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Multivalued second order differential problem

Abstract. Let K be a closed convex cone with nonempty interior in a real Banach space and let $F, G, H\colon K \to cc(K)$ be three given continuous additive set-valued functions. We study the existence and uniqueness of a solution of the second order differential problem

$$D^2\Phi(t,x) = \Phi(t, H(x)), \quad \Phi(0,x) = F(x), \quad D\Phi(t,x)|_{t=0} = G(x)$$

for $t \geq 0$ and $x \in K$, where $D\Phi(t,x)$ and $D^2\Phi(t,x)$ denote the Hukuhara derivative and the second Hukuhara derivative of $\Phi(t,x)$ with respect to t.

Let X be a normed linear space. By $n(X)$ we denote the set of all nonempty subsets of X and by $b(X)$ the set of all nonempty and bounded subsets of X, whereas $c(X)$ stands for the set of all compact members of $n(X)$ and $cc(X)$ stands for the set of all convex members of $c(X)$.

We introduce addition and multiplication by scalars as follows

$$A + B = \{a + b : \ a \in A,\, b \in B\}, \qquad \lambda A = \{\lambda a : \ a \in A\}$$

for $A, B \in n(X)$ and $\lambda \in \mathbb{R}$.

A subset K of the space X is called a *cone* if $tK \subset K$ for all $t \in [0, \infty)$. We say that a cone is *convex* if it is a convex set.

Unless indicated differently, throughout the paper X denotes a normed linear space and K a convex cone in X. The Hausdorff distance d derived from the norm in X is a metric in the set $c(X)$. Concepts such as the limit of a set-valued function at a point, the continuity of a set-valued function, the integral of a set-valued function and the limit of a sequence of set-valued functions are correlated to this metric. Moreover, all linear spaces are supposed to be real.

A set-valued function $F\colon K \to n(X)$ is said to be *additive* if

$$F(x + y) = F(x) + F(y)$$

for all $x, y \in K$. An additive set-valued function F is *linear* if it is *homogeneous*, i.e.,

$$F(\lambda x) = \lambda F(x)$$

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for all $x \in K$, $\lambda \geq 0$. An additive and continuous set-valued function with convex closed and bounded values is linear.

For two set-valued functions $F\colon K \to n(X)$, $G\colon K \to n(K)$ we define a composition $(F \circ G)(x) = F(G(x)) := \bigcup\{F(y) :\ y \in G(x)\}$.

Let A, B, C be sets of $cc(X)$. We say that a set C is the *Hukuhara difference* of A and B, i.e., $C = A - B$, if $B + C = A$. If this difference exists, then it is unique (see Lemma 1 in [12]).

Let $[a, b] \subset \mathbb{R}$ be a fixed interval, $F\colon [a, b] \to cc(X)$ and assume that the Hukuhara differences $F(t) - F(s)$ exist for all $a \leq s < t \leq b$. The *Hukuhara derivative* of F at $t \in (a, b)$ is defined by the formula

$$DF(t) = \lim_{s \to t^+} \frac{F(s) - F(t)}{s - t} = \lim_{s \to t^-} \frac{F(t) - F(s)}{t - s},$$

whenever both of these limits exist. Furthermore,

$$DF(a) = \lim_{s \to a^+} \frac{F(s) - F(a)}{s - a}, \qquad DF(b) = \lim_{s \to b^-} \frac{F(b) - F(s)}{b - s}.$$

The aim of this paper is to study existence and uniqueness of a linear with respect to the second variable solution $\Phi\colon [0, \infty) \times K \to cc(K)$ of the following differential problem

$$D^2\Phi(t, x) = \Phi(t, H(x)), \quad \Phi(0, x) = F(x), \quad D\Phi(t, x)|_{t=0} = G(x), \tag{1}$$

where $F, G, H\colon K \to cc(K)$ are given continuous linear set-valued functions and $D\Phi(t, x)$ and $D^2\Phi(t, x)$ denote the Hukuhara derivative and the second Hukuhara derivative of $\Phi(t, x)$ with respect to t.

The differential problem

$$D\Phi(t, x) = \Phi(t, G(x)), \quad \Phi(0, x) = F(x),$$

where $G, F\colon K \to cc(K)$ are given continuous linear set-valued functions was studied in [15], while the second order differential problem

$$D^2\Phi(t, x) = \Phi(t, G(x)), \quad \Phi(0, x) = F(x), \quad D\Phi(t, x)|_{t=0} = \{0\},$$

where $G, F\colon K \to cc(K)$ are given continuous linear set-valued functions was investigated in [10].

Now we assume that X is a Banach space. Dinghas in [3] and Hukuhara in [4] introduced the Riemann type integral

$$\int_a^b F(t)\, dt$$

for set-valued functions. If there exists the integral of a function $F\colon [a, b] \to cc(X)$, then F is said to be *integrable*. It is known that if $F\colon \mathbb{R} \to cc(X)$ is continuous, then it is integrable on each interval $[a, b] \subset \mathbb{R}$ (cf. [4], p. 212).

