FOLIA 122

Annales Universitatis Paedagogicae Cracoviensis

Studia Mathematica XI (2012)

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Rank function equations

Abstract. The purpose of this paper is to introduce the notion of rank function equation, and to present some results on such equations. In particular, we find all sequences $(A_1, \dots, A_k, B)$ of nonzero nilpotent $n \times n$ matrices satisfying condition

$$\forall m \in \{1, \dots, n\} : \sum_{i=1}^{k} r_{A_i}(m) = r_B(m),$$

and give a characterization of all sequences $(A_1, \dots, A_k, B)$ of nilpotent $n \times n$ matrices such that

$$\forall m \in \{1, \dots, n\} : \sum_{i=1}^{k} f(r_{A_i}(m)) = r_B(m),$$

where $f\colon \mathbb{R} \supset [0, \infty) \to \mathbb{R}$ is a function with certain natural properties. We also provide a geometric characterization of some solutions to rank function equations.

Throughout this paper we assume that $\mathbb{F}$ is an arbitrary field of characteristic zero. We denote by $\mathbb{N}$ the set of all non-negative integers. For $n \in \mathbb{N} \setminus \{0\}$ we define $M_{n\times n}(\mathbb{F})$ to be the set of all $n \times n$ matrices whose entries are elements of the field $\mathbb{F}$. The set of all nonsingular $n \times n$ matrices over $\mathbb{F}$ will be denoted by $GL(n, \mathbb{F})$. The conjugacy class $\mathcal{O}(A)$ of a matrix $A \in M_{n\times n}(\mathbb{F})$ is defined by

$$\mathcal{O}(A) = \{U^{-1}AU : \ U \in GL(n, \mathbb{F})\}.$$

We denote by O_n the zero $n \times n$ matrix.

We refer to [1] for matrix theory and to [3] for algebraic geometry.

The purpose of the present note is to introduce a new object in the geometry of $GL(n, \mathbb{F})$-stable sets of matrices, which will be referred to as the rank function equation. We will construct solutions to some rank function equations.

By a non-trivial solution we mean throughout a solution consisting of nonzero matrices.

AMS (2000) Subject Classification: 15A24, 14M12.

1. Preliminaries

In this section we introduce the notion of rank function equation. Before formulating the definition, we give an outline of basic facts related to the rank functions. For details we refer to [4].

Definition 1.1
The function $r_A\colon \mathbb{N} \to \mathbb{N}$ defined by

$$r_A(m) = \mathrm{rk}(A^m)$$

is called the rank function of a matrix $A \in M_{n\times n}(\mathbb{F})$.

Proposition 1.2
The following properties hold for a matrix $A \in M_{n\times n}(\mathbb{F})$:

(i) $r_A(0) = n$,

(ii) *the function r_A is weakly decreasing,*

(iii) *A is nilpotent iff $r_A(n) = 0$,*

(iv) *if $r_A(m_0) = r_A(m_0+1)$ for some integer $m_0 \in \mathbb{N}$, then $r_A(m_0) = r_A(m_0+i)$ for every $i \in \mathbb{N}$,*

(v) *$r_{U^{-1}AU}(m) = r_A(m)$ for every $m \in \mathbb{N}$ and every $U \in GL(n,\mathbb{F})$,*

(vi) *if $A = A_1 \oplus A_2$, where $A_i \in M_{n_i\times n_i}(\mathbb{F})$, $i = 1,2$, and $\oplus$ is the standard direct sum of matrices, then $r_A(m) = r_{A_1}(m) + r_{A_2}(m)$ for all $m \in \mathbb{N}$.*

The following theorem has been proved in [4].

