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Ma's identity and its applications

Abstract. In the paper we distinguish the, so called, Ma's polynomials and we introduce connections of these polynomials with the classic Cauchy polynomials and the Ferrers-Jackson's polynomials. Presented connections enable to obtain certain interesting divisibility relations for all these three types of polynomials and some other symmetric polynomials. Application of the discussed identities for determining the limits of quotients of the respective polynomials in two variables are also presented here.

1. Introduction

Xinrong Ma in [1], with help of the Riordan's group, has proved the following identity

$$x^n + y^n + z^n = \sum_{k=0}^{\lfloor n/3 \rfloor} \frac{n}{n-2k} \binom{n-2k}{k} (x+y+z)^{n-3k} (xyz)^k \tag{1}$$

for every $x, y, z \in \mathbb{R}$ satisfying the condition

$$xy + yz + zx = 0.$$

Since we have $z = -\frac{xy}{x+y}$ from the last condition, the relation (1) can take the equivalent form

$$\begin{aligned} M_n(x,y) &= (x+y)^n(x^n+y^n) + (-xy)^n \\ &= \sum_{k=0}^{\lfloor n/3 \rfloor} (-1)^k \frac{n}{n-2k} \binom{n-2k}{k} (x^2+xy+y^2)^{n-3k} (xy(x+y))^{2k}. \end{aligned} \tag{2}$$

Polynomials $M_n(x,y)$, for $n \in \mathbb{N}$, will be called the Ma's polynomials.

Identity (2) is, in some sence, an alternating version of two classic identities formulated for the Cauchy polynomials

$$p_n(x,y) := (x+y)^{2n+1} - x^{2n+1} - y^{2n+1}$$

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and for the Ferrers-Jackson polynomials

$$q_n(x,y) := (x+y)^{2n} + x^{2n} + y^{2n}.$$

In [3] it is inductively proved that

$$p_n(x,y) = \sum_{k=0}^{\lfloor (n-1)/3 \rfloor} \frac{2n+1}{n-k} \binom{n-k}{2k+1} (xy(x+y))^{2k+1} (x^2+xy+y^2)^{n-1-3k} \tag{3}$$

and

$$q_n(x,y) = \sum_{k=0}^{\lfloor n/3 \rfloor} \frac{2n}{n-k} \binom{n-k}{2k} (xy(x+y))^{2k} (x^2+xy+y^2)^{n-3k}. \tag{4}$$

Paolo Ribenboim in [2] has presented the other decompositions of these polynomials (see chapter VII in [2]) together with their applications (for solutions of some special cases of the Fermat's Last Theorem).

Similarity of identity (2) to identities (3) and (4) seems to be more evident if we pay more attention to the algebraic connections between polynomials p_n, q_n and M_n. Theorem, written below, describes those connections.

THEOREM 1
The following identities hold

(A) $p_n(x,y)q_n(x,y) = p_{2n}(x,y) + xy(x+y)M_{2n-1}(x,y)$,

(B) $q_n^2(x,y) = q_{2n}(x,y) + 2M_{2n}(x,y)$,

(C) $p_n^2(x,y) = q_{2n+1}(x,y) - 2M_{2n-1}(x,y)$,

(D) $$\begin{aligned} M_n^2(x,y) &= M_{2n}(x,y) + 2(xy(x+y))^n[(x+y)^n + (-1)^n(x^n+y^n)] \\ &= M_{2n}(x,y) + 2(xy(x+y))^n \times \begin{cases} p_{\frac{n-1}{2}}(x,y) & \text{for } n \in 2\mathbb{N}-1, \\ q_{\frac{n}{2}}(x,y) & \text{for } n \in 2\mathbb{N}, \end{cases} \end{aligned}$$

(E) $M_{2n+1}(x,y) = (x^{2n+1}+y^{2n+1})p_n(x,y) + x^{2(2n+1)} + (xy)^{2n+1} + y^{2(2n+1)}$,

(F) $M_{2n}(x,y) = (x^{2n}+y^{2n})q_n(x,y) - x^{4n} - (xy)^{2n} - y^{4n}$.

All proofs of the above identities can be obtained with the aid of simple algebra. Therefore we present only few of them.