Following lemmas introduce some important properties of this integral.

Lemma 1 ([4, p. 212])
If $F\colon [a,b]\to cc(X)$ is continuous and $a<c<b$, then

$$\int_a^b F(t)\,dt=\int_a^c F(t)\,dt+\int_c^b F(t)\,dt.$$

Lemma 2 ([4, p. 211])
If $F,G\colon [a,b]\to cc(X)$ are continuous, then

$$d\Big(\int_a^b F(t)\,dt,\int_a^b G(t)\,dt\Big)\le\int_a^b d(F(t),G(t))\,dt.$$

Lemma 3 ([4, p. 211])
If $F\colon [a,b]\to cc(X)$ is continuous, then

$$\Big\|\int_a^b F(t)\,dt\Big\|\le\int_a^b \|F(t)\|\,dt.$$

Lemma 4 ([9, Lemma 10])
If $F\colon [a,b]\to cc(X)$ is continuous, then the set-valued function

$$H(t)=\int_a^t F(u)\,du \qquad \text{for } t\in[a,b]$$

is continuous.

Lemma 5 ([15, Lemma 4])
If $F\colon [a,b]\to cc(X)$ is continuous and $H(t)=\int_a^t F(u)\,du$, then $DH(t)=F(t)$ for $t\in[a,b]$.

Lemma 6 ([15, Lemma 5])
If $F,G\colon [a,b]\to cc(X)$ are two differentiable set-valued functions such that $DF(t)=DG(t)$ for $t\in[a,b]$ and $F(a)=G(a)$, then

$$F(t)=G(t) \qquad \text{for } t\in[a,b].$$

Definition 1
Let X be a Banach space and let set-valued functions $F,G,H\colon K\to cc(K)$ be continuous and additive. A map $\Phi\colon [0,\infty)\times K\to cc(K)$ is said to be a solution of problem (1) if it is continuous, twice differentiable with respect to t and it satisfies the differential equation from (1) in $[0,\infty)\times K$ and the initial conditions in K.

To the problem (1) we associate the following integral equation

$$\Phi(t,x)=F(x)+tG(x)+\int_0^t\Big(\int_0^s \Phi(u,H(x))\,du\Big)\,ds \tag{2}$$

for $(t,x) \in [0,\infty) \times K$, where $F, G, H\colon K \to cc(K)$ are given continuous linear set-valued maps.

Definition 2
Let X be a Banach space and let set-valued functions $F, G, H\colon K \to cc(K)$ be continuous and additive. A map $\Phi\colon [0,\infty) \times K \to cc(K)$ is said to be a solution of (2) if it is continuous and satisfies (2) in $[0,\infty) \times K$.

The proofs of the next two theorems are based on ideas from the proofs of Proposition and Theorem 1 in [15] and Theorems 1, 2 in [10]. We repeat them with inevitable changes for the reader's convenience.

Theorem 1
Let X be a Banach space and let set-valued functions $F, G, H\colon K \to cc(K)$ be continuous and additive. Set-valued function $\Phi\colon [0,\infty) \times K \to cc(K)$ is a solution of problem (1) *if and only if it is a solution of* (2)*.*

Proof. 1° Suppose that a set-valued function $\Phi\colon [0,\infty) \times K \to cc(K)$ is a solution of (2). Then Φ is continuous in $[0,\infty) \times K$. Hence, since H is continuous in K, from Theorems 1 and 1' in [1, Chap. VI, p. 113] we get continuity of a map $(u,x) \mapsto \Phi(u, H(x))$ in $[0,\infty) \times K$. In particular, for every $x \in K$ a set-valued function

$$u \mapsto \Phi(u, H(x))$$

is continuous in $[0,\infty)$. Thus by Lemmas 4 and 5 the set-valued function

$$\Psi(t,x) = F(x) + tG(x) + \int_0^t \left(\int_0^s \Phi(u, H(x))\, du \right) ds \tag{3}$$

is twice differentiable with respect to t,

$$D\Psi(t,x) = G(x) + D\int_0^t \left(\int_0^s \Phi(u, H(x))\, du \right) ds = G(x) + \int_0^t \Phi(s, H(x))\, ds,$$

and

$$D^2\Psi(t,x) = D\int_0^t \Phi(s, H(x))\, ds = \Phi(t, H(x)).$$

By (2) we have $\Phi(t,x) = \Psi(t,x)$ for all $(t,x) \in [0,\infty) \times K$, therefore

$$D^2\Phi(t,x) = \Phi(t, H(x)), \quad \Phi(0,x) = F(x) \quad \text{and} \quad D\Phi(t,x)|_{t=0} = G(x).$$

Hence Φ satisfies (1).