Theorem 1.3
A function $r\colon \mathbb{N} \to \mathbb{N}$ with $r(0) = n$ is the rank function of a matrix $A \in M_{n\times n}(\mathbb{F})$ iff it is weakly decreasing and satisfies the convexity condition

$$\forall m \in \mathbb{N}:\ \ r(m) + r(m+2) \geq 2r(m+1).$$

Definition 1.4
Let $k, n \in \mathbb{N}\setminus\{0\}$. Consider functions $f, g\colon \mathbb{N} \to \mathbb{N}$ and a nonempty set $S \subseteq \mathbb{N}\setminus\{0\}$. By the rank function equation for unknown $A_1, \ldots, A_k, B \in M_{n\times n}(\mathbb{F})$ we mean the condition

$$\forall m \in S:\ \ f(r_{A_1}(m)) + \ldots + f(r_{A_k}(m)) = g(r_B(m)). \tag{1}$$

Observe that Proposition 1.2 (v) immediately implies the following important property.

Proposition 1.5
If $(A_1, \ldots, A_k, B)$ is a solution to (1) and $U_1, \ldots, U_k, V \in GL(n,\mathbb{F})$, then $(U_1^{-1}A_1U_1, \ldots, U_k^{-1}A_kU_k, V^{-1}BV)$ is also a solution to (1).

In other words, the solution set of (1) is invariant under the action of $\underbrace{GL(n,\mathbb{F})\times\ldots\times GL(n,\mathbb{F})}_{\text{k+1}}$ on $\underbrace{M_{n\times n}(\mathbb{F})\times\ldots\times M_{n\times n}(\mathbb{F})}_{\text{k+1}}$ defined by

$$((A_1,\ldots,A_k,B),(U_1,\ldots,U_k,V))\longmapsto(U_1^{-1}A_1U_1,\ldots,U_k^{-1}A_kU_k,V^{-1}BV).$$

In the sequel, this action is referred to as the action by conjugation.

We start our investigations with nilpotent matrices.

2. Nilpotent case with $f(m)=g(m)=m$

Our first aim is to find all non-trivial solutions $(A_1,\ldots,A_k,B)$ to the rank function equation

$$r_{A_1}(m)+\ldots+r_{A_k}(m)=r_B(m) \tag{2}$$

with a reasonably chosen set S, which consist of nilpotent matrices.

We will need a few standard definitions and facts.

Definition 2.1
The matrix

$$N_k=\begin{pmatrix}0&1&0&\ldots&0&0\\0&0&1&\ldots&0&0\\0&0&0&\ldots&0&0\\\vdots&\vdots&\vdots&&\vdots&\vdots\\0&0&0&\ldots&0&1\\0&0&0&\ldots&0&0\end{pmatrix}\in M_{k\times k}(\mathbb{F})$$

is called the Jordan nilpotent block of size k.

Observe that the Jordan nilpotent block of size 1 is equal to 0, and that

$$r_{N_k}(m)=\begin{cases}k & \text{for } m=0,\\ k-m & \text{for } 1\le m\le k,\\ 0 & \text{for } m>k.\end{cases}$$

Theorem 2.2
Let $A\in M_{n\times n}(\mathbb{F})$ be a nilpotent matrix. Then there exist $U\in GL(n,\mathbb{F})$, $\ell\in\mathbb{N}\setminus\{0\}$, and a weakly decreasing sequence $(k_1,\ldots,k_\ell)$ of positive integers such that $U^{-1}AU=N_{k_1}\oplus\ldots\oplus N_{k_\ell}$. Moreover, ℓ and $(k_1,\ldots,k_\ell)$ are uniquely determined by the matrix A.

The matrix $N_{k_1}\oplus\ldots\oplus N_{k_\ell}$ is referred to as the Jordan canonical form of A.

Definition 2.3
The sequence $(k_1,\ldots,k_\ell)$ from Theorem 2.2 is called the Jordan partition of the matrix A, and denoted by $\mathrm{jp}(A)$.

