)^{\frac{1}{2}}.$$

Let $\mathscr{C}_{c,e}(\mathbb{R}^2)$ be the space of continuous functions on $\mathbb{R}^2$, even with respect to the first variable and with compact support.

In the following, we need the coming result.

Lemma 4.5
Let g be a non negative function, $g \in \mathscr{C}_{c,e}(\mathbb{R}^2)$; $\mathrm{supp}(g) \subset B_a$. For every ε; $0 < \varepsilon < 1$, there exists a non negative function $f \in \mathscr{D}_e(\mathbb{R}^2)$ such that

$$\forall (r,x) \in \mathbb{R}^2;\ 0 \leqslant f(r,x) - g(r,x) \leqslant \varepsilon,$$

with $\mathrm{supp}(f) \subset B_{a+2}$.

Proof. It is well known that for every non negative function h; $h \in \mathscr{C}_{c,e}(\mathbb{R}^2)$, $\mathrm{supp}(h) \subset B_a$ and for every $\eta > 0$, there is a non negative function $f \in \mathscr{D}_e(\mathbb{R}^2)$, $\mathrm{supp}(f) \subset B_{a+1}$ such that

$$\forall (r,x) \in \mathbb{R}^2;\ -\eta \leqslant f(r,x) - h(r,x) \leqslant \eta. \tag{4.39}$$

Let g be a non negative function in $\mathscr{C}_{c,e}(\mathbb{R}^2)$, $\mathrm{supp}(g) \subset B_a$ and let $\varepsilon \in \mathbb{R}$, $0 < \varepsilon < 1$. We define the function θ by

$$\theta(r,x) = \begin{cases} g(r,x) + \dfrac{\varepsilon}{2}, & \text{if } r^2 + x^2 \leqslant a^2; \\ -\sqrt{r^2+x^2} + a + \dfrac{\varepsilon}{2}, & \text{if } a^2 \leqslant r^2 + x^2 \leqslant \Big(a + \dfrac{\varepsilon}{2}\Big)^2; \\ 0, & \text{if } r^2 + x^2 \geqslant \Big(a + \dfrac{\varepsilon}{2}\Big)^2. \end{cases}$$

Then θ is a non negative function, θ belongs to the space $\mathscr{C}_{c,e}(\mathbb{R}^2)$ and $\mathrm{supp}(\theta) \subset B_{a+1}$.

From relation (4.39), there exists a non negative function $f \in \mathscr{D}_e(\mathbb{R}^2)$ such that $\mathrm{supp}(f) \subset B_{a+2}$, and

$$\forall (r,x) \in \mathbb{R}^2;\ -\frac{\varepsilon}{4} \leqslant f(r,x) - \theta(r,x) \leqslant \frac{\varepsilon}{4}.$$

Thus, the function f satisfies

$$\forall (r,x) \in \mathbb{R}^2;\ 0 \leqslant f(r,x) - g(r,x) \leqslant \varepsilon,$$

with $\mathrm{supp}(f) \subset B_{a+2}$.

Proposition 4.6
For every $p \in]1,2]$, and for every function $f \in \mathscr{D}_e(\mathbb{R}^2)$, the function $g(f)$ belongs to the space $L^p(d\nu_\alpha)$ and we have

$$\|g(f)\|_{p,\nu_\alpha} \leqslant 2\frac{2^{\frac{2-p}{2}}}{p}\Big(\frac{p}{p-1}\Big)^{\frac{1}{p}}\|f\|_{p,\nu_\alpha}.$$

Proof. Let f be a non negative function; $f \in \mathscr{D}_e(\mathbb{R}^2)$. From relation (4.34), we have

$$|\nabla(\mathscr{U}(f))(r,x,t)|^2 = \frac{1}{p(p-1)}(\mathscr{U}(f))^{2-p}(r,x,t)\Delta_\alpha(\mathscr{U}^p(f))(r,x,t).$$

For $p = 2$ and using relation (4.33), we obtain

$$\begin{aligned}\int_0^\infty\int_{\mathbb{R}} g^2(f)(r,x)\,d\nu_\alpha(r,x) &= \int_0^\infty\int_{\mathbb{R}}\left(\int_0^\infty |\nabla(\mathscr{U}(f))(r,x,t)|^2 t\,dt\right)d\nu_\alpha(r,x)\\ &= \frac{1}{2}\int_0^\infty\int_{\mathbb{R}}\int_0^\infty \Delta_\alpha(\mathscr{U}^2(f))(r,x,t)t\,dt\,d\nu_\alpha(r,x)\\ &= \frac{1}{2}\int_0^\infty\int_{\mathbb{R}}(f(r,x))^2\,d\nu_\alpha(r,x).\end{aligned}$$