2° Now assume that $\Phi\colon [0,\infty) \times K \to cc(K)$ is a solution of (1) and let Ψ be defined by equation (3) for $(t,x) \in [0,\infty) \times K$. By Lemmas 4 and 5 we get

$$D\Psi(t,x) = G(x) + \int_0^t \Phi(u, H(x))\, du$$

and

$$D^2\Psi(t,x) = \Phi(t,H(x)).$$

Since $D^2\Psi(t,x) = D^2\Phi(t,x)$ and $D\Psi(t,x)|_{t=0} = G(x) = D\Phi(t,x)|_{t=0}$, by Lemma 6 we obtain

$$D\Psi(t,x) = D\Phi(t,x) \qquad \text{for } (t,x) \in [0,\infty) \times K.$$

Thus, since $\Psi(0,x) = F(x) = \Phi(0,x)$, similarly we obtain

$$\Psi(t,x) = \Phi(t,x) \qquad \text{for } (t,x) \in [0,\infty) \times K.$$

Therefore Φ satisfies (2).

Let K be a closed convex cone in X and Y be a normed linear space. The functional

$$F \mapsto \|F\| := \sup_{x\in K,\, x\neq 0} \frac{\|F(x)\|}{\|x\|}$$

is finite for every continuous linear set-valued function $F\colon K \to c(Y)$. This functional will be called a *norm* (cf. [13]).

Next lemmas will be used in the proof of Theorem 2.

Lemma 7 ([16, Theorem 3], [13, Lemma 4])
Let Y be a normed linear space. Suppose that $\{F_i : i \in I\}$ is a family of continuous linear set-valued functions $F_i\colon K \to n(Y)$. If K is of the second category in K and $\bigcup_{i\in I} F_i(x) \in b(Y)$ for all $x \in K$, then there exists a positive constant M such that

$$\sup_{i\in I} \|F_i(x)\| \le M\|x\| \qquad \text{for } x \in K.$$

Lemma 8 ([13, Lemma 5])
Let Y be a normed linear space and let d be the Hausdorff distance derived from the norm in Y. Suppose that K is a convex cone with nonempty interior in X. Then there exists a positive constant M_0 such that for every linear continuous set-valued function $F\colon K \to c(Y)$ the inequality

$$d(F(x),F(y)) \le M_0\|F\|\|x-y\|$$

holds for all $x, y \in K$.

Assume that X is a Banach space and $\operatorname{int} K \neq \emptyset$. Let T be a positive real number and let $\mathcal{E}$ be the set of all continuous set-valued functions $\Phi\colon [0,T]\times K \to cc(K)$, which are linear with respect to the second variable. Define a functional ρ in $\mathcal{E}\times\mathcal{E}$ by

$$\rho(\Phi,\Psi) = \sup\{d(\Phi(t,A),\Psi(t,A)) : t \in [0,T],\ A \in cc(K),\ \|A\| \le 1\}$$

for $\Phi, \Psi \in \mathcal{E}$ (see proof of Theorem 1 in [15] and proof of Theorem 2 in [10]). Sets

$$\Phi([0,T],x) = \bigcup_{t\in[0,T]} \Phi(t,x)$$

are compact for $\Phi \in \mathcal{E}$ and $x \in K$ by Theorem 3 in [1, Chap. VI, p. 110], thus they are bounded. Therefore by Lemma 7, for every Φ there exists a positive constant M_Φ such that

$$\|\Phi(t,x)\| \le M_\Phi \|x\|$$

for $t \in [0,T]$ and $x \in K$. Hence

$$\begin{aligned} d(\Phi(t,A),\Psi(t,A)) &\le d(\Phi(t,A),\{0\}) + d(\{0\},\Psi(t,A)) = \|\Phi(t,A)\| + \|\Psi(t,A)\| \\ &\le M_\Phi + M_\Psi \end{aligned}$$

for $t \in [0,T]$ and $A \in cc(K)$ with $\|A\| \le 1$. Thus

$$\rho(\Phi,\Psi) \le M_\Phi + M_\Psi < \infty,$$

so the functional ρ is finite. It is easy to verify that ρ is a metric in $\mathcal{E}$.

Since the space $(cc(K),d)$ is complete (see [2]), $(\mathcal{E},\rho)$ is a complete metric space.

Theorem 2

Let K be a closed convex cone with nonempty interior in a Banach space and let set-valued functions $F, G, H\colon K \to cc(K)$ be continuous and additive. Then there exists exactly one solution of problem (1). *Moreover, this solution is linear with respect to the second variable.*

Proof. Fix $T > 0$ arbitrarily. On $\mathcal{E}$ we introduce a map Γ which values are set-valued functions defined by

$$(\Gamma\Phi)(t,x) := F(x) + tG(x) + \int_0^t \left(\int_0^s \Phi(u,H(x))\,du \right) ds$$

for $(t,x) \in [0,T] \times K$. It is easy to see that every set $(\Gamma\Phi)(t,x)$ belongs to $cc(K)$.