Definition 2.4
Let $A \in M_{n\times n}(\mathbb{F})$ be a nilpotent matrix. The conjugate of the Jordan partition $\mathrm{jp}(A) = (k_1, \ldots, k_\ell)$ is the sequence $\mathrm{jp}(A)^* = (q_1, \ldots, q_{k_1})$ defined by $q_j = \#\{i \in \{1, \ldots, \ell\} : k_i \geq j\}$.

Notice that

$$\begin{cases} k_1 = \min\{m \in \mathbb{N} : r_A(m) = 0\}, \\ q_j = r_A(j-1) - r_A(j) \qquad \text{for } j = 1, \ldots, k_1. \end{cases} \tag{3}$$

We are in a position to formulate and prove two key results.

Proposition 2.5
Let $k, n, n_1, \ldots, n_k \in \mathbb{N} \setminus \{0\}$, and let $B_i \in M_{n_i \times n_i}(\mathbb{F})$, $i = 1, \ldots, k$, be nilpotent matrices whose Jordan canonical forms do not contain blocks of size 1. Assume that $n_1 + \ldots + n_k \leq n$. Then $A_i = B_i \oplus O_{n-n_i}$, $i = 1, \ldots, k$, and $B = B_1 \oplus \ldots \oplus B_k \oplus O_{n-(n_1+\ldots+n_k)}$ form a solution to equation (2) with $S = \mathbb{N} \setminus \{0\}$.

Proof. Straightforward verification, based on Proposition 1.2 (vi).

The matrices $A_1, \ldots, A_k, B$ from the above proposition are nonzero and nilpotent. In the sequel, we write "nilpotent solution" instead of "solution consisting of nilpotent $n \times n$ matrices over $\mathbb{F}$".

Theorem 2.6
Each non-trivial nilpotent solution to (2) with $S = \{1, \ldots, n\}$ has, up to conjugation, the form described in Proposition 2.5.

Proof. Let $(A_1, \ldots, A_k, B)$ be a non-trivial nilpotent solution to (2) with $S = \{1, \ldots, n\}$. For $i = 1, \ldots, k$ define $B_i \in M_{n_i \times n_i}(\mathbb{F})$ to be the direct sum of all nonzero blocks contained in the Jordan canonical form of A_i. Moreover, define $C \in M_{p\times p}(\mathbb{F})$ to be the direct sum of all nonzero blocks contained in the Jordan canonical form of B. (The blocks are ordered from largest to smallest). It is clear that

$$\exists U_1, \ldots, U_k, V \in GL(n, \mathbb{F}) : \begin{cases} U_i^{-1} A_i U_i = B_i \oplus O_{n-n_i} \qquad (i = 1, \ldots, k), \\ V^{-1} B V = C \oplus O_{n-p}. \end{cases}$$

The proof will be completed, if we show that $n_1 + \ldots + n_k = p$ and $W^{-1}CW = B_1 \oplus \ldots \oplus B_k$ for a certain $W \in GL(p, \mathbb{F})$. Observe that

$$\begin{aligned} r_{B_1\oplus\ldots\oplus B_k}(m) &= \sum_{i=1}^{k} r_{B_i}(m) = \sum_{i=1}^{k} r_{U_i^{-1}A_iU_i}(m) = \sum_{i=1}^{k} r_{A_i}(m) \\ &= r_B(m) = r_{V^{-1}BV}(m) \\ &= r_C(m) \end{aligned}$$

for all $m \in S$, and that $r_{B_1\oplus\ldots\oplus B_k}(n) = r_C(n) = 0$. Consider now the conditions

$$\begin{cases} \forall m \in S : \ r_{B_1\oplus\ldots\oplus B_k}(m) = r_C(m), \\ r_{B_1\oplus\ldots\oplus B_k}(n) = r_C(n) = 0, \end{cases}$$

formulae (3), and the fact that $B_1 \oplus \ldots \oplus B_k$ and C do not contain Jordan blocks of size 1. These imply that $\mathrm{jp}(B_1 \oplus \ldots \oplus B_k)^* = \mathrm{jp}(C)^*$. Consequently, $B_1 \oplus \ldots \oplus B_k$ and C may differ only in the order of blocks. Thus, $n_1 + \ldots + n_k = p$ and $B_1 \oplus \ldots \oplus B_k = W^{-1}CW$ for a certain $W \in GL(p, \mathbb{F})$.

