FOLIA 102
Annales Universitatis Paedagogicae Cracoviensis
Studia Mathematica X (2011)

Report of Meeting

14th International Conference on Functional Equations and Inequalities, Będlewo, September 11-17, 2011

Contents

The *Fourteenth International Conference on Functional Equations and Inequalities* was held from September 11 to 17, 2011 in Będlewo, Poland. The series of ICFEI meetings has been organized by the *Department of Mathematics of the Pedagogical University in Cracow* since 1984. For the fourth time, the conference was organized jointly with the *Stefan Banach International Mathematical Center* and hosted by the *Mathematical Research and Conference Center* in Będlewo.

This year's conference was dedicated to the memory of Professor Marek Kuczma, the founder of the *Polish School of Functional Equations and Inequalities*, who died 20 years ago.

The *Organizing Committee* of the 14th ICFEI consisted of Janusz Brzdęk as Chairman, Zbigniew Leśniak as Vice-Chairman, Anna Bahyrycz, Magdalena Piszczek, Paweł Solarz, Janina Wiercioch and Krzysztof Ciepliński, who also acted as Vice-Chairman and Scientific Secretary.

The *Scientific Committee* consisted of Professors: Dobiesław Brydak as Honorary Chairman, Janusz Brzdęk as Chairman, Nicole Brillouët-Belluot, Jacek Chmieliński, Bogdan Choczewski, Roman Ger, Hans-Heinrich Kairies, László Losonczi, Zsolt Páles and Marek Cezary Zdun.

The 60 participants came from 12 countries: Australia, Austria, Denmark, France, Germany, Hungary, India, Iran, Israel, Russia, Serbia and Poland.

The conference was opened on Monday, September 12 by Professor Janusz Brzdęk – the Chairman of the Scientific and Organizing Committees, who welcomed the participants on behalf of the Organizing Committee and read a letter to them from Professor Władysław Błasiak, the Dean of the *Faculty of Mathe-*

matics, Physics and Technical Science of the Pedagogical University. The opening address was given by Professor Jacek Chmieliński, the Head of the Department of Mathematics. The opening ceremony was followed by the first scientific session chaired by Professor Roman Ger. Altogether, during 22 scientific sessions 3 lectures (given by Professors: Karol Baron – opening lecture, Stevo Stević and Boris Paneah) and 51 talks were delivered. They focused on functional equations in a single variable and in several variables, functional inequalities, stability theory, convexity, multifunctions, means and other topics. Several contributions have been made during special *Problems and Remarks* sessions.

On Tuesday, September 13, a picnic was organized. On the next day afternoon the participants visited Rogalin Palace with its gallery of paintings from the 19th and 20th centuries, and collection of horse-drawn vehicles. In the evening the piano recital was performed by Professor Marek Czerni. On Thursday, September 15, a banquet was held at the Palace in Będlewo. At the start of the banquet, George Gershwin's "Summertime" was performed by Professor Ewelina Mainka-Niemczyk.

The conference was closed on Friday, September 16 by Professor Janusz Brzdęk. He announced that Professors Ekaterina Shulman and Lászlo Székelyhidi had agreed to join the Scientific Committee. The 15th ICFEI will be organized in the south of Poland, in the spring of 2013.

The following part of the report contains the abstracts of the talks, the problems and remarks, and a list of the participants with their addresses.

Abstracts of Talks

Anna Bahyrycz *On the solutions of Wilson first generalization of d'Alembert's functional equation on some set*

Let A be a non-empty subset of an Abelian group. In the talk, under some simple additional assumptions on the set A, we deal with functions $f, g\colon A \to \mathbb{C}$ satisfying the equation

$$f(x+y) + f(x-y) = 2f(x)g(y)$$

for $x, y \in A$ such that $x+y, x-y \in A$.

Karol Baron *Marek Kuczma*

The scientific output of Marek Kuczma consists of 179 papers (listed in [4]) published within the years 1958–1993 and three books [6, 7, 8] still used and quoted. Professor Marek Kuczma created and developed the theory of iterative functional equations but his name is also connected with important results on functional equations in several variables, in particular on Cauchy's equation and Jensen's inequality. In fact Marek Kuczma has founded a mathematical school: he supervised 13 Ph.D. dissertations, 10 his students have already their habilitation and 6 of them became full professors (cf. also [5]). In the talk I would like to present more information about this great teacher and, making also use of [1, 2, 3], some results of this outstanding mathematician.

[1] K. Baron, *M. Kuczma's papers on iterative functional equations*, Selected topics in functional equations and iteration theory (Graz, 1991), 1–6, Grazer Math. Ber. **316**, Karl-Franzens-Univ. Graz, Graz, 1992.

[2] B. Choczewski, *Papers of Marek Kuczma written in the last decade of his life*, Selected topics in functional equations and iteration theory (Graz, 1991), 7–16, Grazer Math. Ber. **316**, Karl-Franzens-Univ. Graz, Graz, 1992.

[3] R. Ger, *M. Kuczma's papers on functional equations in several variables*, Selected topics in functional equations and iteration theory (Graz, 1991), 17–28, Grazer Math. Ber. **316**, Karl-Franzens-Univ. Graz, Graz, 1992.

[4] R. Ger, *Marek Kuczma, 1935-1991*, Aequationes Math. **44** (1992), 1–10.

[5] R. Ger, *Functional equations and inequalities* (Polish), *Half a century of mathematics in Upper Silesia* (Polish), 223–251, Pr. Nauk. Uniw. Śl. Katow. **2196**, Wydawn. Uniw. Śląskiego, Katowice, 2003.

[6] M. Kuczma, *Functional equations in a single variable*, Monografie Matematyczne **46**, Państwowe Wydawnictwo Naukowe, Warszawa, 1968.

[7] M. Kuczma, *An introduction to the theory of functional equations and inequalities. Cauchy's equation and Jensen's inequality*, Pr. Nauk. Uniw. Śl. Katow. **489**, Uniwersytet Śląski, Katowice; Państwowe Wydawnictwo Naukowe, Warszawa, 1985 [Second edition: Edited and with a preface by Attila Gilányi, Birkhaüser Verlag, Basel, 2009].

[8] M. Kuczma, B. Choczewski, R. Ger, *Iterative functional equations*, Encyclopedia of Mathematics and its Applications **32**, Cambridge University Press, Cambridge, 1990.

Janusz Brzdęk *Stability of linear equations of higher orders*
(joint work with **B. Xu** and **W. Zhang**)

We present some fixed point results, which provide a general method for investigations of the Hyers-Ulam stability of the linear operator equations of higher orders. In numerous cases, the Hyers-Ulam stability of such an equation is a consequence of a similar property of the corresponding first order equations. We describe some particular examples of applications for differential, integral, and functional equations.

Jacek Chmieliński *On approximate parallelogram identity in normed spaces*

Suppose that a norm in a real or complex space X approximately satisfies the parallelogram law, i.e.,

$$\big|\|x+y\|^2+\|x-y\|^2-2\|x\|^2-2\|y\|^2\big| \le \Phi(x,y), \qquad x,y\in X$$

with a given mapping $\Phi\colon X\to\mathbb{R}_+$. We show that for some control mappings Φ the above property yields that X is equivalent to an inner product space.

Jacek Chudziak *On continuous solutions of a composite functional equation*

Inspired by some problems concerning invariant utility functions we consider continuous solutions of the following functional equation

$$f(k(t)x+l(t)) = a(t)f(x)+b(t).$$

Krzysztof Ciepliński *A fixed point approach to the stability of functional equations in non-Archimedean metric spaces*
(joint work with **J. Brzdęk**)

In the talk we present a fixed point theorem for complete non-Archimedean metric spaces and apply it to obtain the Hyers-Ulam stability of a quite wide class of functional equations in a single variable.

Marek Czerni *On a generalization of the problem of D. Brydak*

Let I be a real interval of the form $[0,a)$, where $0 < a \leq \infty$. Let $\psi\colon I \to \mathbb{R}$ be a continuous solution of the linear nonhomogeneous functional inequality

$$\psi[f(x)] \leq g(x)\psi(x) + h(x).$$

We assume the following hypotheses about given functions f, g and h:

(H_1) the function $f\colon I \to \mathbb{R}$ is continuous and strictly increasing. Moreover, $0 < f(x) < x$ for $x \in I^\star = I \setminus \{0\}$,

(H_2) the function $g\colon I \to \mathbb{R}$ is continuous and $g(x) > 0$ for $x \in I^\star$,

(H_3) the function $h\colon I \to \mathbb{R}$ is continuous and $h(0) = 0$,

(H_4) the functional sequence $G_n(x) = \prod_{i=0}^{n-1} g[f^i(x)]$ converges to zero almost uniformly in $I^\star$,

(H_5) the functional sequence $\varphi_n^\star(x) = \sum_{i=0}^{n-1} \frac{h[f^i(x)]}{G_{i+1}(x)}$ converges almost uniformly in $I^\star$.

In the talk we give partial answer to the following question: does there always exist a continuous solution $\varphi\colon I \to \mathbb{R}$ of the functional equation

$$\varphi[f(x)] = g(x)\varphi(x) + h(x)$$

or

$$\varphi[f(x)] = g(x)\varphi(x)$$

such that the finite limit $\lim_{x\to 0^+} \frac{\psi(x)}{\varphi(x)}$ exists?

At the 3rd International Symposium on Functional Equations and Inequalities in Noszvaj (Hungary) in September 1986 D. Brydak put the similar problem for linear homogeneous inequality (see [1]). This problem was solved in [2].

[1] *Report on the Third International Symposium on Functional Equations and Inequalities. Abstracts from the symposium held in Noszvaj, September 21-27, 1986*, Publ. Math. Debrecen **38** (1991), 1–38.

[2] M. Czerni, *On a problem of D. Brydak*, Publ. Math. Debrecen **44** (1994), 243–248.