Let $\Phi \in \mathcal{E}$. We shall prove that $\Gamma\Phi$ is continuous. Fix $x, y \in K$. As above, by Lemma 7 there exists a positive constant M_Φ such that

$$\|\Phi(u,a)\| \le M_\Phi \|a\| \tag{4}$$

for $u \in [0,T]$ and $a \in K$. Hence

$$\|\Phi(u,H(x))\| \le M_\Phi \|H(x)\|$$

for $u \in [0,T]$. Let $0 \le t_1 \le t_2 \le T$. By Lemma 3

$$\begin{aligned} &\left\| \int_{t_1}^{t_2} \left(\int_0^s \Phi(u,H(x))\,du \right) ds \right\| \\ &\qquad \le \int_{t_1}^{t_2} \left(\int_0^s \|\Phi(u,H(x))\|\,du \right) ds \le \int_{t_1}^{t_2} \left(\int_0^s M_\Phi \|H(x)\|\,du \right) ds \\ &\qquad = \int_{t_1}^{t_2} s M_\Phi \|H(x)\|\,ds = M_\Phi \|H(x)\| \frac{t_2^2 - t_1^2}{2} \\ &\qquad \le (t_2 - t_1) T M_\Phi \|H(x)\|. \end{aligned} \tag{5}$$

From Lemma 8 and (4) there exists a positive constant M_0 such that

$$d(\Phi(u,a),\Phi(u,b)) \le M_0\|\Phi(u,\cdot)\|\|a-b\| \le M_0 M_\Phi \|a-b\|$$

for $u \in [0,T]$ and $a,b \in K$. This implies that

$$\Phi(u,a) \subset \Phi(u,b) + M_0 M_\Phi \|a-b\| S$$

for $u \in [0,T]$ and $a,b \in K$, where S is the closed unit ball centered at zero in X.
Let $a \in H(x)$. There exists $b \in H(y)$ for which

$$\|a-b\| = \inf\{\|a-u\| : \ u \in H(y)\}.$$

Consequently, for every $a \in H(x)$ there exists $b \in H(y)$ such that

$$\begin{aligned}\Phi(u,a) &\subset \Phi(u,b) + M_0 M_\Phi d(H(x),H(y))S \\ &\subset \Phi(u,H(y)) + M_0 M_\Phi d(H(x),H(y))S,\end{aligned}$$

whence

$$\Phi(u,H(x)) \subset \Phi(u,H(y)) + M_0 M_\Phi d(H(x),H(y))S$$

for every $u \in [0,T]$. Since $x,y \in K$ are arbitrary, we obtain

$$d(\Phi(u,H(x)),\Phi(u,H(y))) \le M_0 M_\Phi d(H(x),H(y))$$

for every $u \in [0,T]$. Therefore by Lemma 2

$$\begin{aligned} d\Big(\int_0^{t_1}\Big(\int_0^s \Phi(u,H(x))\,du\Big)\,ds, \int_0^{t_1}\Big(\int_0^s \Phi(u,H(y))\,du\Big)\,ds\Big) \\ \le \int_0^{t_1}\Big(\int_0^s d(\Phi(u,H(x)),\Phi(u,H(y))\,du\Big)\,ds \\ \le \int_0^{t_1}\Big(\int_0^s M_0 M_\Phi d(H(x),H(y))\,du\Big)\,ds \\ = \frac{t_1^2}{2} M_0 M_\Phi d(H(x),H(y)). \end{aligned} \tag{6}$$

Using Lemma 1 and properties of the Hausdorff distance we get

$$\begin{aligned} &d((\Gamma\Phi)(t_1,x),(\Gamma\Phi)(t_2,y)) \\ &\quad\le d(F(x),F(y)) + d(t_1 G(x), t_2 G(y)) \\ &\quad\quad + d\Big(\int_0^{t_1}\Big(\int_0^s \Phi(u,H(x))\,du\Big)\,ds, \int_0^{t_2}\Big(\int_0^s \Phi(u,H(y))\,du\Big)\,ds\Big) \\ &\quad\le d(F(x),F(y)) + t_1 d(G(x),G(y)) + (t_2-t_1)\|G(y)\| \end{aligned}$$

$$+ d\Big(\int_0^{t_1}\Big(\int_0^s \Phi(u,H(x))\,du\Big)\,ds, \int_0^{t_1}\Big(\int_0^s \Phi(u,H(y))\,du\Big)\,ds\Big)$$
$$+ \Big\|\int_{t_1}^{t_2}\Big(\int_0^s \Phi(u,H(y))\,du\Big)\,ds\Big\|,$$

hence from inequalities (5) and (6)

$$\begin{aligned} &d((\Gamma\Phi)(t_1,x),(\Gamma\Phi)(t_2,y)) \\ &\quad\le d(F(x),F(y)) + t_1 d(G(x),G(y)) + (t_2-t_1)\|G(y)\| \\ &\qquad + \frac{t_1^2}{2}M_0M_\Phi d(H(x),H(y)) + (t_2-t_1)TM_\Phi\|H(y)\|. \end{aligned}$$

Since F, G and H are continuous, this shows that $\Gamma\Phi$ is a continuous set-valued function. It is easily seen that $x \mapsto (\Gamma\Phi)(t,x)$ are linear for all $t \in [0,T]$. This implies that $\Gamma(\mathcal{E}) \subset \mathcal{E}$.