Notice that $n \geq 2k$ whenever equation (2) with $S = \{1, \ldots, n\}$ has a non-trivial nilpotent solution.

To conclude the section, let us take a look at a harder rank function equation. Namely, we will find some non-trivial nilpotent solutions to the rank function equation

$$[r_{A_1}(m)]^2 + [r_{A_2}(m)]^2 = [r_B(m)]^2. \tag{4}$$

Example 2.7
Let $A_1, A_2, B \in M_{7\times 7}(\mathbb{F})$ be nilpotent matrices whose Jordan partitions are $(2,2,2,1)$, $(5,1,1)$, $(5,2)$, respectively. Then (A_1, A_2, B) is a solution to (4) with $S = \mathbb{N} \setminus \{0\}$.

Example 2.8
Let $A_1, A_2, B \in M_{5(\ell+1)\times 5(\ell+1)}(\mathbb{F})$, where $\ell \in \mathbb{N} \setminus \{0\}$, be nilpotent matrices such that

$$\begin{aligned}
\mathrm{jp}(A_1) &= (\ell+1, \ell+1, \ell+1, \ell+1, \underbrace{1, \ldots, 1}_{\ell+1}),\\
\mathrm{jp}(A_2) &= (\ell+1, \ell+1, \ell+1, \underbrace{1, \ldots, 1}_{2(\ell+1)}),\\
\mathrm{jp}(B) &= (\ell+1, \ell+1, \ell+1, \ell+1, \ell+1).
\end{aligned}$$

Then (A_1, A_2, B) is a solution to (4) with $S = \{1, \ldots, \ell+1\}$.

Problem 2.9
Describe all non-trivial nilpotent solutions to (4) *with* $S = \{1, \ldots, n\}$.

3. Nilpotent case with a general f and $g(m) = m$

We start with the following mixed case of the rank function equation:

$$[r_{A_1}(m)]^2 + [r_{A_2}(m)]^2 = r_B(m). \tag{5}$$

Example 3.1
If $B_1 \in M_{7\times 7}(\mathbb{F})$, $B_2 \in M_{9\times 9}(\mathbb{F})$ and $B \in M_{94\times 94}(\mathbb{F})$ are nilpotent matrices such that $\mathrm{jp}(B_1) = (3,2,2)$, $\mathrm{jp}(B_2) = (4,3,2)$ and $\mathrm{jp}(B) = (4, \underbrace{3, \ldots, 3}_{8}, \underbrace{2, \ldots, 2}_{33})$, then the matrices $A_1 = B_1 \oplus O_{87}$, $A_2 = B_2 \oplus O_{85}$ and B form a solution to (5) with $S = \mathbb{N} \setminus \{0\}$.

The function $\mathbb{R} \supset [0, \infty) \ni x \mapsto x^2 \in \mathbb{R}$ is strictly increasing and convex. Moreover, it maps every non-negative integer to a non-negative integer.

Lemma 3.2
Let $k, n_1, \ldots, n_k \in \mathbb{N} \setminus \{0\}$, and let $A_i \in M_{n_i \times n_i}(\mathbb{F})$ for $i = 1, \ldots, k$. Consider a strictly increasing convex function $f : \mathbb{R} \supset [0, \infty) \to \mathbb{R}$. Assume that $f(\mathbb{N}) \subseteq \mathbb{N}$. Then

$$\exists p \in \mathbb{N} \setminus \{0\}\ \exists D \in M_{p \times p}(\mathbb{F})\ \forall m \in \mathbb{N} :\ r_D(m) = f(r_{A_1}(m)) + \ldots + f(r_{A_k}(m)).$$