Włodzimierz Fechner *Functional equations with exotic addition*

S. Northshield in [1] introduced the term exotic addition for two types of operations he dealt with. One of them was the following "sine-type" addition on the real line

$$x \oplus y := xf(y) + yf(x)$$

with $f\colon \mathbb{R} \to \mathbb{R}$ enjoying some regularity properties. The name "sine-type" addition which appears in [1] is justified by the fact that if one takes mapping $f\colon [-1,1] \to \mathbb{R}$ given by $f(x) = \sqrt{1-x^2}$ for $x \in [-1,1]$, then in this particular case the sine function acts as a homomorphism between the real line with ordinary addition and the interval $[-1,1]$ with "exotic" addition $\oplus$. In other words, we have

$$\sin(x+y) = \sin(x) \oplus \sin(y), \qquad x, y \in \mathbb{R}.$$

Northshield provided several conditions equivalent to the associativity of $\oplus$ (see Theorem 4 and Corollary 1 in [1]). From these results it follows that the associativity seems to be a fairly restrictive assumption since it implies a particular form of the mapping f (given in an implicit form involving solutions of some ODE's).

We will deal with operation $\oplus$ without assuming its associativity and for mapping f defined on an arbitrary interval. We solve "exotic" modifications of some functional equations, including the equation of derivations, with ordinary addition replaced by $\oplus$.

[1] S. Northshield, *On two types of exotic addition*, Aequationes Math. **77** (2009), 1–23.

Żywilla Fechner *On some integral generalizations of trigonometric functional equations*

Let $(G,+)$ be a locally compact Abelian group, $\mathcal{B}(G)$ the space of all Borel subsets of G and $\mu\colon \mathcal{B}(G) \to \mathbb{C}$ a bounded regular measure. The following equation

$$\int_G \{f(x+y-s) + f(x-y+s)\}\, d\mu(s) = f(x)f(y), \qquad x, y \in G,$$

where $f\colon G \to \mathbb{C}$ is essentially bounded, was introduced and solved by Z. Gajda in [3]. We are going to present some other possible generalizations of this functional equation.

[1] Ż. Fechner, *A generalization of Gajda's equation*, J. Math. Anal. Appl. **354** (2009), 584–593.

[2] Ż. Fechner, *A note on a modification of Gajda's equation*, Aequationes Math. **82** (2011), 135–141.

[3] Z. Gajda, *A generalization of d'Alembert's functional equation*, Funkcial. Ekvac. **33** (1990), 69–77.

[4] L. Székelyhidi, *Convolution type functional equations on topological abelian groups*, World Scientific Publishing Co., Inc., Teaneck, NJ, 1991.

Roman Ger *On a subsequent problem of Roger Cuculière*

In the May 2011 issue of *The American Mathematical Monthly* (**118**, Problems and Solutions, p. 464) the following problem was proposed by Roger Cuculière:

Let E be a real normed vector space of dimension at least 2. Let f be a mapping from E into E, bounded on the unit sphere $\{x \in E : \ \|x\| = 1\}$, such that whenever

x and y are in E, $f(x+f(y)) = f(x)+y$. Prove that f is a continuous, linear involution on E. (Problem **11578**).

We shall present the general solution of the functional equation in question (in much more general setting) from which the proof spoken of will be obtained as a corollary.

Dorota Głazowska *Uniformly bounded composition operators in the space of functions of bounded φ-variation with weight in the sense of Riesz*
(joint work with **J. Matkowski**)

We prove that if a uniformly bounded (or equidistantly bouned) Nemytskij operator maps a space of functions of bounded φ-variation with weight function in the sense of Riesz into another space of the same type and its generator function is continuous with respect to the second variable, then this generator function is an affine function in the second variable.

Moshe Goldberg *Submultiplicativity and stability of sup norms on homotonic algebras*

An algebra $\mathcal{A}$ of real or complex valued functions defined on a set S shall be called *homotonic* if $\mathcal{A}$ is closed under forming of absolute values, and if for all f and g in $\mathcal{A}$, the product $f \times g$ satisfies $|f \times g| \leq |f| \times |g|$. Our purpose in this talk is to offer several examples of homotonic algebras and provide a simple inequality which characterizes submultiplicativity and strong stability for weighted sup norms on such algebras.

Niyati Gurudwan *Strong convergence theorem for finite family of m-accretive operators in Banach spaces*
(joint work with **B.K. Sharma**)

The purpose of this presentation is to propose a composite iterative scheme for approximating a common solution of a finite family of m-accretive (nonlinear) operators in a strictly convex Banach space having a uniformly Gateaux differentiable norm. As a consequence, the strong convergence of the scheme for a common fixed point of a finite family of pseudocontractive mappings is also obtained. The results presented herein improve and extend the corresponding results of Kim and Xu, Qin and Su, Xu, and Zegeye and Shahzad (see [1, 2, 3, 4] and the references given there) to a finite family of operators in a strictly convex Banach space.

[1] T.-H. Kim, H.-K. Xu, *Strong convergence of modified Mann iterations*, Nonlinear Anal. **61** (2005), 51–60.

[2] X. Qin, Y. Su, *Approximation of a zero point of accretive operator in Banach spaces*, J. Math. Anal. Appl. **329** (2007), 415–424.

[3] H.-K. Xu, *Strong convergence of an iterative method for nonexpansive and accretive operators*, J. Math. Anal. Appl. **314** (2006), 631–643.

[4] H. Zegeye, N. Shahzad, *Strong convergence theorems for a common zero for a finite family of m-accretive mappings*, Nonlinear Anal. **66** (2007), 1161–1169.

Attila Házy *On (α, β, a, b)-convex functions*

Bernstein and Doetsch (see [1]) proved that the local upper boundedness of a Jensen-convex function yields its local boundedness and continuity as well on the whole domain, which implies the convexity of the function.

In this talk we present some Bernstein-Doetsch type results for (α, β, a, b)-convex functions, which were intoduced by Maksa and Páles (see [4]) in the following way:

Let X be a real or complex topological vector space, $D \subset X$ be a nonempty open (α, β)-convex (that is, $\alpha(t)x + \beta(t)y \in D$ whenever $x, y \in D$ and $t \in [0,1]$) set, and $\alpha, \beta, a, b\colon [0,1] \to \mathbb{R}$ be given functions. The function f is called (α, β, a, b)-convex function if

$$f(\alpha(t)x + \beta(t)y) \le a(t)f(x) + b(t)f(y), \qquad x, y \in D,\ t \in [0,1]$$

holds. To avoid the trivialities and the unimportant cases, we suppose that there exists an element t_0 such that $\alpha(t_0)\beta(t_0)a(t_0)b(t_0) \neq 0$.

[1] F. Bernstein, G. Doetsch, *Zur Theorie der konvexen Funktionen*, Math. Ann. **76** (1915), 514–526.

[2] P. Burai, A. Házy, *On approximately h-convex functions*, J. Convex Anal. **18** (2011), 447–454.

[3] P. Burai, A. Házy, T. Juhász, *Bernstein-Doetsch type results for s-convex functions*, Publ. Math. Debrecen **75** (2009), 23–31.

[4] Gy. Maksa, Zs. Páles, *The equality case in some recent convexity inequalities*, Opuscula Math. **31** (2011), 269–277.

[5] A. Házy, *Bernstein-Doetsch type results for h-convex functions*, Math. Inequal. Appl. **14** (2011), 499–508.

[6] S. Varošanec, *On h-convexity*, J. Math. Anal. Appl. **326** (2007), 303–311.

Eliza Jabłońska *On the pexiderized Gołąb-Schinzel equation*

Let X be a linear space over a commutative field $\mathbb{K}$. We characterize a general solution $f, g, h, k\colon X \to \mathbb{K}$ of the pexiderized Gołąb-Schinzel equation

$$f(x + g(x)y) = h(x)k(y),$$

as well as, in the case $\mathbb{K} = \mathbb{R}$, continuous on rays solutions of the equation.

Justyna Jarczyk *On some equality problem connected with conjugate means* (joint work with **J. Dascăl**)

Let $I \subset \mathbb{R}$ be an open interval and $p, q \in (0,1)$. We present some partial results on solutions (φ, ψ) of the functional equation

$$\varphi^{-1}\Big(\varphi(x) + \varphi(y) - \frac{1}{2}(\varphi(px + (1-p)y) + \varphi(qx + (1-q)y))\Big) = \psi^{-1}\Big(\frac{\psi(x) + \psi(y)}{2}\Big),$$

where $\varphi, \psi\colon I \to \mathbb{R}$ are four times continuously differentiable functions.

Witold Jarczyk *Note on an equation occurring in a problem of Nicole Brillouët-Belluot*
(joint work with **J. Morawiec**)

We study the functional equation

$$f(x)f^{-1}(x) = x^2$$

imposing no continuity assumptions on its bijective solutions defined on an interval. All the continuous bijections of an interval were determined in [2] when solving a problem posed by N. Brillouët-Belluot (see [1]).

[1] N. Brillouët-Belluot, *Problem posed during the Forty-nine International Symposium on Functional Equations, June 19–26, 2011, Graz-Mariatrost, Austria.*

[2] J. Morawiec, *On a problem of Nicole Brillouët-Belluot,* Aequationes Math. (in print).

Vyacheslav Kalnitsky *Solution of the Kuczma equation*

In the works of M. Kuczma and J. Sándor (see [1, 2]) was noted that all monotone convex (vertex) solutions of a special class of Stamate-type equation

$$\frac{f(x)-f(y)}{x-y} = \varphi(h(x), h(y)),$$

where $\varphi(u,v) = u+v, uv, \frac{1}{u+v}$, are arcs of cone sections. Assuming $h, \varphi \in C$, $\varphi(u,u)$ being invertible, the above equation is reducible to the form

$$\frac{f(x)-f(y)}{x-y} = K(f'(x), f'(y)), \tag{1}$$

where K is necessarily generalized mean. In this form Kuczma's result can be formulated in the following way: all solutions for arithmetical, geometrical and harmonical means are arcs of vertical paraboles, vertical hyperbols and horizontal parabols correspondently.