Next we shall prove that Γ has exactly one fixed point. Fix $\Phi, \Psi \in \mathcal{E}$ arbitrarily. By Lemma 2 we have

$$\begin{aligned} &d((\Gamma\Phi)(t,x),(\Gamma\Psi)(t,x)) \\ &\quad= d\Big(\int_0^t\Big(\int_0^s \Phi(u,H(x))\,du\Big)\,ds, \int_0^t\Big(\int_0^s \Psi(u,H(x))\,du\Big)\,ds\Big) \qquad (7) \\ &\quad\le \int_0^t\Big(\int_0^s d(\Phi(u,H(x)),\Psi(u,H(x))\,du\Big)\,ds \end{aligned}$$

for $t \in [0,T]$ and $x \in K$, which implies that

$$d((\Gamma\Phi)(t,x),(\Gamma\Psi)(t,x)) \le \frac{t^2}{2}\|H(x)\|\rho(\Phi,\Psi) \qquad (8)$$

for $t \in [0,T]$ and $x \in K$ and consequently

$$\rho(\Gamma\Phi,\Gamma\Psi) \le \frac{T^2}{2}\|H\|\rho(\Phi,\Psi).$$

Let

$$\Phi_1(t,x) := (\Gamma\Phi)(t,x), \qquad \Psi_1(t,x) := (\Gamma\Psi)(t,x).$$

From (7) we have

$$\begin{aligned} d((\Gamma^2\Phi)(t,x),(\Gamma^2\Psi)(t,x)) &= d((\Gamma\Phi_1)(t,x),(\Gamma\Psi_1)(t,x)) \\ &\le \int_0^t\Big(\int_0^s d(\Phi_1(u,H(x)),\Psi_1(u,H(x)))\,du\Big)\,ds \end{aligned}$$

for $t \in [0,T]$ and $x \in K$, whereas from (8)

$$d(\Phi_1(u,y),\Psi_1(u,y)) \le \frac{u^2}{2}\|H(y)\|\rho(\Phi,\Psi)$$

for all $y \in H(x)$, thus

$$d(\Phi_1(u,y),\Psi_1(u,y)) \le \frac{u^2}{2}\|H(H(x))\|\rho(\Phi,\Psi).$$

It is easy to verify that this implies that

$$d(\Phi_1(u,H(x)),\Psi_1(u,H(x))) \le \frac{u^2}{2}\|H^2(x)\|\rho(\Phi,\Psi),$$

and therefore we get

$$\begin{aligned} d((\Gamma^2\Phi)(t,x),(\Gamma^2\Psi)(t,x)) &\le \int_0^t \Big(\int_0^s \frac{u^2}{2}\|H^2(x)\|\rho(\Phi,\Psi)\,du\Big)\,ds \\ &= \frac{t^4}{4!}\|H^2(x)\|\rho(\Phi,\Psi) \end{aligned}$$

for $t \in [0,T]$ and $x \in K$, thus

$$\rho(\Gamma^2\Phi,\Gamma^2\Psi) \le \frac{T^4}{4!}\|H^2\|\rho(\Phi,\Psi).$$

By induction we can prove that

$$\rho(\Gamma^n\Phi,\Gamma^n\Psi) \le \frac{T^{2n}}{(2n)!}\|H\|^n\rho(\Phi,\Psi)$$

for every positive integer n. Since T is a positive constant, there is $n \in \mathbb{N}$ such that $\frac{T^{2n}}{(2n)!}\|H\|^n < 1$. From the Banach's fixed point Theorem Γ^n has exactly one fixed point Φ. But

$$\Gamma^n(\Gamma\Phi) = \Gamma(\Gamma^n\Phi) = \Gamma\Phi.$$

Since Φ is a unique fixed point of Γ^n, we get $\Gamma\Phi = \Phi$. If Φ was not unique, Γ^n would also have more than one fixed point. Therefore we obtain existence and uniqueness of $\Phi \in \mathcal{E}$ satisfying the differential equation from (1) in $[0,T]\times K$ and the initial conditions in K. Since T was arbitrary, this finishes the proof.

Let $\{F_t : t \ge 0\}$ be a family of set-valued functions $F_t\colon K \to n(X)$, $t \ge 0$. A family $\{E_t : t \ge 0\}$ of set-valued functions $E_t\colon K \to n(K)$, $t \ge 0$, is called a *sine family associated with family* $\{F_t : t \ge 0\}$, if

$$E_{t+s}(x) = E_{t-s}(x) + 2F_t(E_s(x)) \tag{9}$$

for $0 \le s \le t$ and $x \in K$.

A sine family $\{E_t : t \ge 0\}$ of set-valued functions with compact values is called *regular* if $\lim_{t\to 0^+} \frac{E_t(x)}{t} = \{x\}$ (cf. [5]).