Proof. By definitions of p_n, q_n and M_n we get

$$\begin{aligned} p_n(x,y)q_n(x,y) &= (x+y)^{4n+1} - (x+y)^{2n}(x^{2n+1}+y^{2n+1}) \\ &\quad + (x+y)^{2n+1}(x^{2n}+y^{2n}) - (x^{2n}+y^{2n})(x^{2n+1}+y^{2n+1}) \\ &= p_{2n}(x,y) + (x+y)^{2n}(xy^{2n}+yx^{2n}) - x^{2n}y^{2n+1} - y^{2n}x^{2n+1} \\ &= p_{2n}(x,y) + xy(x+y)M_{2n-1}(x,y) \end{aligned}$$

which is the (A) identity,

$$\begin{aligned}p_n^2(x,y) &= (x+y)^{4n+2} - 2(x^{2n+1}+y^{2n+1})(x+y)^{2n+1}\\ &\quad + x^{4n+2} + y^{4n+2} + 2(xy)^{2n+1}\\ &= q_{2n+1}(x,y) - 2M_{2n-1}(x,y)\end{aligned}$$

which is the (C) identity and

$$\begin{aligned}&(x^{2n+1}+y^{2n+1})p_n(x,y)\\ &\quad = (x^{2n+1}+y^{2n+1})(x+y)^{2n+1} - (x^{2n+1}+y^{2n+1})^2\\ &\quad = (x^{2n+1}+y^{2n+1})(x+y)^{2n+1} - (xy)^{2n+1} - (x^{4n+2}+(xy)^{2n+1}+y^{4n+2})\\ &\quad = M_{2n+1}(x,y) - x^{4n+2} - (xy)^{2n+1} - y^{4n+2}\end{aligned}$$

which implies (E).

Our paper is devoted in principle to the application of the identities (2)–(4) and (A)–(F) (obviously, not all of them because of the size of paper).

So, in Section 2 there are considered the divisibility relations connected with the discussed in this paper polynomials in two variables.

In the last section of this paper we present one more important, in our opinion, application of identities (2)–(4) for calculating the limits of quotients of the respective polynomials in two variables. Let us emphasize the fact that exactly this analytical nature of identities (2)–(4) was one of the main impulses for preparing this paper.

2. Divisibility relations

Our next three results concern some special divisibility relations. The first one refers to the polynomials p_n, q_n and M_n, whereas the two others are formulated for polynomials of the type $x^{2n}+(xy)^n+y^{2n}$ for $n\in\mathbb{N}$.

We note that Theorem 2, given below, can be easily deduced from all three decompositions (2), (3) and (4). The detailed proof will be omitted here.

Theorem 2
From identity (3) *we get*

$$(x^2\pm xy+y^2)\mid p_n(x,\pm y) \iff 3\nmid(n-1),$$

however, from (2) *and* (4) *we receive*

$$\begin{aligned}(x^2\pm xy+y^2)\mid M_n(x,\pm y) &\iff 3\nmid n,\\ (x^2\pm xy+y^2)\mid q_n(x,\pm y) &\iff 3\nmid n.\end{aligned}$$

Moreover, if $n\equiv 1\,(\mathrm{mod}\,3)$, *then*

$$\begin{aligned}&(x^2+xy+y^2)^4 \mid \Big(M_n(x,y) - (-1)^{\frac{n-1}{3}}n(x^2+xy+y^2)(xy(x+y))^{\frac{2(n-1)}{3}}\Big),\\ &(x^2+xy+y^2)^4 \mid \Big(q_n(x,y) - 2n(x^2+xy+y^2)(xy(x+y))^{\frac{2(n-1)}{3}}\Big),\\ &(x^2+xy+y^2)^3 \mid \Big(p_n(x,y) - 3(xy(x+y))^{\frac{2n+1}{3}}\Big).\end{aligned}$$

If $n \equiv 2 \pmod 3$, *then*

$$(x^2+xy+y^2)^2 \mid M_n(x,y), \quad (x^2+xy+y^2)^2 \mid q_n(x,y), \quad (x^2+xy+y^2) \mid p_n(x,y)$$

and

$$(x^2+x\,y+y^2)^5 \mid \Big(M_n(x,y) - (-1)^{\frac{n-2}{3}} \frac{n(n+1)}{6}(x^2+xy+y^2)^2 \times (xy(x+y))^{\frac{2(n-2)}{3}}\Big),$$

$$(x^2+xy+y^2)^5 \mid \Big(q_n(x,y) - \frac{n(2n-1)}{3}(x^2+xy+y^2)^2(xy(x+y))^{\frac{2(n-2)}{3}}\Big),$$

$$(x^2+xy+y^2)^4 \mid \Big(p_n(x,y) - (2n+1)(x^2+xy+y^2)(xy(x+y))^{\frac{2n-1}{3}}\Big).$$

If $3 \mid n$, *then*

$$(x^2+xy+y^2)^2 \mid p_n(x,y)$$

and

$$(x^2+xy+y^2)^5 \mid \Big(p_n(x,y) - \frac{1}{3}n(2n+1)(x^2+xy+y^2)^2(xy(x+y))^{\frac{2n-3}{3}}\Big).$$

Furthermore, by using identities (E) and (F) from Theorem 1 and by applying Theorem 2 we obtain two following results.