If for a mean K there is the function f solving equation (1), then we call it *solvable mean.* The simple criteria of solvability is proven: the nondegenerate mean $K \in C^1([\alpha] \times [\beta])$ is solvable if and only if for any four numbers $a, b, c, d \in [\alpha, \beta]$,

$$\begin{vmatrix} K(a,b) & K(a,b) & K(c,d) \\ K(c,a) & K(a,d) & K(c,d) \\ K(c,b) & K(b,d) & K(c,d) \end{vmatrix} = \begin{vmatrix} K(a,d) & K(b,d) & K(b,d) \\ K(a,d) & K(a,b) & K(c,b) \\ K(c,a) & K(c,a) & K(c,b) \end{vmatrix}.$$

Equation (1) has relations with different disciplines such as economics, operational research and low-dimensional geometry (see [3, 4, 5]).

[1] M. Kuczma, *On some functional equations with conic sections as solutions*, Rocznik Nauk.-Dydakt. Prace Mat. **13** (1993), 197–213.

[2] J. Sándor, *On certain functional equations*, Itinerant Seminar on Functional Equations, Approximation and Convexity (Cluj-Napoca, 1988), 285–288, Preprint, 88-6, Univ. "Babes-Bolyai", Cluj-Napoca, 1988.

[3] E. Akerman, V. Kalnitsky, *The purpose function in the problem of the effective distribution of resurces*, Works of Int. Sci. School "MA SRQ 2001", 18-22 June 2001, St.P., Russia.

[4] V. Kalnitsky, *The reconstruction operator of two-argument function by its section*, Proc. of Int. Sci. Conf. "Lobachevsky readings-2001", Kazan.

[5] V. Kalnitsky, *Arithmetic properties of the Lagrange equation solutions*, Proc. of ICM'2002, August 20-28, Beijing, China.

Tomasz Kochanek *Steinhaus' lattice points problem for Banach spaces*

A classical property of the Euclidean plane, which goes back to H. Steinhaus, asserts that for any $n \in \mathbb{N}$ one may find a circle surrounding exactly n lattice points. P. Zwoleński generalized this result to the setting of Hilbert spaces replacing the set of lattice points by a *quasi-finite* set, i.e., a countably infinite set such that every ball contains only finitely many of its points.

We extend his result by giving a geometrical characterization (involving only the shape of the unit ball) of what we shall call a *Steinhaus property* of a given Banach space X:

(S) for any quasi-finite set $A \subset X$ there exists a dense set $Y \subset X$ such that for any $y \in Y$ and $n \in \mathbb{N}$ there exists a ball B centered at y with $|A \cap B| = n$.

It turns out that every strictly convex Banach space shares this property, but in any dimension greater than 2 property (S) is weaker than strict convexity (e.g., as we will see, the space $L_1(0,1)$ satisfies (S), nonetheless it is not strictly convex). We will give some positive and negative examples for property (S) and discuss its connection with the existence of an equivalent strictly convex norm.

Barbara Koclęga-Kulpa *On a functional equation connected to Hermite quadrature rule*
(joint work with **T. Szostok**)

In the talk we deal with the functional equation

$$F(y) - F(x) = (y-x)\Big[\alpha f(x) + \beta f\Big(\frac{x+y}{2}\Big) + \alpha f(y)\Big] + (y-x)^2[g(y) - g(x)],$$

which is connected to Hermite quadrature rule. It is easy to note that particular cases of this equation generalize many well-known functional equations connected to quadrature rules and mean value theorems. Thus the set of solutions is too complicated to be described completely and therefore we prove that (under some assumptions) all solutions of the above equation have to be polynomials.

We obtain the aforementioned result using a lemma proved by M. Sablik, however this lemma works only in case $\beta \neq 0$. Taking $\beta = 0$, we obtain the following equation

$$F(y) - F(x) = (y-x)[f(x) + f(y)] + (y-x)^2[g(y) - g(x)],$$

which will also be solved in the talk.

Zygfryd Kominek *On pexiderized Jensen-Hosszú functional equation on the unit interval*

We solve the functional equation of the form

$$2f\Big(\frac{x+y}{2}\Big) = g(x+y-xy) + h(xy)$$

in the class of real functions defined on the unit interval $[0,1]$. We prove that it is not stable, but if two functions from the triple $\{f,g,h\}$ coincide the analogue equation is stable in the Hyers-Ulam sense.

Dawid Kotrys *Hermite-Hadamard inequality for convex stochastic processes*

In 1980 K. Nikodem introduced convex stochastic processes and investigated their regularity properties. In 1992 A. Skowroński obtained some further results on convex stochastic processes which generalize some known properties of convex functions. The aim of this talk is to extend the classical Hermite-Hadamard inequality to convex stochastic processes.

Grażyna Łydzińska *On iterative roots of some multifunctions with a unique set-value point*

In [1] and [2] the authors considered the problem of the existence of square iterative roots of multifunctions with exactly one set-value point. In this talk we present a generalization of some results from these papers.

[1] L. Li, J. Jarczyk, W. Jarczyk, W. Zhang, *Iterative roots of mappings with a unique set-value point*, Publ. Math. Debrecen **75** (2009), 203–220.

[2] W. Jarczyk, W. Zhang, *Also set-valued functions do not like iterative roots*, Elem. Math. **62** (2007), 73–80.

Ewelina Mainka-Niemczyk *Set-valued sine families*

Let K be a convex cone in a normed linear space X and let $F_t\colon K\to n(X)$, $E_t\colon K\to n(K)$ for $t\geq 0$. A family $\{E_t:\ t\geq 0\}$ is called a *sine family associated with family* $\{F_t:\ t\geq 0\}$ if

$$E_{t+s}(x) = E_{t-s}(x) + 2F_t(E_s(x)), \qquad 0\leq s\leq t,\ x\in K.$$

Our primary objective in the talk is to show some properties of sine families, such as continuity and correlation with cosine families. Moreover, an integral representation of sine families is given.

Judit Makó *Implications between approximate convexity properties and approximate Hermite-Hadamard inequalities*
(joint work with **Zs. Páles**)

In this talk, the connection between the functional inequalities

$$f\Big(\frac{x+y}{2}\Big) \leq \frac{f(x)+f(y)}{2} + \alpha_J(x-y), \qquad x,y\in D$$

and

$$\int_0^1 f(tx+(1-t)y)\rho(t)\,dt \le \lambda f(x)+(1-\lambda)f(y)+\alpha_H(x-y), \qquad x,y\in D$$

is investigated, where D is a convex subset of a linear space, $f\colon D\to\mathbb{R}$, $\alpha_H,\alpha_J\colon D-D\to\mathbb{R}$ are even functions, $\lambda\in[0,1]$, and $\rho\colon[0,1]\to\mathbb{R}_+$ is an integrable nonnegative function with $\int_0^1\rho(t)\,dt=1$.

Bartosz Micherda *On some inequalities of Hermite-Hadamard-Fejér type for (k,h)-convex functions*
(joint work with **T. Rajba**)

Let k,h be two given real functions defined on the interval $(0,1)$, and choose a nonempty set $D\subset\mathbb{R}$. Then a function $f\colon D\to\mathbb{R}$ will be called (k,h)-*convex* if, for all $x,y\in D$ and $t\in(0,1)$, $k(t)x+k(1-t)y\in D$ and

$$f(k(t)x+k(1-t)y)\le h(t)f(x)+h(1-t)f(y). \tag{1}$$

Condition (1), for conveniently chosen mappings k and h, produces various families of well-known functions, e.g. s-Orlicz convex functions, h-convex functions, subadditive functions and starshaped functions.

In our talk we present two new inequalities of Hermite-Hadamard-Fejér type for (k,h)-convex functions, and we apply them to some special kinds of mappings. This extends results given e.g. in [1] and [2].

[1] M. Bombardelli, S. Varošanec, *Properties of h-convex functions related to the Hermite-Hadamard-Fejér inequalities*, Comput. Math. Appl. **58** (2009), 1869–1877.

[2] S.S. Dragomir, S. Fitzpatrick, *Hadamard's inequality for s-convex functions in the first sense and applications*, Demonstratio Math. **31** (1998), 633–642.

[3] B. Micherda, T. Rajba, *On some Hermite-Hadamard-Fejér inequalities for (k,h)-convex functions* (preprint).

Krzysztof Misztal *Midconvexity for finite sets*
(joint work with **Jacek Tabor** and **Józef Tabor**)

Motivated by increasing role of computers we introduce two definitions of midconvexity for a finite subset X of $\mathbb{R}^N$:

Definition 1
We say that $W\subset X$ is X-*midconvex* if

$$\frac{v+w}{2}\in X \implies \frac{v+w}{2}\in W, \qquad v,w\in W.$$

Definition 2
We say that $W\subset X$ is *function X-midconvex* if there exists a function $f\colon X\to\mathbb{R}_+$ such that

$$\frac{x_1+x_2}{2}\in X \implies f\Big(\frac{x_1+x_2}{2}\Big)\le\frac{f(x_1)+f(x_2)}{2}, \qquad x_1,x_2\in X$$

and $W=f^{-1}(0)$.

The properties of such notions are investigated, and the analogues of some classical results are shown. In particular we show that, for the second definition, an analogue of the theorem stating that compact convex set in $\mathbb{R}^N$ is a convex hull of its extremal points is valid.

Janusz Morawiec *On a problem of Nicole Brillouët-Belluot*

We solve the problem posed by Nicole Brillouët-Belluot during the 49th International Symposium on Functional Equations determining all continuous bijections $f\colon I \to I$ satisfying

$$f(x)f^{-1}(x) = x^2, \qquad x \in I,$$

where I is an arbitrary subinterval of the real line.

Marek Niezgoda *Schur-convexity and similar separability of vectors*

Let G be a compact group acting on an inner product space $(V, \langle\cdot,\cdot\rangle)$. A vector $y \in V$ is said to be *G-majorized* by a vector $x \in V$, written as $y \prec_G x$, if y lies in the the convex hull of the orbit $\{gx : \ g \in G\}$. A function $F\colon V \to \mathbb{R}$ is called *G-increasing* if for $x, y \in V$, $y \prec_G x$ implies $F(y) \leq F(x)$.

A group G acting on V is said to be a *reflection group*, if G is the closure of a subgroup of the orthogonal group $O(V)$ generated by a set of reflections in the form $S_r x = x - 2\langle x, r\rangle r$ for $x \in V$, where $r \in V$, $\|r\| = 1$.