Lemma 9 ([5, Proposition 1])
Assume that $\{F_t:\ t\geq 0\}$ *and* $\{E_t:\ t\geq 0\}$ *are families of set-valued functions* $F_t\colon K\to n(X)$, $E_t\colon K\to n(X)$ *such that* F_0 *is continuous linear,* $F_0(x)\in c(K)$, $E_0(x)\in cc(K)$, $x\in F_0(x)$ *for* $x\in K$. *If* $\{E_t:\ t\geq 0\}$ *is a sine family associated with the family* $\{F_t:\ t\geq 0\}$, *then* $E_0(x)=\{0\}$ *for* $x\in K$.

Lemma 10 ([5, Theorem 3], [6, Theorem 3])
Let X *be a Banach space,* K *a closed convex cone with nonempty interior in* X *and let* $\{F_t:\ t\geq 0\}$ *and* $\{E_t:\ t\geq 0\}$ *be families of continuous additive set-valued functions* $F_t\colon K\to cc(K)$, $E_t\colon K\to cc(K)$, $F_0(x)=\{x\}$ *for* $x\in K$ *and* $x\in F_t(x)$ *for* $x\in K$ *and* $t>0$. *Assume that* $\{E_t:\ t\geq 0\}$ *is a regular sine family associated with* $\{F_t:\ t\geq 0\}$. *Then the set-valued function* $u\mapsto F_u(x)$ *is continuous for every* $x\in K$ *and*

$$E_t(x)=\int_0^t F_u(x)du,\qquad t\geq 0,\ x\in K.$$

A family $\{F_t:\ t\geq 0\}$ of set-valued functions $F_t\colon K\to n(K)$ is called a *cosine family*, if

$$F_0(x)=\{x\} \tag{10}$$

for all $x\in K$ and

$$F_{t+s}(x)+F_{t-s}(x)=2F_t(F_s(x)), \tag{11}$$

whenever $0\leq s\leq t$ and $x\in K$.

A cosine family $\{F_t:\ t\geq 0\}$ of set-valued functions with compact values is called *regular* if $\lim_{t\to 0^+}F_t(x)=\{x\}$ (cf. [13]).

Lemma 11 ([8, Theorem])
Let K *be a closed convex cone with nonempty interior in a Banach space* X. *Suppose that* $\{F_t:\ t\geq 0\}$ *is a regular cosine family of continuous linear set-valued functions* $F_t\colon K\to cc(K)$, $x\in F_t(x)$ *for all* $x\in K$, $t>0$ *and* $F_t\circ F_s=F_s\circ F_t$ *for all* $s,t>0$. *Then this cosine family is twice differentiable and*

$$D^2F_t(x)=F_t(H(x))\quad\textit{and}\quad DF_t(x)|_{t=0}=\{0\} \tag{12}$$

for $x\in K$, $t\geq 0$, *where* $DF_t(x)$ *and* $D^2F_t(x)$ *denote the Hukuhara derivative and the second Hukuhara derivative of* $F_t(x)$ *with respect to* t, *respectively, and* $H(x)=D^2F_t(x)|_{t=0}$.

We shall need some further properties of the Hukuhara derivative and of the Riemann integral.

Lemma 12 ([14, Lemma 3])
Let K *be a closed convex cone in a linear space* X. *Assume that* $F\colon K\to cc(K)$ *is a continuous additive set-valued function and* $A,B\in cc(K)$. *If there exists the difference* $A-B$, *then there exists* $F(A)-F(B)$ *and* $F(A)-F(B)=F(A-B)$.

Lemma 13

Let K be a closed convex cone in X and $[a,b]\subset\mathbb{R}$ be a given interval. Let $F\colon K\to cc(K)$ be a continuous additive set-valued function and $G\colon[a,b]\to cc(K)$ be a differentiable set-valued function. Then $D(F\circ G(t))$ exists and $D(F\circ G(t)) = F\circ DG(t)$.

Proof. By the definition of the Hukuhara derivative and Lemma 12

$$D(F\circ G)(t)=\lim_{s\to t^+}\frac{F\circ G(s)-F\circ G(t)}{s-t}=\lim_{s\to t^+}\frac{F[G(s)-G(t)]}{s-t}.$$

Since F is linear and continuous we have

$$\lim_{s\to t^+}\frac{F[G(s)-G(t)]}{s-t}=F\Big(\lim_{s\to t^+}\frac{G(s)-G(t)}{s-t}\Big)=F\circ DG(t)$$

(see Lemma 8 and [13, Lemma 6]). We use the same reasoning when s converges to t from the left.