Proof. In virtue of Proposition 1.2 (vi), it is enough to prove the lemma for $k = 1$. Consider the composite function $\tilde{f} = f \circ r_{A_1} : \mathbb{N} \to \mathbb{N}$. Since r_{A_1} is weakly decreasing, so is $\tilde{f}$. By the monotonicity of f, we have $p := \tilde{f}(0) = f(r_{A_1}(0)) = f(n_1) \in \mathbb{N} \setminus \{0\}$. In virtue of Theorem 1.3, it remains to prove that

$$\forall m \in \mathbb{N} :\ \tilde{f}(m) + \tilde{f}(m + 2) \geq 2\tilde{f}(m + 1).$$

Since the function f is convex,

$$f\Big(\frac{1}{2} r_{A_1}(m) + \frac{1}{2} r_{A_1}(m + 2)\Big) \leq \frac{1}{2} f(r_{A_1}(m)) + \frac{1}{2} f(r_{A_1}(m + 2))$$

for all $m \in \mathbb{N}$. Consequently,

$$2f\Big(\frac{1}{2} r_{A_1}(m) + \frac{1}{2} r_{A_1}(m + 2)\Big) \leq \tilde{f}(m) + \tilde{f}(m + 2).$$

On the other hand, the monotonicity of f and the fact that

$$\forall m \in \mathbb{N} :\ 2r_{A_1}(m + 1) \leq r_{A_1}(m) + r_{A_1}(m + 2)$$

yield

$$2f(r_{A_1}(m + 1)) \leq 2f\Big(\frac{1}{2} r_{A_1}(m) + \frac{1}{2} r_{A_1}(m + 2)\Big).$$

Finally,

$$2\tilde{f}(m + 1) = 2f(r_{A_1}(m + 1)) \leq 2f\Big(\frac{1}{2} r_{A_1}(m) + \frac{1}{2} r_{A_1}(m + 2)\Big) \leq \tilde{f}(m) + \tilde{f}(m + 2).$$

The proof is complete.

Notice that the matrix D is nilpotent iff so are $A_1, \ldots, A_k$ and $f(0) = 0$. (It follows from the monotonicity of f, and the fact that $A_j^{\max\{n_1, \ldots, n_k\}} = O_{n_j}$ whenever A_j is nilpotent for a certain $j \in \{1, \ldots, k\}$).

In the sequel of the section, $f : \mathbb{R} \supset [0, \infty) \to \mathbb{R}$ is a strictly increasing convex function such that $f(\mathbb{N}) \subseteq \mathbb{N}$ and $f(0) = 0$.

Consider the rank function equation

$$f(r_{A_1}(m)) + \ldots + f(r_{A_k}(m)) = r_B(m). \tag{6}$$

Equations (2) and (5) are special cases of (6). The next key result of the paper reads as follows.

Theorem 3.3
Let $A_1, \ldots, A_k \in M_{n \times n}(\mathbb{F})$ be nilpotent matrices. For $m \in \mathbb{N}$ define $r(m) = f(r_{A_1}(m)) + \ldots + f(r_{A_k}(m))$. Then the following conditions are equivalent:

(•) *there exists a matrix* $B \in M_{n\times n}(\mathbb{F})$ *such that* $(A_1,\dots,A_k,B)$ *is a solution to* (6) *with* $S=\{1,\dots,n\}$,

(••) $2r(1)-r(2)\le n$.