A differential characterization of G-increasing functions, due to Eaton and Perlman, is as follows.

Let G be a reflection group acting on V with $\dim V < \infty$. Assume $F\colon V \to \mathbb{R}$ is a G-invariant function, i.e., $F(gx) = F(x)$ for $x \in V$ and $g \in G$. If F is differentiable on V, then a necessary and sufficient condition that F be G-increasing on V is

$$\langle x, r\rangle \cdot \langle \nabla F(x), r\rangle \geq 0$$

for $x \in V$ and $r \in V$ such that $S_r \in G$, where $\nabla F(x)$ stands for the gradient of F at x.

In the particular situation when G is the permutation group $\mathbb{P}_n$ acting on $V = \mathbb{R}^n$, the preorder $\prec_G$ reduces to the classical majorization $\prec$ on $\mathbb{R}^n$. In this case, the G-increasing functions are called *Schur-convex* functions.

A differential characterization of Schur-convex functions is included in the following Schur-Ostrowski's Theorem.

If $F\colon \mathbb{R}^n \to \mathbb{R}$ is a symmetric differentiable function, then a necessary and sufficient condition that F be a Schur-convex function on $\mathbb{R}^n$ is

$$(x_i - x_j)\Big(\frac{\partial F}{\partial x_i}(x) - \frac{\partial F}{\partial x_j}(x)\Big) \geq 0, \qquad x \in \mathbb{R}^n,\ i, j = 1, 2, \ldots, n.$$

The aim of this talk is to present some extensions of the sufficiency part of Schur-Ostrowski and Eaton-Perlman's Theorems from majorized vectors to similarly separable vectors. A generalized Schur-Ostrowski's condition is introduced.

The obtained results are applied for cone orderings and group-induced cone orderings.

Agata Nowak *On a generalization of the Gołąb-Schinzel equation*

Inspired by a problem posed by J. Matkowski in [1] we investigate the equation

$$f\big(p(x,y)(xf(y)+y)+(1-p(x,y))(yf(x)+x))\big)=f(x)f(y), \qquad x,y\in\mathbb{R},$$

where functions $f\colon\mathbb{R}\to\mathbb{R},\ p\colon\mathbb{R}^2\to\mathbb{R}$ are assumed to be continuous.

[1] J. Matkowski, *A generalization of the Gołąb-Schinzel functional equation*, Aequationes Math. **80** (2010), 181–192.

Andrzej Olbryś *On some derivatives and (s,t)-convex functions*

In our talk we consider some kinds of derivatives and investigate their connections with (s,t)-convexity.

Zsolt Páles *On the generalization of the lower Hermite-Hadamard inequality and Korovkin type theorems*

We investigate functions that satisfy an approximate or strenghtened version of the lower Hermite-Hadamard inequality. Under certain assumptions we deduce that they are also approximately or strongly convex in an appropriate sense. The approach involves certain Korovkin type approximation theorems.

Boris Paneah *On the general theory of multidimensional functional operators: new problems and new approaches*

At first, using the frameworks of the classical triad: where we are? who we are? where do we go? we discuss the notion "General theory of MFO". The most meaningful part here is undoubtedly nonphilosophical second part. Up to recently the only solvability of the Ulam–stability problem for new functional operators traditionally considered as an advance in the general theory of MFO. But now such approach can not be treated as progressive and, moreover, it is harmful one.

The following problem is deep, very interesting, and plays an important role in applications:

given an MFO $\mathcal{P}$*, to describe asymptotic behavior of solutions to nonhomogeneous* MFE

$$\mathcal{P}F=H_\varepsilon(x), \qquad x\in D\subset\mathbb{R}^n$$

with $H_\varepsilon(x)=\mathcal{O}(\varepsilon)$*, as* $\varepsilon\to 0$.

In his book Ulam guessed the answer for the Cauchy operator, and Hyers verified this. The answer is:

$$F(t)=\lambda t+\mathcal{O}(\varepsilon),$$

where λ is an arbitrary real number. Thus, the function $\varphi(t)=\lambda t$ is the main term of the asymptotic of the solution to the equation

$$\mathcal{P}F=H_\varepsilon, \quad \text{as } H_\varepsilon\to 0 \text{ for } \varepsilon\to 0.$$

The excellent question, excellent answer, and … no mystical stability.

In 2006 it was established something cardinally new in searching asymptotic behavior of the same function F. Namely, to identify the main term of the asymptotic we do not need to know the function $\mathcal{P}F$ on the full domain D. It suffices to know it only at the points of a curve Γ (Γ-asymptotic). This result generates very actual problem. Given an operator $\mathcal{P}$ to describe a set of submanifolds Γ for which the Γ-asymptotic problem

$$\mathcal{P}F\,|_\Gamma = H_\varepsilon(x),\ x \in D,\ H_\varepsilon(x) = \mathcal{O}(\varepsilon) \implies F = \psi + \mathcal{O}(\varepsilon)$$

is solvable.

It is not difficult to formulate a series important technical problems leading to the significant extension of the class MFO of operators $\mathcal{P}$ generating for some Γ a solvable Γ-asymptotic problem.

Example: $a = b + c$.

The following problems are formulated in a general form for the first time, although some particular cases have been considered earlier by the speaker.

Inverse problem. To describe a class of MFO in a domain D such that any operator from this class can be uniquely determined by its asymptotic behavior.

It is a surprising result reminding the famous inverse problem in the spectral theory of the differential operators. The possibility to reconstruct an MFO operator using only the asymptotic behavior of the solution to the equation $\mathcal{P}F = H_\varepsilon$ must find many important applications.

Uniqueness problem. Given an MFO operator $\mathcal{P}$ in a domain $D \subset \mathbb{R}^n$, to find a curve $\Gamma \subset D$ such that

$$\text{if } \mathcal{P}F\,|_\Gamma = 0, \text{ then } \mathcal{P}F = 0 \text{ in } D.$$

It is an analogue of the famous uniqueness theorem in the theory of holomorphic functions.

Magdalena Piszczek *The properties of functional inclusions and Hyers-Ulam stability*

We show the properties of some inclusions, especially we prove that a set-valued function satisfying these inclusions admits, in appropriate conditions, a unique selection. As a consequence we obtain a result on the Hyers-Ulam stability of the functional equation

$$\Psi \circ f \circ a = f,$$

where $\Psi\colon Y \to Y$, $f\colon K \to Y$, $a\colon K \to K$, K is a nonempty set and Y is a complete metric space.

Wolfgang Prager *On a system of inhomogeneous linear functional equations* (joint work with **J. Schwaiger**)

Given $a, A \in \mathbb{R}$ with $aA \neq 0$, and an additive function $\phi\colon \mathbb{R} \to \mathbb{R}$, we give the possible additive solution(s) of the equation

$$\alpha(ax) - A\alpha(x) = \phi(x). \tag{1}$$

Imposing the same assumptions as above on b, B, ψ, we consider (1) together with

$$\alpha(bx) - B\alpha(x) = \psi(x) \tag{2}$$

and investigate solvability of the system (1), (2) within the set of additive functions. Finally, the role of system (1), (2) in our efforts to find necessary and sufficient conditions for solvability of the inhomogeneous general linear functional equation will be discussed.

Ludwig Reich *Reversible power series and generalized Abel equations*
(joint work with **P. Kahlig**)

An invertible formal power series F with complex coefficients is called *reversible* if there exists an invertible series T such that

$$F^{-1} = T^{-1} \circ F \circ T \tag{1}$$

holds. If, in particular, we can choose $T(X) = \eta X$ in (1), then (1) yields the generalized Legendre-Gudermann equation

$$F^{-1}(X) = \frac{1}{\eta} F(\eta X) \tag{2}$$

for F.

We are interested here in solutions F with $F(X) = X + \dots$, $F(X) \neq X$. We characterize the values of η for which (2) has such a solution, then we construct the set of all solutions of (2) using ideas of J. Haneczok and we discuss the following connections with (generalized) Abel equations.

Theorem
(i) $F(X) = X + \dots$, $F(X) \neq X$, is reversible if and only if there exists a non-constant Laurent series and a Möbius transformation L such that

$$V(F(X)) = L(V(X))$$

holds.
(ii) A formal series F as in (i) is a solution of (2) if and only if there exists a Laurent series V and a constant $C \in \mathbb{C} \setminus \{0\}$ such that

$$V(\eta X) = -V(X),$$
$$V(F(X)) = V(X) + C$$

holds.

This theorem can also be used to construct reversible series.

Maciej Sablik *Functional equations characterizing future life-time*

In our talk we will present a source of functional equations appearing in actuarial mathematics. The analytic form of future life-time has been an essential concept for actuaries since it makes calculations easier. Another facilitation of calculations is used in the procedure of "group insurance" where many lives are replaced by one artificial, usually aggregated. The method leads to some functional equations which in turn characterize models of de Moivre, Gompertz, Makeham and Weibull among others.

Jens Schwaiger *On the construction of functional equations with prescribed solutions of a certain type*

In the literature one may find certain functional equations such that their general solution is a homogeneous polynomial of degree n where $n \leq 5$. One example is the equation

$$f(kx+y)+f(ky-y) = k^2(f(x+y)+f(x-y))+2k^2(k^2-1)f(x)-2(k^2-1)f(y).$$

This equation was considered in [1] for $k \in \mathbb{N}$, $k \geq 2$. One can motivate the special form of the coefficients not only for this equation but for much more general cases. For example the following holds true.

Theorem
Let $n \in \mathbb{N}$ and let $\rho_1, \rho_2, \ldots, \rho_n$ be rationals different from 0 such that the squares ρ_i^2 are different in pairs. Then for any given rational number k a certain linear system of equations has a unique solution $(\alpha_0, \alpha_1, \ldots, \alpha_n)$. If, moreover, V, W are non-trivial rational vector spaces, then all generalized homogeneous polynomials $f\colon V \to W$ of degree $2n$ satisfy

$$f(kx+y)+f(kx-y)+\sum_{i=1}^{n-1}\alpha_i(f(x+\rho_i y)+f(x-\rho_i y))+\alpha_0 f(x)+\alpha_n f(y) = 0,\ x, y \in V.$$

Provided that k is not a zero of a certain polynomial with rational coefficients this equation does not have other solutions.