Lemma 14

Let K be a convex cone with nonempty interior in a Banach space and let $\{F_t:\ t\geq 0\}$ be a regular cosine family of continuous additive set-valued functions $F_t\colon K\to cc(K)$ such that $F_t\circ F_s=F_s\circ F_t$ for all $s,t\geq 0$. Assume that $\Phi\colon K\to cc(K)$ is continuous additive and $F_t\circ\Phi=\Phi\circ F_t$ for all $t\geq 0$. Then

$$\Big(\int_0^s F_u(\cdot)\,du\Big)(\Phi(x))=\int_0^s F_u(\Phi(x))\,du.$$

Proof. First we shall prove that

$$\big(F_{\frac{s}{4}}+F_{\frac{3s}{4}}\big)(\Phi(x))=F_{\frac{s}{4}}(\Phi(x))+F_{\frac{3s}{4}}(\Phi(x)).$$

Indeed, since $\{F_t:\ t\geq 0\}$ is a cosine family, by (11) and the commutativity of Φ and all F_t we have

$$\begin{aligned}\big(F_{\frac{s}{4}}+F_{\frac{3s}{4}}\big)(\Phi(x)) &= \big(F_{\frac{s}{2}-\frac{s}{4}}+F_{\frac{s}{2}+\frac{s}{4}}\big)(\Phi(x))=2\big(F_{\frac{s}{2}}\circ F_{\frac{s}{4}}\big)(\Phi(x))\\ &=\Phi\big(2F_{\frac{s}{2}}\circ F_{\frac{s}{4}}(x)\big)=\Phi\big(F_{\frac{s}{4}}(x)+F_{\frac{3s}{4}}(x)\big)\\ &=\Phi\big(F_{\frac{s}{4}}(x)\big)+\Phi\big(F_{\frac{3s}{4}}(x)\big)\\ &=F_{\frac{s}{4}}(\Phi(x))+F_{\frac{3s}{4}}(\Phi(x)).\end{aligned}$$

Let us now assume that for some positive integer n and for all $x\in K$ we have

$$\Big(\sum_{i=0}^{2^n-1}F_{(\frac{1}{2^{n+1}}+\frac{i}{2^n})s}\Big)(\Phi(x))=\sum_{i=0}^{2^n-1}F_{(\frac{1}{2^{n+1}}+\frac{i}{2^n})s}(\Phi(x)).\tag{13}$$

Then, by (11) we obtain

$$\begin{aligned}
&\left(\sum_{i=0}^{2^{n+1}-1} F_{(\frac{1}{2^{n+2}}+\frac{i}{2^{n+1}})s}\right)(\Phi(x)) \\
&\quad= \left(\sum_{i=0}^{2^n-1} \left[F_{(\frac{1}{2^{n+2}}+\frac{2i+1}{2^{n+1}})s} + F_{(\frac{1}{2^{n+2}}+\frac{2i}{2^{n+1}})s}\right]\right)(\Phi(x)) \\
&\quad= \left(\sum_{i=0}^{2^n-1} \left[F_{(\frac{1}{2^{n+1}}+\frac{i}{2^n}+\frac{1}{2^{n+2}})s} + F_{(\frac{1}{2^{n+1}}+\frac{i}{2^n}-\frac{1}{2^{n+2}})s}\right]\right)(\Phi(x)) \\
&\quad= \left(\sum_{i=0}^{2^n-1} 2\left[F_{(\frac{1}{2^{n+1}}+\frac{i}{2^n})s} \circ F_{\frac{s}{2^{n+2}}}\right]\right)(\Phi(x)).
\end{aligned}$$

From the commutativity of F_t, F_s and Φ and (13), on account of the fact that Φ and F_t are additive we have

$$\begin{aligned}
&\left(\sum_{i=0}^{2^n-1} 2\left[F_{(\frac{1}{2^{n+1}}+\frac{i}{2^n})s} \circ F_{\frac{s}{2^{n+2}}}\right]\right)(\Phi(x)) \\
&\quad= F_{\frac{s}{2^{n+2}}}\left[\left(\sum_{i=0}^{2^n-1} 2F_{(\frac{1}{2^{n+1}}+\frac{i}{2^n})s}\right)(\Phi(x))\right] = F_{\frac{s}{2^{n+2}}}\left[\sum_{i=0}^{2^n-1} 2F_{(\frac{1}{2^{n+1}}+\frac{i}{2^n})s}(\Phi(x))\right] \\
&\quad= \Phi\left[\sum_{i=0}^{2^n-1} 2F_{(\frac{1}{2^{n+1}}+\frac{i}{2^n})s} \circ F_{\frac{s}{2^{n+2}}}(x)\right].
\end{aligned}$$