Moreover, the matrix B *is nilpotent and unique up to the usual conjugation*

$$M_{n\times n}(\mathbb{F})\times GL(n,\mathbb{F})\ni (X,U)\longmapsto U^{-1}XU\in M_{n\times n}(\mathbb{F}).$$

Proof. In virtue of Theorem 1.3 and the fact that $r(n)=0$, condition (•) holds true iff the function $\tilde r\colon \mathbb{N}\to\mathbb{N}$ defined by

$$\tilde r(m)=\begin{cases} n & \text{for } m=0,\\ r(m) & \text{for } m\in S,\\ 0 & \text{for } m>n\end{cases}$$

is weakly decreasing and such that $\tilde r(m)+\tilde r(m+2)\ge 2\tilde r(m+1)$ for all $m\in\mathbb{N}$. By Lemma 3.2, the function r is weakly decreasing and such that $r(m)+r(m+2)\ge 2r(m+1)$ for all $m\in\mathbb{N}$. Observe that in fact $r(m)=0$ whenever $m\ge n$. Thus, (•) is satisfied iff $n\ge r(1)$ and $n+r(2)\ge 2r(1)$. By the monotonicity of r, the last two inequalities hold iff $n-r(1)\ge r(1)-r(2)$, and this is condition (••). The nilpotency of B is obvious. The uniqueness follows from the equality $r_B=\tilde r$.

Example 3.4

Consider equation (5) and the matrices $A_1,A_2\in M_{94\times 94}(\mathbb{F})$ from Example 3.1. Under the notations of Theorem 3.3, we have

$$r(m)=\begin{cases} 52 & \text{for } m=1,\\ 10 & \text{for } m=2,\\ 1 & \text{for } m=3,\\ 0 & \text{for } m\ge 4.\end{cases}$$

Consequently, $2r(1)-r(2)=104-10=94$. So, by Theorem 3.3, there exists a nilpotent matrix $B\in M_{94\times 94}(\mathbb{F})$ such that the triple (A_1,A_2,B) is a solution to (5) with $S=\{1,\dots,94\}$. Since the rank function of B is defined by

$$r_B(m)=\begin{cases} 94 & \text{for } m=0,\\ r(m) & \text{for } m\in S,\\ 0 & \text{for } m>94,\end{cases}$$

we have $\mathrm{jp}(B)^*=(94-52,52-10,10-1,1-0)=(42,42,9,1)$. This yields $\mathrm{jp}(B)=(4,\underbrace{3,\dots,3}_{8},\underbrace{2,\dots,2}_{33})$.

Example 3.5

Consider once more equation (5) and the matrix A_2 from the previous example. Let $A_1\in M_{94\times 94}(\mathbb{F})$ be a nilpotent matrix such that $\mathrm{jp}(A_1)=(4,4,2,\underbrace{1,\dots,1}_{84})$.

Then (we keep the notations of Theorem 3.3) $2r(1)-r(2) = 2\cdot 85-25 = 145 > 94$. Consequently, there is no matrix $B \in M_{94\times 94}(\mathbb{F})$ such that (A_1, A_2, B) is a solution to (5) with $S = \{1, \ldots, 94\}$.

Let us notice that Theorem 3.3 can be proven in another way, based on the following lemma which seems to be of separate interest.

Lemma 3.6
Let $A \in M_{n\times n}(\mathbb{F})$ be a nonzero nilpotent matrix and let $C \in M_{p\times p}(\mathbb{F})$ be the direct sum of all nonzero blocks contained in the Jordan canonical form of A (the order of the blocks does not matter). Then

(i) $p = 2r_A(1) - r_A(2)$,

(ii) $r_C(m) = r_A(m)$ *for all* $m \in \mathbb{N} \setminus \{0\}$.

Proof. For $j \in \mathbb{N}\setminus\{0\}$ denote by ℓ_j the number of all blocks of size j contained in the Jordan canonical form of A, and observe that

$$2r_A(1) - r_A(2) = 2\sum_{j=2}^{\infty}(j-1)\ell_j - \sum_{j=2}^{\infty}(j-2)\ell_j = \sum_{j=2}^{\infty} j\ell_j = p.$$

Property (ii) is obvious.