[1] M. Eshaghi Gordji, Ch. Park, M.B. Savadkouhi, *The stability of a quartic type functional equation with the fixed point alternative*, Fixed Point Theory **11** (2010), 265–272.

Ekaterina Shulman *Subadditive set-functions on groups and applications to functional equations*

Let G be a group and Ω be an arbitrary set. A map $F\colon G \to 2^{\Omega}$ is called *subadditive* if $F(gh) \subset F(g) \cup F(h)$ for all $g, h \in G$. Let us denote by $|M|$ the number of elements of a subset $M \subset \Omega$. It will be shown that

$$\Big|\bigcup_{g\in G} F(g)\Big| \leq 4 \sup_{g \in G} |F(g)|.$$

We also establish the extensions of this inequality to maps with values in measurable subsets of a measure space and to maps with values in subspaces of a linear space. We apply this technique to the functional equation

$$f(g_1 g_2 \ldots g_n) = \sum_E \sum_{j=1}^{N_E} u_j^E v_j^E, \tag{1}$$

where E runs through all proper non-empty subsets of $\{1, 2, \ldots, n\}$, $N_E \in \mathbb{N}$ and for each E, the functions u_j^E only depend on variables g_i with $i \in E$, while the v_j^E only depend on g_i with $i \notin E$. Namely, we prove that any bounded continuous function f on G satisfying (1), for an $n > 2$, is a matrix element of a continuous finite-dimensional representation of G. Earlier this was known only for topologically finitely generated G.

Dhiraj Kumar Singh *On three sum form functional equations*
(joint work with **P. Nath**)

The general solutions of three sum form functional equations, without imposing any regularity conditions on any of the mappings appearing in these equations, have been obtained.

Barbara Sobek *Wilson's functional equation on a restricted domain*

Assume that X is a real or complex linear topological space and D is a non-empty, open and connected subset of $X \times X$. Let

$$\begin{aligned} D_+ &:= \{x + y : \ (x, y) \in D\}, \\ D_- &:= \{x - y : \ (x, y) \in D\}, \\ D_1 &:= \{x : \ (x, y) \in D \text{ for a } y \in X\} \end{aligned}$$

and

$$D_2 := \{y : \ (x, y) \in D \text{ for an } x \in X\}.$$

We study the equation

$$f(x + y) + g(x - y) = h(x)k(y), \qquad (x, y) \in D,$$

where $f\colon D_+ \to \mathbb{C}$, $g\colon D_- \to \mathbb{C}$, $h\colon D_1 \to \mathbb{C}$ and $k\colon D_2 \to \mathbb{C}$ are unknown functions. We investigate the problem of existence and uniqueness of extensions of the solutions and determine the general solution of this equation. Some results concerning conditional d'Alembert's equation are also presented.

Przemysław Spurek *Strict numerical verification of optimality condition for approximately midconvex functions*
(joint work with **Jacek Tabor**)

Let X be a normed space and V be a convex subset of X. Let $\alpha\colon \mathbb{R}_+ \to \mathbb{R}_+$. A function $f\colon V \to \mathbb{R}$ is called *α-midconvex* if

$$f\Big(\frac{x + y}{2}\Big) - \frac{f(x) + f(y)}{2} \le \alpha(\|x - y\|), \qquad x, y \in V.$$

It can be shown that every continuous α-midconvex function satisfies the following estimation:

$$f(tx + (1 - t)y) - tf(x) - (1 - t)f(y) \le \sum_{k=0}^{\infty} \frac{1}{2^k}\alpha(d(2^{kt}\|x - y\|)), \qquad t \in [0, 1],$$

where $d(t) := 2\mathrm{dist}(t, \mathbb{Z})$ for $t \in [0, 1]$.

An important problem lies in verifying for which functions α the above estimation is optimal. The conjecture of Zs. Páles that this is the case for functions of type $\alpha(r) = r^p$ for $p \in (0,1)$, was proved by J. Makó and Zs. Páles in [1].

In this paper we present a computer assisted method to verifying optimality of this estimation in the class of piecewise linear functions α.

[1] J. Makó, Zs. Páles, *Approximate convexity of Takagi type function*, J. Math. Anal. Appl. **369** (2010), 545–554.

Henrik Stetkær *Levi-Civitá's functional equation*

Let G be a group. Levi-Civitá's functional equation

$$f(xy) = \sum_{l=1}^{N} g_l(x)h_l(y), \qquad x, y \in G, \tag{1}$$

where $f, g_1, \dots, g_N, h_1, \dots, h_N\colon G \to \mathbb{C}$ are unknown, has been thoroughly studied on abelian groups by Székelyhidi in [1]. But little is known about its solutions on non-abelian groups, even for N as small as 2.

We shall briefly discuss the general structure of the solutions of (1) on any group G, before we concentrate on a choice example, viz.

$$f(xy) = f(x)h(y) + f(y), \qquad x, y \in G,$$

$f, h\colon G \to \mathbb{C}$ being the unknown functions. It turns out that h is multiplicative on any group G, if $f \neq 0$. We will prove that f is central on all nilpotent groups, and give an example of a non-central f on the $(ax+b)$-group. We conclude that the solution formulas for f are the same on nilpotent groups as on abelian, and that a new phenomenon occurs on the $(ax+b)$-group.

[1] L. Székelyhidi, *Convolution type functional equations on topological abelian groups*, World Scientific Publishing Co., Inc., Teaneck, NJ, 1991.

Stevo Stević *On some nonlinear recurrences*

Studying nonlinear difference equations and systems has attracted considerable attention in the last few decades. Usually such equations cannot be solved so that the behavior of their solutions is investigated by various analytic methods. Here we present several classes of difference equations and systems whose solutions can be explicitly found.

First we present some classical methods for solving the nonhomogeneous linear difference equation of the first order

$$x_{n+1} = p_n x_n + q_n, \qquad n \in \mathbb{N}_0,$$

where $(p_n)_{n\in\mathbb{N}_0}$ and $(q_n)_{n\in\mathbb{N}_0}$ are arbitrary real sequences and $x_0 \in \mathbb{R}$. Then we emphasize the role of the equation by giving numerous applications of it, for example in: getting Cauchy-Binet formula, solving nonhomogeneous second order difference equation with constant coefficients, solving some homogeneous second order difference equations with nonconstant coefficients, solving Beverton-Holt difference equation and studying its periodic solutions etc.

Further we show that the system of difference equations

$$u_{n+1} = \frac{w_n}{1+s_n}, \qquad v_{n+1} = \frac{t_n}{1+r_n},$$

where w_n, s_n, t_n and r_n are some of the sequences u_n or v_n, with $u_0, v_0 \in \mathbb{R}$, can be solved in many cases.

Finally we show that the system of difference equations

$$x_{n+1} = \frac{a x_{n-1}}{b y_n x_{n-1} + c}, \quad y_{n+1} = \frac{\alpha y_{n-1}}{\beta x_n y_{n-1} + \gamma}, \qquad n \in \mathbb{N}_0,$$

where parameters $a, b, c, \alpha, \beta, \gamma$ and initial values x_{-1}, x_0, y_{-1}, y_0 are real numbers, can be also solved.

[1] S. Stević, *More on a rational recurrence relation*, Appl. Math. E-Notes **4** (2004), 80–85.

[2] S. Stević, *On a system of difference equations*, Appl. Math. Comput. (to appear).

[3] S. Stević, *On some solvable systems of difference equations*, Appl. Math. Comput. (to appear).

László Székelyhidi *Polynomial functions on Abelian groups*

Polynomial functions on Abelian groups play a basic role in the theory of functional equations and spectral analysis. In this paper we investigate the ring-structure of polynomial functions on topological Abelian groups. We show that polynomial functions form a Noetherian ring if and only if the linear space of continuous homomorphisms of the group into the additive group of complex numbers is finite dimensional. In the case of discrete Abelian groups this is equivalent to the presence of spectral synthesis.

Jacek Tabor *New approach to entropy*
(joint work with **M. Śmieja**)

The classical approach to entropy lies in the division of the given measure space into pairwise disjoint sets. We show that we can equivalently use a partition of the measure into measures with not necessarily disjoint supports.

The basic role in our proof plays the classical Hardy-Littlewood-Polya Theorem.

Józef Tabor *Uniform convexity*
(joint work with **Jacek Tabor**)

We present some convenient tools to compute the modulus of uniform convexity of a given convex function $f\colon I \to \mathbb{R}$, where I is a subinterval of $\mathbb{R}$.

We first show that if f' is convex or concave then the modulus of uniform convexity of f equals to the Bergman distance at a respective endpoint of I. Then, in our main result, we give an estimation from below the modulus of uniform convexity of f by applying moduli of uniform convexities of f restricted to intervals J, K such that I is the union of J and K.

Jointly the two above mentioned results allow to estimate the moduli of uniform convexity for a large class of convex functions.

Jörg Tomaschek *On the solvability of generalized Dhombres functional equations*

The generalized Dhombres functional equation in the complex domain was introduced in [1] and is given by

$$f(zf(z)) = \varphi(f(z)), \tag{1}$$

where f is an unknown function and φ is a known one. In [2] it is shown that (1) is equivalent to the transformed generalized Dhombres functional equation

$$g(w_0 z + zg(z)) = \tilde{\varphi}(g(z)).$$

We discuss solutions f of (1) with $f(0) = w_0$, where w_0 is a root of unity of order $l \geq 2$, and we characterize those equations (1) which have non-trivial solutions. After that an example where the given function $\tilde{\varphi}$ is a Möbius transformation is computed.

[1] L. Reich, J. Smítal, M. Štefánková, *Local analytic solutions of the generalized Dhombres functional equation I*, Österreich. Akad. Wiss. Math.-Natur. Kl. Sitzungsber. II **214** (2005), 3–25.

[2] L. Reich, J. Smítal, M. Štefánková, *Local analytic solutions of the generalized Dhombres functional equation II*, J. Math. Anal. Appl. **355** (2009), 821–829.