Again using (11) and the commutativity we get

$$\begin{aligned}
&\Phi\left[\sum_{i=0}^{2^n-1} 2F_{(\frac{1}{2^{n+1}}+\frac{i}{2^n})s} \circ F_{\frac{s}{2^{n+2}}}(x)\right] \\
&\quad= \Phi\left[\sum_{i=0}^{2^n-1} F_{(\frac{1}{2^{n+1}}+\frac{i}{2^n}+\frac{1}{2^{n+2}})s}(x) + F_{(\frac{1}{2^{n+1}}+\frac{i}{2^n}-\frac{1}{2^{n+2}})s}(x)\right] \\
&\quad= \sum_{i=0}^{2^n-1} \left[F_{(\frac{1}{2^{n+2}}+\frac{2i+1}{2^{n+1}})s}(\Phi(x)) + F_{(\frac{1}{2^{n+2}}+\frac{2i}{2^{n+1}})s}(\Phi(x))\right] \\
&\quad= \sum_{i=0}^{2^{n+1}-1} F_{(\frac{1}{2^{n+2}}+\frac{i}{2^{n+1}})s}(\Phi(x)).
\end{aligned}$$

Hence we have proved equality (13) for all positive integers n. Multiplying both sides of it by $\frac{s}{2^n}$ we get

$$\left(\sum_{i=0}^{2^n-1} \frac{s}{2^n} F_{(\frac{1}{2^{n+1}}+\frac{i}{2^n})s}\right)(\Phi(x)) = \sum_{i=0}^{2^n-1} \frac{s}{2^n} F_{(\frac{1}{2^{n+1}}+\frac{i}{2^n})s}(\Phi(x))$$

Since $\{F_t : t \geq 0\}$ is regular, it is continuous (cf. Theorem 2 in [13]) and therefore integrable. Hence letting n tend to infinity in the above equality, by Lemma 5 in [11] we obtain the result.

The following theorem gives the solution of the problem (1) in two special cases.

THEOREM 3

Let K be a closed convex cone with nonempty interior in a Banach space. Let $\{F_t: t \geq 0\}$ be a regular cosine family of continuous additive set-valued functions $F_t: K \to cc(K)$ such that $x \in F_t(x)$ for all $x \in K$, $t \geq 0$, $F_t \circ F_s = F_s \circ F_t$ for all $s, t \geq 0$ and $H(x)$ is the second Hukuhara derivative of $F_t(x)$ at $t = 0$.

(a) *Assume that there is $G(x) = \{0\}$ in problem (1). Then $\Phi(t, x) = F \circ F_t(x)$, $(t, x) \in [0, \infty) \times K$ is the unique solution of this problem.*

(b) *Let $\{E_t: t \geq 0\}$ be a regular sine family of continuous additive set-valued functions $E_t: K \to cc(K)$ associated with $\{F_t: t \geq 0\}$. Assume that $F_t \circ H = H \circ F_t$ for all $t \geq 0$ and there is $F(x) = \{0\}$ in problem (1). Then $\Phi(t, x) = G \circ E_t(x)$, $(t, x) \in [0, \infty) \times K$ is the unique solution of this problem.*

Proof. (a) From Lemmas 11 and 13 the set-valued function Φ fulfills equality (1). The initial conditions

$$\Phi(0, x) = F(x), \qquad D\Phi(t, x)|_{t=0} = \{0\}$$

are satisfied on account of (10) and (12). By Theorem 2 this solution is unique.

(b) First we shall prove that the set-valued function $(t, x) \mapsto E_t(x)$ satisfies (1). From Lemma 10 we have $DE_t(x) = F_t(x)$ and therefore

$$D^2 E_t(x) = DF_t(x) =: G_t(x), \qquad H_t(x) := D^2 F_t(x) = DG_t(x).$$

Since $G_0(x) = \{0\}$ and $H_t(x) = F_t(H(x))$ (cf. Lemma 11), from Lemma 14 we obtain

$$\begin{aligned} D^2 E_t(x) = G_t(x) &= \int_0^t H_u(x)\, du = \int_0^t F_t(H(x))\, du = \Big(\int_0^t F_t(\cdot)\, du\Big)(H(x)) \\ &= E_t(H(x)). \end{aligned}$$

By Lemma 13

$$D^2\Phi(t, x) = D^2(G \circ E_t(x)) = G \circ D^2 E_t(x) = G \circ E_t(H(x)) = \Phi(t, H(x)).$$

Of course

$$\Phi(0, x) = G \circ E_0(x) = \{0\}$$

and

$$D\Phi(t, x)|_{t=0} = G \circ DE_t(x)|_{t=0} = G(F_0(x)) = G(x).$$

A simple corollary of Theorem 3 is the following

Corollary 1
Under assumptions of Theorem 3*, if* $F_t \circ H = H \circ F_t$ *for all* $t \geq 0$ *and the map* $H(x)$ *is single-valued, then the set-valued function* $\Phi(t,x) = F \circ F_t(x) + G \circ E_t(x)$ *is the unique solution of problem* (1)*.*

Proof. By Theorem 3 we have

$$D^2\Phi(t,x) = D^2(F \circ F_t(x)) + D^2(G \circ E_t(x)) = F \circ F_t(H(x)) + G \circ E_t(H(x)).$$

Therefore, since H is single-valued, we obtain

$$D^2\Phi(t,x) = (F \circ F_t + G \circ E_t)(H(x)) = \Phi(t, H(x)).$$

It is easy to see that Φ fulfills also the initial conditions.

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