This means that

$$\|g(f)\|_{2,\nu_\alpha} = \frac{1}{\sqrt{2}}\|f\|_{2,\nu_\alpha}.$$

For $p \in]1,2[$, we have

$$\begin{aligned}&\int_0^\infty\int_{\mathbb{R}}(g(f))^p(r,x)\,d\nu_\alpha(r,x)\\ &\quad= \int_0^\infty\int_{\mathbb{R}}\left(\int_0^\infty |\nabla(\mathscr{U}(f))(r,x,t)|^2 t\,dt\right)^{\frac{p}{2}} d\nu_\alpha(r,x)\\ &\quad= \Big(\frac{1}{p(p-1)}\Big)^{\frac{p}{2}}\int_0^\infty\int_{\mathbb{R}}\left(\int_0^\infty \mathscr{U}^{2-p}(f)(r,x,t)\Delta_\alpha(\mathscr{U}^p(f))(r,x,t)t\,dt\right)^{\frac{p}{2}} d\nu_\alpha(r,x)\\ &\quad\leqslant \Big(\frac{1}{p(p-1)}\Big)^{\frac{p}{2}}\int_0^\infty\int_{\mathbb{R}}(f^*(r,x))^{(2-p)\frac{p}{2}}\left(\int_0^\infty \Delta_\alpha(\mathscr{U}^p(f))(r,x,t)t\,dt\right)^{\frac{p}{2}} d\nu_\alpha(r,x),\end{aligned}$$

where f^* is the maximal function defined by relation (3.24).

Using Hölder's inequality and relation (4.33), we get

$$\begin{aligned}&\int_0^\infty\int_{\mathbb{R}^n}(g(f))^p(r,x)\,d\nu_\alpha(r,x)\\ &\quad\leqslant \Big(\frac{1}{p(p-1)}\Big)^{\frac{p}{2}}\|f^*\|_{p,\nu_\alpha}^{p\frac{(2-p)}{2}}\left(\int_0^\infty\int_{\mathbb{R}}\int_0^\infty \Delta_\alpha(\mathscr{U}^p(f))(r,x,t)t\,dt\,d\nu_\alpha(r,x)\right)^{\frac{p}{2}}\\ &\quad= \Big(\frac{1}{p(p-1)}\Big)^{\frac{p}{2}}\|f^*\|_{p,\nu_\alpha}^{p\frac{(2-p)}{2}}\|f\|_{p,\nu_\alpha}^{p\frac{p}{2}},\end{aligned}$$

and by means of relation (3.25),

$$\int_0^\infty\int_{\mathbb{R}^n}(g(f))^p(r,x)\,d\nu_\alpha(r,x)\leqslant\Big(\frac{1}{p(p-1)}\Big)^{\frac{p}{2}}\Big(2\Big(\frac{p}{p-1}\Big)^{\frac{1}{p}}\Big)^{p\frac{(2-p)}{2}}\|f\|^p_{p,\nu_\alpha},$$

in other words,

$$\|g(f)\|_{p,\nu_\alpha}\leqslant\frac{2^{\frac{2-p}{2}}}{p}\Big(\frac{p}{p-1}\Big)^{\frac{1}{p}}\|f\|_{p,\nu_\alpha}. \tag{4.40}$$

Let $f\in\mathscr{D}_e(\mathbb{R}^2)$; $\mathrm{supp}(f)\subset B_a$ and let $f^+=\frac{f+|f|}{2}$, $f^-=\frac{-f+|f|}{2}$. Then f^+ is a non negative function, $f^+\in\mathscr{C}_{c,e}(\mathbb{R}^2)$. From Lemma 4.5, for every $\varepsilon\in\mathbb{R}$, $0<\varepsilon<1$, there is a non negative function $h_1\in\mathscr{D}_e(\mathbb{R}^2)$, $\mathrm{supp}(h_1)\subset B_{a+2}$ and

$$\forall(r,x)\in\mathbb{R}^2;\ 0\leqslant h_1(r,x)-f^+(r,x)\leqslant\varepsilon. \tag{4.41}$$

Now, the function

$$h_2=h_1-f=h_1-f^++f^-$$

is non negative, belongs to the space $\mathscr{D}_e(\mathbb{R}^2)$ with $\mathrm{supp}(h_2)\subset B_{a+2}$. Moreover

$$\forall(r,x)\in\mathbb{R}^2;\ 0\leqslant h_2(r,x)-f^-(r,x)=h_1(r,x)-f^+(r,x)\leqslant\varepsilon,$$

and we have $f=h_1-h_2$.