Hamid Vaezi *Fuzzy approximation of an additive functional equation*

In this paper, we investigate the generalized Hyers-Ulam-Rassias stability of the functional equation

$$\sum_{i=1}^{m} f\Big(mx_i + \sum_{j=1, j\neq i}^{m} x_j\Big) + f\Big(\sum_{i=1}^{m} x_i\Big) = 2f\Big(\sum_{i=1}^{m} mx_i\Big)$$

in fuzzy Banach spaces. Some applications of our results to the stability of the above equation in the case when f is a mapping from a normed space to a Banach space will also be presented.

Szymon Wąsowicz *Spline approximation method in higher-order convexity business*

It is well-known that a continuous convex function $f\colon [a,b] \to \mathbb{R}$ can be uniformly approximated on $[a,b]$ by convex polygonal functions. This property allows us to give easy proofs of many linear inequalities involving (continuous) convex functions, among others of the celebrated Hermite-Hadamard inequality

$$f\Big(\frac{a+b}{2}\Big) \leqslant \frac{1}{b-a}\int_a^b f(x)\,dx \leqslant \frac{f(a)+f(b)}{2}. \tag{1}$$

Our considerations will be based on the observation that the left hand side inequality of (1) gives the better estimate of the integral mean value from the right one

$$\frac{1}{b-a}\int_a^b f(x)\,dx - f\Big(\frac{a+b}{2}\Big) \leqslant \frac{f(a)+f(b)}{2} - \frac{1}{b-a}\int_a^b f(x)\,dx, \tag{2}$$

whenever f is convex on $[a,b]$.

The similar approximation property holds for convex functions of higher order. Namely, every continuous n-convex function $f\colon [a,b] \to \mathbb{R}$ can be uniformly approximated on $[a,b]$ by n-convex spline functions of order n (Bojanic, Roulier, 1974). In the talk we will show an aplication of this result to prove some counterparts of (2) for convex functions of higher order.

Alfred Witkowski *Interpolations of Schwab-Borchardt mean*

For positive numbers x, y the pair of sequences

$$x_{n+1} = \frac{x_n+y_n}{2}, \quad y_{n+1} = \sqrt{y_n\frac{x_n+y_n}{2}}, \quad x_0 = x, \quad y_0 = y \tag{1}$$

converges to a common limit called the Schwab-Borchardt mean

$$SB(x,y) = \begin{cases} \dfrac{\sqrt{y^2-x^2}}{\arccos\frac{x}{y}}, & x<y, \\ \dfrac{\sqrt{x^2-y^2}}{\operatorname{arccosh}\frac{x}{y}}, & y<x, \\ x, & x=y. \end{cases}$$

Algorithm (1) was known to Gauss but has been rediscovered by Borchradt and named after him.

Two means introduced by Seiffert

$$P(x,y) = \begin{cases} \dfrac{x-y}{2\arcsin\frac{x-y}{x+y}}, & x\neq y, \\ x, & x=y, \end{cases}$$

$$T(x,y) = \begin{cases} \dfrac{x-y}{2\arctan\frac{x-y}{x+y}}, & x\neq y, \\ x, & x=y \end{cases}$$

are of great interest for many mathematicians. Neuman and Sándor proved that both are particular cases of the Schwab-Borchardt means, namely

$$P(x,y) = SB\Big(\sqrt{xy}, \frac{x+y}{2}\Big) \quad \text{and} \quad T(x,y) = SB\Big(\frac{x+y}{2}, \sqrt{\frac{x^2+y^2}{2}}\Big).$$

Interesting inequalities between P, T, arithmetic, geometric, logarithmic, identric and power means were obtained by many authors using analytic approach or properties of the Schwab-Borchardt algorithm.

In this talk we use geometric properties of the "upper" part of SB to generalize those results and to obtain some new estimates. In particular we show some new interpolations of the Seiffert means.

David Yost *Pseudolinear functions, Banach spaces and polyhedra*

We begin with F-spaces (which are not necessarily locally convex) and a class of mappings between them defined by a certain functional inequality. We briefly describe how their study led us to some results about Minkowski decomposability of finite-dimensional convex sets.

Marek C. Zdun *On some applications of Kuczma's ideas to Schröder's equation in multidimensional case*

Let $U \subset \mathbb{R}^N$ be a neighbourhood of the origin, a function $F\colon U \to U$ be of class C^2 and $0 \in \operatorname{Int} U$ be an attractive fixed point of F. We consider a problem when a regular solution φ of Schröder's equation

$$\varphi(F(x)) = S\varphi(x),$$

where $S = \mathrm{d}F(0)$, is given by

$$\varphi(x) = \lim_{n\to\infty} S^{-n}F^n(x).$$

We give some sufficient conditions for truthfulness of this formula as well as some conditions which imply its falsehood.

Marek Żołdak *Approximately convex functions on Abelian topological groups*

Let $(G,+)$ be an Abelian topological group and let $\alpha\colon G \to \mathbb{R}_+$ be an even function. A function $f\colon D \to \mathbb{R}$, where D is a subset of G, is called *α-convex* if

$$f(z) \le \frac{f(x)+f(y)}{2} + \alpha(x-y)$$

for all $x,y,z \in D$ such that $x+y=2z$.

Our main result is that if $\alpha(0)=0$, α is continuous at zero, D is open and connected, f is α-convex and locally bounded above at a point, then f is locally uniformly continuous. The same is true if we replace the assumption that f is locally bounded above at a point by assumption that f is Haar measurable or Baire measurable.

Problems and Remarks

1. Remark.

We consider the Sierpiński Carpet $\mathcal{L}$ (defined in [1]), which is the ICFEI-Logo-Set. For convenience, we work with the set shifted and sized according to the requirement

$$\operatorname{conv}\{\mathcal{L}\} = \Big[-\frac{1}{2},\frac{1}{2}\Big]^2.$$

From the symmetry and invariance properties of this set we obtain the following covering of $\mathcal{L}$ by its eight subsets (the *self-similarity equation*)

$$\mathcal{L} = \frac{1}{3}\mathcal{A} + \frac{1}{3}\mathcal{L} = \bigcup_{a\in\mathcal{A}} \Big(\frac{1}{3}a + \frac{1}{3}\mathcal{L}\Big), \tag{1}$$

where $\mathcal{A} = \{-1, 0, 1\}^2 \setminus \{(0,0)\}$. Accordingly, we seek a suitably invariant Borel probability measure $\mu_{\mathcal{L}}$ concentrated on $\mathcal{L}$. In terms of the random variables

- L is a 2-dimensional random variable (r.v.) with the probability distribution (p.d.) $P_L = \mu_{\mathcal{L}}$,
- A is an independent 2-dimensional r.v. with the classical p.d. $P_A = P_{\mathcal{A}}^{\text{class}}$ on the set $\mathcal{A}$,

the expected invariance of $\mu_{\mathcal{L}}$ is expressed as follows (see (1))

$$P_L = \frac{1}{8} \sum_{a \in \mathcal{A}} P_{\frac{1}{3}a + \frac{1}{3}L} = P_{\frac{1}{3}A + \frac{1}{3}L} = P_{\frac{1}{3}A} * P_{\frac{1}{3}L}, \tag{2}$$

where $*$ stands for the convolution of measures. Now, property (2) is equivalent to the following *Poincaré equation* for the characteristic function of L

$$\varphi_L(t) = \varphi_{\frac{1}{3}A}(t) \cdot \varphi_{\frac{1}{3}L}(t) = \varphi_A\Big(\frac{t}{3}\Big) \cdot \varphi_L\Big(\frac{t}{3}\Big). \tag{3}$$

Thus, by iteration procedure, with the use of continuity at 0 only ($\lim_{x \to 0} \varphi_L(x) = 1$), we arrive at the unique solution of (3)

$$\varphi_L(t) = \prod_{n=1}^{\infty} \varphi_A\Big(\frac{t}{3^n}\Big) = \varphi_S(t),$$

where almost surely $S = \sum_{n=1}^{\infty} \frac{A_n}{3^n}$ and the random vectors A_n for $n \in \mathbb{N}$ are independent, all with the same classical p.d. on $\mathcal{A}$. Since the intersection of $\mathcal{L}$ and the open square $(-\frac{1}{6}, \frac{1}{6})^2$ is empty, and the same property has the set of values of the series S, with the use of the symmetry of $\mathcal{L}$ we obtain the following well-known result

Corollary
The Borel measure $\mu_{\mathcal{L}} = P_L$ concentrated on $\mathcal{L}$ and satisfying (2) exists and is unique. Moreover, all possible values of the infinite sums of vectors $\sum_{n=1}^{\infty} \frac{a_n}{3^n}$ with $a_n \in \mathcal{A} = \{-1, 0, 1\}^2 \setminus \{(0,0)\}$ form a set of full measure $\mu_{\mathcal{L}}$.

[1] W. Sierpiński, *On a curve which contains the image of any curve* (Russian), Mat. Sb. **30** (1916), 267–287.

Joachim Domsta

2. Remark.
Let d denote the distance function from integer numbers defined by

$$d(x) := \inf\{|x - k|,\ k \in \mathbb{Z}\}.$$

Then one can see that d is nonnegative and even. It is not difficult to prove that

d is also subadditive. It easily follows from these properties that

$$|d(x) - d(y)| \leq d(x - y), \qquad x, y \in \mathbb{R}. \tag{1}$$

As an application of these properties we have the following result.

THEOREM
Let $a, b, a_1, b_1, \ldots, a_k, b_k$ be positive numbers such that the rectangle of sides a, b is the union of rectangles of sides a_i, b_i $(i = 1, \ldots, k)$ such that these rectangles have no interior points in common. Then

$$d(a)d(b) \leq \sum_{i=1}^{k} d(a_i)d(b_i). \tag{2}$$

Proof. Assume that $I = [0, a] \times [0, b]$ and $I_i = [x_i, x_i + a_i] \times [y_i, y_i + b_i]$ for some points $(x_i, y_i) \in I$. Let

$$f(x, y) := d'(x)d'(y), \qquad (x, y) \in I,$$

and compute the integral of f over I in two ways.