Theorem 3

(1) *If* $3 \mid n$, *then there exists a polynomial* $\theta_n(x,y) \in \mathbb{Q}[x,y]$ *such that*

$$x^{4n} + (xy)^{2n} + y^{4n} = (x^2+xy+y^2)^3\theta_n(x,y) + 3(xy(x+y))^{\frac{2n}{3}}(x^{2n}+y^{2n} - (xy(x+y))^{\frac{2n}{3}}).$$

Additionally, if we assume that $x^2+xy+y^2=0$, *then we have*

$$x^{4n} + (xy)^{2n} + y^{4n} = 3(xy(x+y))^{\frac{2n}{3}}(x^{2n}+y^{2n} - (xy(x+y))^{\frac{2n}{3}}) = 3x^{4n} = 3y^{4n}. \tag{5}$$

(2) *If* $n \equiv 1 \pmod 3$, *then*

$$(x^2+xy+y^2) \mid (x^{4n}+(xy)^{2n}+y^{4n}), \tag{6}$$

$$(x^2+xy+y^2)^2 \mid \Big(x^{4n}+(xy)^{2n}+y^{4n} - 2n(x^{2n}+y^{2n}) \times (x^2+xy+y^2)(xy(x+y))^{\frac{2}{3}(n-1)}\Big), \tag{7}$$

$$(x^2+xy+y^2)^4 \mid \Big(x^{4n}+(xy)^{2n}+y^{4n} - 2n(x^2+xy+y^2) \times (xy(x+y))^{\frac{2}{3}(n-1)}\Big(x^{2n}+y^{2n} - \frac{2n+1}{6}(x^2+xy+y^2) \times (xy(x+y))^{\frac{2}{3}(n-1)}\Big)\Big). \tag{8}$$

(3) *If $n \equiv 2 \pmod 3$, then*

$$(x^2+xy+y^2) \mid (x^{4n}+(xy)^{2n}+y^{4n}), \tag{9}$$

$$(x^2+xy+y^2)^2 \mid \Big(x^{4n}+(xy)^{2n}+y^{4n}-2n(x^2+xy+y^2) \times (xy(x+y))^{\frac{2}{3}(2n-1)}\Big), \tag{10}$$

$$(x^2+xy+y^2)^4 \mid \Big(x^{4n}+(xy)^{2n}+y^{4n}-2n(x^2+xy+y^2) \times (xy(x+y))^{\frac{2}{3}(2n-1)} -\frac{1}{3}n(2n-1)(x^2+xy+y^2)^2(xy(x+y))^{\frac{2}{3}(n-2)}\Big). \tag{11}$$

Proof. (1) From (F) we obtain

$$\begin{aligned}
x^{4n}&+(xy)^{2n}+y^{4n}\\
&= (x^{2n}+y^{2n})q_n(x,y)-M_{2n}(x,y)\\
&\overset{(2),(4)}{=} (x^{2n}+y^{2n})\Big(3(xy(x+y))^{\frac{2n}{3}}\\
&\quad +2\Big(\frac{n}{3}\Big)^2\Big(\frac{2n}{3}-1\Big)(xy(x+y))^{\frac{2n-6}{3}}(x^2+xy+y^2)^3+\dots\Big)\\
&\quad -3(xy(x+y))^{\frac{4n}{3}}+2\Big(\frac{n}{3}\Big)^2\Big(\frac{2n}{3}+1\Big)(xy(x+y))^{\frac{4n-6}{3}}(x^2+xy+y^2)^3-\dots\\
&= 3(xy(x+y))^{\frac{2n}{3}}\Big(x^{2n}+y^{2n}-(xy(x+y))^{\frac{2n}{3}}\Big)+(x^2+xy+y^2)^3\theta_n(x,y),
\end{aligned}$$

where $\theta_n(x,y)$ is a certain polynomial belonging to the family $\mathbb{Q}[x,y]$.

For proving relation (5) we will need the following lemma.

Lemma 4

If $x^2+xy+y^2=0$, then $x^{2n}+y^{2n}=2(xy)^n\cos(\frac{2}{3}\pi n)$ for $n\in\mathbb{N}$. Moreover, if $y=e^{i\frac{2}{3}\pi}x$, then $x+y=e^{i\frac{\pi}{3}}x$ and $xy=(e^{i\frac{\pi}{3}}x)^2$.

Proof. First, let us set $x^{2n}+y^{2n}=G_n(xy)^n$, where $G_n\in\mathbb{C}$ for $n\in\mathbb{N}$. Thus we obtain

$$x^2+y^2=-(xy),\ x^4+2(xy)^2+y^4=(xy)^2 \implies x^4+y^4=-(xy)^2,$$

i.e., $G_1=G_2=-1$.