Since the mapping $f\longmapsto g(f)$ is sub-linear in the sense that $g(f_1+f_2)\leqslant g(f_1)+g(f_2)$; we deduce that

$$g(f)\leqslant g(h_1)+g(h_2),$$

and applying inequality (4.40), we get

$$\|g(f)\|_{p,\nu_\alpha}\leqslant\|g(h_1)\|_{p,\nu_\alpha}+\|g(h_2)\|_{p,\nu_\alpha}\leqslant\frac{2^{\frac{2-p}{2}}}{p}\Big(\frac{p}{p-1}\Big)^{\frac{1}{p}}(\|h_1\|_{p,\nu_\alpha}+\|h_2\|_{p,\nu_\alpha}).$$

On the other hand, from relation (4.41), we obtain

$$\begin{aligned}\|h_1\|_{p,\nu_\alpha}&=\Bigg(\iint_{B^+_{a+2}}(h_1(r,x))^p\,d\nu_\alpha(r,x)\Bigg)^{\frac{1}{p}}\\&\leqslant\Bigg(\iint_{B^+_{a+2}}(f^+(r,x))^p\,d\nu_\alpha(r,x)\Bigg)^{\frac{1}{p}}+\varepsilon(\nu_\alpha(B^+_{a+2}))^{\frac{1}{p}}\\&\leqslant\|f\|_{p,\nu_\alpha}+\varepsilon(\nu_\alpha(B^+_{a+2}))^{\frac{1}{p}}.\end{aligned}$$

As the same way,

$$\|h_2\|_{p,\nu_\alpha}\leqslant\|f\|_{p,\nu_\alpha}+\varepsilon(\nu_\alpha(B^+_{a+2}))^{\frac{1}{p}}.$$

This means that for every $\varepsilon \in \mathbb{R}$, $0 < \varepsilon < 1$,

$$\|g(f)\|_{p,\nu_\alpha} \leqslant 2\frac{2^{\frac{2-p}{2}}}{p}\Big(\frac{p}{p-1}\Big)^{\frac{1}{p}}(\|f\|_{p,\nu_\alpha} + \varepsilon(\nu_\alpha(B^+_{a+2}))^{\frac{1}{p}}),$$

and consequently,

$$\|g(f)\|_{p,\nu_\alpha} \leqslant 2\frac{2^{\frac{2-p}{2}}}{p}\Big(\frac{p}{p-1}\Big)^{\frac{1}{p}}\|f\|_{p,\nu_\alpha}.$$

The precedent Proposition allows us to prove the followings Theorem, that is the main result of this paper.

Theorem 4.7
For every $p \in]1,2]$; the mapping $f \longmapsto g(f)$ can be extended to the space $L^p(d\nu_\alpha)$ and for every $f \in L^p(d\nu_\alpha)$, we have

$$\|g(f)\|_{p,\nu_\alpha} \leqslant 2\frac{2^{\frac{2-p}{2}}}{2}\Big(\frac{p}{p-1}\Big)^{\frac{1}{p}}\|f\|_{p,\nu_\alpha}.$$

Proof. Let $f \in L^p(d\nu_\alpha)$, then there exists a sequence $(f_k)_k \subset \mathscr{D}_e(\mathbb{R}^2)$ such that

$$\lim_{k\to+\infty}\|f_k - f\|_{p,\nu_\alpha} = 0.$$

Since the mapping $f \longmapsto g(f)$ is sub-linear; then for every $(k,l) \in \mathbb{N}^2$; we have

$$\begin{aligned}\|g(f_{k+l}) - g(f_k)\|_{p,\nu_\alpha} &\leqslant \|g(f_{k+l} - f_k)\|_{p,\nu_\alpha}\\ &\leqslant 2\frac{2^{\frac{2-p}{2}}}{2}\ \Big(\frac{p}{p-1}\Big)^{\frac{1}{p}}\|f_{k+l} - f_k\|_{p,\nu_\alpha}.\end{aligned}$$

Consequently, the sequence $(g(f_k))_k$ is a Cauchy one in $L^p(d\nu_\alpha)$. We put

$$g(f) = \lim_{k\to+\infty} g(f_k)$$

in $L^p(d\nu_\alpha)$.

It is clear that $g(f)$ is independent of the choice of the sequence $(f_k)_k$ and we have

$$\|g(f)\|_{p,\nu_\alpha} = \lim_{k\to+\infty}\|g(f_k)\|_{p,\nu_\alpha} \leqslant 2\frac{2^{\frac{2-p}{2}}}{p}\Big(\frac{p}{p-1}\Big)^{\frac{1}{p}}\|f\|_{p,\nu_\alpha}.$$

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