First, using Fubini's Theorem,

$$\begin{aligned} \int_I f &= \int_0^a \left(\int_0^b f(x, y)\, dy \right) dx = \int_0^a d'(x)\, dx \cdot \int_0^b d'(y)\, dy \\ &= (d(a) - d(0)) \cdot (d(b) - d(0)) = d(a) \cdot d(b). \end{aligned}$$

Secondly, also using the additivity of the integral and (1),

$$\begin{aligned} \int_I f &= \sum_{i=1}^{k} \int_{I_i} f = \sum_{i=1}^{k} \int_{x_i}^{x_i + a_i} \left(\int_{y_i}^{y_i + b_i} f(x, y)\, dy \right) dx \\ &= \sum_{i=1}^{k} \int_{x_i}^{x_i + a_i} d'(x)\, dx \cdot \int_{y_i}^{y_i + b_i} d'(y)\, dy \\ &= \sum_{i=1}^{k} (d(x_i + a_i) - d(x_i)) \cdot (d(y_i + b_i) - d(y_i)) \\ &\leq \sum_{i=1}^{k} |d(x_i + a_i) - d(x_i)| \cdot |d(y_i + b_i) - d(y_i)| \\ &\leq \sum_{i=1}^{k} d(a_i) \cdot d(b_i). \end{aligned}$$

Thus, the proof is complete.

The following consequence is a well-known result.

Corollary

Let $a, b, a_1, b_1, \dots, a_k, b_k$ be positive numbers such that the rectangle of sides a, b can be decomposed as the (almost disjoint) union of rectangles of sides a_i, b_i ($i = 1, \dots, k$). Assume that, for all i, either a_i or b_i is an integer number. Then a or b is also an integer.

Proof. If a_i or b_i is an integer then $d(a_i) \cdot d(b_i) = 0$ for all i. Thus, by (2), we get $d(a) \cdot d(b) = 0$. Therefore, $d(a) = 0$ or $d(b) = 0$ holds. This shows that a or b is an integer.

Zsolt Páles

3. Problem.

Let $g\colon \mathbb{C} \to \mathbb{C}$ be an entire function, and assume that

$$g(z) = \frac{1}{2} g\Big(\frac{z}{2}\Big) + \frac{1}{2} g\Big(\frac{z+1}{2}\Big), \qquad z \in \mathbb{C}$$

holds. Then F. Schottky and G. Herglotz showed that g is constant.

What is known about entire solutions g of functional equations of the form

$$p_0(z)g(z) = \sum_{j=1}^{N} p_j(z) g(\alpha_j z + \beta_j) + R(z), \qquad z \in \mathbb{C},$$

where $p_0, \dots, p_N$ are slowly growing entire functions (e.g. polynomials), R is a given entire function and α_j, β_j satisfy appopriate conditions?

Ludwig Reich

4. Problems.

1. Let V denote a translation invariant linear subspace in the space of complex polynomials in k variables. Suppose that $(p_n)_{n\in\mathbb{N}}$ is a sequence in V which converges pointwise to the polynomial p. Does it follow that p is in V? In the case $k = 1$ the answer is “yes”. (This problem has been presented at the 49th ISFE, Graz-Mariatrost, 2011.)

2. Does there exist a strictly descending infinite chain of translation invariant linear spaces of complex polynomials in k variables? In the case $k = 1$ the answer is “no”.

László Székelyhidi

5. Remark.

The classical Hermite-Hadamard inequality

$$f\Big(\frac{a+b}{2}\Big) \overset{(1)}{\leqslant} \frac{1}{b-a} \int_a^b f(x)\, dx \overset{(2)}{\leqslant} \frac{f(a)+f(b)}{2}$$

holds for all convex functions $f\colon [a, b] \to \mathbb{R}$. It is well-known that inequality (1) gives the better estimate of the integral mean value than inequality (2). After

Sz. Wąsowicz's talk M. Goldberg asked the speaker the question what about the multivariate case. Below we give the negative answer.

If $S \subset \mathbb{R}^n$ is a simplex with vertices $p_0, p_1, \ldots, p_n$, then the following Hermite-Hadamard type inequality holds:

$$f\Big(\frac{1}{n+1}\sum_{i=0}^{n} p_i\Big) \leqslant \frac{1}{\mathrm{vol(S)}}\int\limits_S f(\mathbf{x})\,d\mathbf{x} \leqslant \frac{1}{n+1}\sum_{i=0}^{n} f(p_i),$$

whenever $f\colon S \to \mathbb{R}$ is a convex function (cf. [1, 2]).

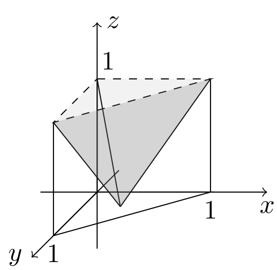
Now let $S = \operatorname{conv}\big\{(0,0),(0,1),(1,0)\big\}$ be the unit simplex in $\mathbb{R}^2$. Then the above inequalities have the form

$$f\Big(\frac{1}{3},\frac{1}{3}\Big) \overset{(3)}{\leqslant} 2\iint\limits_S f(x,y)\,dx\,dy \overset{(4)}{\leqslant} \frac{f(0,0)+f(0,1)+f(1,0)}{3}.$$

For the convex function $f(x,y) = x^2$ we obtain

$$\frac{1}{9} \leqslant \frac{1}{6} \leqslant \frac{1}{3}$$

which means that inequality (3) estimates the integral mean value better than (4). Take now another convex function, whose graph is the surface of a pyramid shown in the picture below.



Then it is easy to observe that for this function inequalities (3) and (4) have the form

$$0 \leqslant 2\Big(\frac{1}{2} - \frac{1}{6}\Big) = \frac{2}{3} \leqslant 1$$

and in this case (4) estimates the integral mean value better than (3).

[1] M. Bessenyei, *The Hermite-Hadamard inequality on simplices*, Amer. Math. Monthly **115** (2008), 339–345.

[2] Sz. Wąsowicz, *Hermite-Hadamard-type inequalities in the approximate integration*, Math. Inequal. Appl. **11** (2008), 693–700.

Szymon Wąsowicz and Alfred Witkowski

6. Problem.

Let us say that a function $f\colon X \to Y$ between normed spaces has the property $\mathcal{P}_n$ (for a fixed $n \in \mathbb{N}$) if it is homogenous and satisfies the functional inequality

$$\left\| f\Big(\sum_{i=1}^{n} x_i\Big) - \sum_{i=1}^{n} f(x_i) \right\| \leq K \sum_{i=1}^{n} \|x_i\|, \qquad x_1, \ldots, x_n \in X.$$

The classic property quasilinearity is simply $\mathcal{P}_2$. It is clear that $\mathcal{P}_{n+1} \Longrightarrow \mathcal{P}_n$ for all n. What about the converse? If we denote by K_n the best constant for which f has $\mathcal{P}_n$, a short calculation shows that $K_{n+1} \leq K_n + K_2$. Can this estimate be improved? For the "worst" example (which has $X = \ell_1$ and $Y = \mathbb{R}$) it is only known that $K_n \geq \log n$. (When the domain X is a so-called $\mathcal{K}$-space, $\mathcal{P}_2$ already implies $\mathcal{P}_n$ for all n with a common value for K. This class includes all super-reflexive spaces, all quotients of $\mathcal{L}_\infty$ spaces, in particular all classical spaces except ℓ_1.) Dropping the homogeneity requirement leads to a very different problem, which may also be interesting.

David Yost

List of Participants

BAHYRYCZ Anna, Instytut Matematyki, Uniwersytet Pedagogiczny, ul. Podchorążych 2, 30-084 KRAKÓW, Poland, e-mail: bah@up.krakow.pl

BARON Karol, Instytut Matematyki, Uniwersytet Śląski, ul. Bankowa 14, 40-007 KATOWICE, Poland, e-mail: baron@us.edu.pl

BATKO Bogdan, Instytut Matematyki, Uniwersytet Pedagogiczny, ul. Podchorążych 2, 30-084 KRAKÓW, Poland, e-mail: bbatko@up.krakow.pl

BRILLOUËT-BELLUOT Nicole, Département d'Informatique et de Mathématiques, Ecole Centrale de Nantes, 1 rue de la Noë, B. P. 92101, 44321 NANTES-Cedex 03, France, e-mail: Nicole.Belluot@ec-nantes.fr

BRZDĘK Janusz, Instytut Matematyki, Uniwersytet Pedagogiczny, ul. Podchorążych 2, 30-084 KRAKÓW, Poland, e-mail: jbrzdek@up.krakow.pl

CHMIELIŃSKI Jacek, Instytut Matematyki, Uniwersytet Pedagogiczny, ul. Podchorążych 2, 30-084 KRAKÓW, Poland, e-mail: jacek@up.krakow.pl

CHUDZIAK Jacek, Instytut Matematyki, Uniwersytet Rzeszowski, ul. Rejtana 16 A, 35-959 RZESZÓW, Poland, e-mail: chudziak@univ.rzeszow.pl

CIEPLIŃSKI Krzysztof, Instytut Matematyki, Uniwersytet Pedagogiczny, ul. Podchorążych 2, 30-084 KRAKÓW, Poland, e-mail: kc@up.krakow.pl

CZERNI Marek, Instytut Matematyki, Uniwersytet Pedagogiczny, ul. Podchorążych 2, 30-084 KRAKÓW, Poland, e-mail: mczerni@up.krakow.pl

DOMSTA Joachim, Wydział Fizyki Technicznej i Matematyki Stosowanej, Politechnika Gdańska, ul. Narutowicza 11/12, 80-952 GDAŃSK, Poland, e-mail: jdomsta@mif.pg.gda.pl

FECHNER Włodzimierz, Instytut Matematyki, Uniwersytet Śląski, ul. Bankowa 14, 40-007 KATOWICE, Poland, e-mail: fechner@math.us.edu.pl

FECHNER Żywilla, Instytut Matematyki, Uniwersytet Śląski, ul. Bankowa 14, 40-007 KATOWICE, Poland, e-mail: zfechner@math.us.edu.pl

FÖRG-ROB Wolfgang, Institut für Mathematik, Universität Innsbruck, Technikerstr. 13, 6020 INNSBRUCK, Austria, e-mail: wolfgang.foerg-rob@uibk.ac.at