Another proof of Theorem 3.3. If $A_i = O_n$ for $i = 1, \ldots, k$, then the assertion is obviously true. So, we assume additionally that $A_j \neq O_n$ for some $j \in \{1, \ldots, k\}$. Observe that this assumption yields $n \geq 2$.

Suppose now that condition (•) is satisfied. Then $r_B(m) = r(m)$ for all $m \in S$. Thus, B is a nonzero nilpotent matrix. (Notice that $r_B(1) \geq f(r_{A_j}(1)) > f(0) = 0$). Let $C \in M_{p\times p}(\mathbb{F})$ be the direct sum of all nonzero blocks contained in the Jordan canonical form of B (the order of the blocks does not matter). In virtue of Lemma 3.6, we have

$$n \geq p = 2r_B(1) - r_B(2) = 2r(1) - r(2).$$

The proof of (•) $\Rightarrow$ (••) is complete.

Suppose that condition (••) is satisfied. In virtue of Lemma 3.2,

$$\exists p \in \mathbb{N} \setminus \{0\}\ \exists D \in M_{p\times p}(\mathbb{F})\ \forall m \in \mathbb{N}:\ r_D(m) = r(m).$$

The matrix D is nonzero and nilpotent. Let $\tilde{B} \in M_{q\times q}(\mathbb{F})$ be the direct sum of all nonzero blocks contained in the Jordan canonical form of D, ordered from largest to smallest. By Lemma 3.6, we have $q = 2r_D(1) - r_D(2) = 2r(1) - r(2) \leq n$. Define $B = \tilde{B} \oplus O_{n-q}$. Then B is a nilpotent matrix belonging to $M_{n\times n}(\mathbb{F})$, and

$$r_B(m) = r_{\tilde{B}}(m) = r_D(m) = r(m) = f(r_{A_1}(m)) + \ldots + f(r_{A_k}(m))$$

for all $m \in S$. The proof of (••) $\Rightarrow$ (•) is complete. The nilpotency and the uniqueness of B are obvious.

4. A geometric remark

Throughout the section, we assume that the field $\mathbb{F}$ is algebraically closed. We start with a well-known theorem due to Gerstenhaber [2]. Define $\overline{\mathcal{O}(A)}$ to be the Zariski closure of the conjugacy class of a matrix A in $M_{n\times n}(\mathbb{F}) = \mathbb{F}^{n^2}$.

Theorem 4.1
Let $A, B \in M_{n\times n}(\mathbb{F})$ be nilpotent matrices. Then the following conditions are equivalent:

(∗) $A \in \overline{\mathcal{O}(B)}$,

(∗∗) $r_A(m) \leq r_B(m)$ *for all* $m \in \mathbb{N}$.

We are in a position to give a geometric characterization of the nilpotent solutions to equation (6).

Proposition 4.2
Suppose that nilpotent matrices $A_1, \ldots, A_k, B \in M_{n\times n}(\mathbb{F})$ form a solution to equation (6) *with $S = \{1, \ldots, n\}$. Then $A_1, \ldots, A_k \in \overline{\mathcal{O}(B)}$.*

Proof. Observe that the function f in (6) must satisfy condition

$$\forall m \in \mathbb{N} : \ f(m) \geq m.$$

Consequently, $r_B(m) \geq f(r_{A_j}(m)) \geq r_{A_j}(m)$ for all $j \in \{1, \ldots, k\}$ and all $m \in \mathbb{N} \setminus \{0\}$. Thus, by the Gerstenhaber theorem, $A_j \in \overline{\mathcal{O}(B)}$.

Similarly, if $A_1, A_2, B \in M_{n\times n}(\mathbb{F})$ form a nilpotent solution to equation (4) with $S = \{1, \ldots, n\}$, then $A_1, A_2 \in \overline{\mathcal{O}(B)}$.

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Received: 30 November 2011; final version: 18 September 2012;
available online: 4 October 2012.