GER Roman, Instytut Matematyki, Uniwersytet Śląski, ul. Bankowa 14, 40-007 KATOWICE, Poland, e-mail: romanger@math.us.edu.pl

GŁAZOWSKA Dorota, Wydział Matematyki, Informatyki i Ekonometrii, Uniwersytet Zielonogórski, ul. Szafrana 4a, 65-516 ZIELONA GÓRA, Poland,
e-mail: D.Glazowska@wmie.uz.zgora.pl

GOLDBERG Moshe, Department of Mathematics, Technion-Israel Institute of Technology, HAIFA 32000, Israel, e-mail: goldberg@math.technion.ac.il

GURUDWAN Niyati, School of Studies in Mathematics, Pt. Ravishankar Shukla University, Raipur, CHHATTISGARH, India, e-mail: niyati.kuhu@gmail.com

HÁZY Attila, Department of Applied Mathematics, University of Miskolc, 3515 MISKOLC-EGYETEMVÁROS, Hungary, e-mail: matha@uni-miskolc.hu

JABŁOŃSKA Eliza, Katedra Matematyki, Politechnika Rzeszowska, ul. Pola 2, 35-959 RZESZÓW, Poland, e-mail: elizapie@prz.rzeszow.pl

JARCZYK Justyna, Wydział Matematyki, Informatyki i Ekonometrii, Uniwersytet Zielonogórski, ul. Szafrana 4a, 65-516 ZIELONA GÓRA, Poland,
e-mail: j.jarczyk@wmie.uz.zgora.pl

JARCZYK Witold, Wydział Matematyki, Informatyki i Ekonometrii, Uniwersytet Zielonogórski, ul. Szafrana 4a, 65-516 ZIELONA GÓRA, Poland,
e-mail: w.jarczyk@wmie.uz.zgora.pl

KALNITSKY Vyacheslav, Mathematics & Mechanics Faculty, Saint Petersburg State University, Universitetsky ave. 28, 198504 SAINT PETERSBURG, Russian Federation,
e-mail: skalnitsky@hotmail.com

KOCHANEK Tomasz, Instytut Matematyki, Uniwersytet Śląski, ul. Bankowa 14, 40-007 KATOWICE, Poland, e-mail: t_kochanek@wp.pl

KOCLĘGA-KULPA Barbara, Instytut Matematyki, Uniwersytet Śląski, ul. Bankowa 14, 40-007 KATOWICE, Poland, e-mail: koclega@ux2.math.us.edu.pl

KOMINEK Zygfryd, Instytut Matematyki, Uniwersytet Śląski, ul. Bankowa 14, 40-007 KATOWICE, Poland, e-mail: zkominek@ux2.math.us.edu.pl

KOTRYS Dawid, Katedra Matematyki i Informatyki, Akademia Techniczno-Humanistyczna, ul. Willowa 2, 43-309 BIELSKO-BIAŁA, Poland, e-mail: dkotrys@ath.bielsko.pl

ŁYDZIŃSKA Grażyna, Instytut Matematyki, Uniwersytet Śląski, ul. Bankowa 14, 40-007 KATOWICE, Poland, e-mail: lydzinska@ux2.math.us.edu.pl

MAINKA-NIEMCZYK Ewelina, Instytut Matematyki, Politechnika Śląska, ul. Kaszubska 23, 44-100 GLIWICE, Poland, e-mail: Ewelina.Mainka@polsl.pl

MAKÓ Judit, Institute of Mathematics, University of Debrecen, Pf. 12, 4010 DEBRECEN, Hungary, e-mail: makoj@science.unideb.hu

MICHERDA Bartosz, Katedra Matematyki i Informatyki, Akademia Techniczno-Humanistyczna, ul. Willowa 2, 43-309 BIELSKO-BIAŁA, Poland,
e-mail: bmicherda@ath.bielsko.pl

MISZTAL Krzysztof, Instytut Informatyki, Uniwersytet Jagielloński, ul.Łojasiewicza 6, 30-348 KRAKÓW, Poland, e-mail: krzysztof.misztal@ii.uj.edu.pl

MORAWIEC Janusz, Instytut Matematyki, Uniwersytet Śląski, ul. Bankowa 14, 40-007 KATOWICE, Poland, e-mail: morawiec@math.us.edu.pl

NIEZGODA Marek, Katedra Zastosowań Matematyki i Informatyki, Uniwersytet Przyrodniczy, ul. Akademicka 13, 20-950 LUBLIN, Poland, e-mail: marek.niezgoda@up.lublin.pl

NOWAK Agata, Instytut Matematyki, Uniwersytet Śląski, ul. Bankowa 14, 40-007 KATOWICE, Poland, e-mail: tnga@poczta.onet.pl

OLBRYŚ Andrzej, Instytut Matematyki, Uniwersytet Śląski, ul. Bankowa 14, 40-007 KATOWICE, Poland, e-mail: andrzej.olbrys@wp.pl

PÁLES Zsolt, Institute of Mathematics, University of Debrecen, Pf. 12, 4010 DEBRECEN, Hungary, e-mail: pales@math.klte.hu

PANEAH Boris, Department of Mathematics, Technion-Israel Institute of Technology, HAIFA 32000, Israel, e-mail: peter@tx.technion.ac.il

PISZCZEK Magdalena, Instytut Matematyki, Uniwersytet Pedagogiczny, ul. Podchorążych 2, 30-084 KRAKÓW, Poland, e-mail: magdap@ap.krakow.pl

PRAGER Wolfgang, Institut für Mathematik und Wissenschaftliches Rechnen, Karl-Franzens-Universität, Heinrichstraße 36, 8010 GRAZ, Austria, e-mail: wolfgang.prager@uni-graz.at

REICH Ludwig, Institut für Mathematik und Wissenschaftliches Rechnen, Karl-Franzens-Universität, Heinrichstraße 36, 8010 GRAZ, Austria, e-mail: ludwig.reich@kfunigraz.ac.at

SABLIK Maciej, Instytut Matematyki, Uniwersytet Śląski, ul. Bankowa 14, 40-007 KATOWICE, Poland, e-mail: mssablik@us.edu.pl

SCHLEIERMACHER Adolf, Rablstr. 18, 81669 MÜNCHEN, Germany, e-mail: adsle@aol.com

SCHWAIGER Jens, Institut für Mathematik und Wissenschaftliches Rechnen, Karl-Franzens-Universität, Heinrichstraße 36, 8010 GRAZ, Austria, e-mail: jens.schwaiger@uni-graz.at

SHULMAN Ekaterina, Department of Mathematics, Vologda State Pedagogical University, 6 Orlova str., VOLOGDA 160000, Russian Federation, e-mail: shulmank@yahoo.com

SINGH Dhiraj Kumar, Zakir Husain College, University of Delhi, Jawaharlal Nehru Marg, DELHI 110002, India, e-mail: dhiraj426@rediffmail.com, dksingh@maths.du.ac.in

SOBEK Barbara, Instytut Matematyki, Uniwersytet Rzeszowski, ul. Rejtana 16 A, 35-959 RZESZÓW, Poland, e-mail: b_sobek@wp.pl

SOLARZ Paweł, Instytut Matematyki, Uniwersytet Pedagogiczny, ul. Podchorążych 2, 30-084 KRAKÓW, Poland, e-mail: psolarz@up.krakow.pl

SPUREK Przemysław, Instytut Informatyki, Uniwersytet Jagielloński, ul.Łojasiewicza 6, 30-348 KRAKÓW, Poland, e-mail: przemyslaw.spurek@ii.uj.edu.pl

STETKAER Henrik, Department of Mathematical Sciences, Aarhus University, Ny Munkegade 118, 8000, AARHUS C, Denmark, e-mail: stetkaer@imf.au.dk

STEVIĆ Stevo, Mathematical Institute, Serbian Academy of Sciences, Knez Mihailova 36/III, 11000 BEOGRAD, Serbia, e-mail: sstevic@ptt.rs

SZÉKELYHIDI László, Institute of Mathematics, University of Debrecen, Pf. 12, 4010 DEBRECEN, Hungary, e-mail: szekely@math.klte.hu

TABOR Jacek, Instytut Informatyki, Uniwersytet Jagielloński, ul.Łojasiewicza 6, 30-348 KRAKÓW, Poland, e-mail: tabor@im.uj.edu.pl

TABOR Józef, Instytut Matematyki, Uniwersytet Rzeszowski, ul. Rejtana 16 A, 35-959 RZESZÓW, Poland, e-mail: tabor@univ.rzeszow.pl

TOMASCHEK Jörg, Institut für Mathematik und Wissenschaftliches Rechnen, Karl-Franzens-Universität, Heinrichstraße 36, 8010 GRAZ, Austria, e-mail: joerg.tomaschek@uni-graz.at

VAEZI HAMID, Deptament of Mathematics, Faculty of Mathematical Sciences, University of Tabriz, TABRIZ, Islamic Republic of Iran, e-mail: hvaezi@tabrizu.ac.ir

WĄSOWICZ Szymon, Katedra Matematyki i Informatyki, Akademia Techniczno-Humanistyczna, ul. Willowa 2, 43-309 BIELSKO-BIAŁA, Poland, e-mail: swasowicz@ath.bielsko.pl

WITKOWSKI Alfred, Instytut Matematyki i Fizyki, Uniwersytet Technologiczno-Przyrodniczy, al. Kaliskiego 7, 85-796 BYDGOSZCZ, Poland, e-mail: Alfred.Witkowski@utp.edu.pl

YOST David, Centre for Informatics and Applied Optimization, School of Science, Information Technology & Engineering, University of Ballarat, BALLARAT, Australia, e-mail: d.yost@ballarat.edu.au

ZDUN Marek Cezary, Instytut Matematyki, Uniwersytet Pedagogiczny, ul. Podchorążych 2, 30-084 KRAKÓW, Poland, e-mail: mczdun@up.krakow.pl

ŻOŁDAK Marek, Instytut Matematyki, Uniwersytet Rzeszowski, ul. Rejtana 16 A, 35-959 RZESZÓW, Poland, e-mail: marek_z2@op.pl

(Compiled by Krzysztof Ciepliński)