Generally, we have

$$\begin{aligned}
x^{2(n+1)}+y^{2(n+1)} &= (x^2+y^2)(x^{2n}+y^{2n})-(xy)^2(x^{2(n-1)}+y^{2(n-1)})\\
&= -xyG_n(xy)^n-(xy)^2G_{n-1}(xy)^{n-1}\\
&= -(G_n+G_{n-1})(xy)^{n+1},
\end{aligned}$$

i.e., $G_{n+1}+G_n+G_{n-1}=0$, which easily implies $G_n=2\cos(\frac{2}{3}\pi n)$ for $n\in\mathbb{N}$.

Next, if $y=e^{i\frac{2}{3}\pi}x$, then $y+x=(1+e^{i\frac{2}{3}\pi})x=2\cos\frac{\pi}{3}e^{i\frac{\pi}{3}}x=e^{i\frac{\pi}{3}}x$ and $xy=(e^{i\frac{\pi}{3}}x)^2$.

Now, let us present the proof of relation (5). If $x^2+xy+y^2=0$ and $y=e^{i\frac{2}{3}\pi}x$, then, by Lemma 4, we get

$$x^{2n}+y^{2n}=2(xy)^n\cos\Big(\frac{2}{3}\pi n\Big)=2(e^{i\frac{\pi}{3}}x)^{2n}=2x^{2n}$$

and

$$(xy(x+y))^{\frac{2n}{3}}=(e^{i\pi}x^3)^{\frac{2n}{3}}=e^{i\frac{2n\pi}{3}}x^{2n}=x^{2n}.$$

Thus we have

$$(xy(x+y))^{\frac{2n}{3}}\Big(x^{2n}+y^{2n}-(xy(x+y))^{\frac{2n}{3}}\Big)=x^{2n}(2x^{2n}-x^{2n})=x^{4n}.$$

(2) Similarly as in the previous case we generate the relation

$$\begin{aligned}
&x^{4n}+(xy)^{2n}+y^{4n}\\
&\quad= (x^{2n}+y^{2n})q_n(x,y)-M_{2n}(x,y)\\
&\quad\overset{(2),(4)}{=} (x^{2n}+y^{2n})\Big(2n(x^2+xy+y^2)(xy(x+y))^{\frac{2}{3}(n-1)}\\
&\qquad+\frac{n}{2}\binom{\frac{2n+1}{3}}{3}(x^2+xy+y^2)^4(xy(x+y))^{\frac{2}{3}(n-4)}+\ldots\Big)\\
&\qquad-\frac{n}{3}(2n+1)(x^2+xy+y^2)^2(xy(x+y))^{\frac{4}{3}(n-1)}\\
&\qquad+\frac{2n}{5}\binom{\frac{2n+7}{3}}{4}(x^2+xy+y^2)^5(xy(x+y))^{\frac{2}{3}(2n-5)}-\ldots
\end{aligned}$$

which easily implies all three divisibility relations (6), (7) and (8).

(3) We have

$$\begin{aligned}
&x^{4n}+(xy)^{2n}+y^{4n}\\
&\quad= (x^{2n}+y^{2n})q_n(x,y)-M_{2n}(x,y)\\
&\quad\overset{(2),(4)}{=} (x^{2n}+y^{2n})\Big(\frac{n}{3}(2n-1)(x^2+xy+y^2)^2(xy(x+y))^{\frac{2}{3}(n-2)}\\
&\qquad+\frac{2n}{5}\binom{\frac{2}{3}(n+1)}{4}(x^2+xy+y^2)^5(xy(x+y))^{\frac{2}{3}(n-5)}+\ldots\Big)\\
&\qquad-2n(x^2+xy+y^2)(xy(x+y))^{\frac{2}{3}(2n-1)}\\
&\qquad+\frac{n}{2}\binom{\frac{2n+5}{3}}{3}(x^2+xy+y^2)^4(xy(x+y))^{\frac{2}{3}(2n-4)}-\ldots
\end{aligned}$$

which implies the relations (9), (10) and (11).

Theorem 5

(1) *If* $n\equiv 1\,(\mathrm{mod}\,3)$, *then there exists a polynomial* $\Psi_n(x,y)\in\mathbb{Q}[x,y]$ *such that*

$$\begin{aligned}
&x^{2(2n+1)}+(xy)^{2n+1}+y^{2(2n+1)}\\
&\quad=(x^2+xy+y^2)^3\Psi_n(x,y)\\
&\qquad-3(xy(x+y))^{\frac{2n+1}{3}}\Big((xy(x+y))^{\frac{2n+1}{3}}+x^{2n+1}+y^{2n+1}\Big).
\end{aligned}$$

Additionally, if we assume that $x^2+xy+y^2=0$ *and* $y=e^{i\frac{2}{3}\pi}x$, *then*

$$\begin{aligned}
&x^{2(2n+1)}+(xy)^{2n+1}+y^{2(2n+1)}\\
&\quad=-3(xy(x+y))^{\frac{2n+1}{3}}\Big((xy(x+y))^{\frac{2n+1}{3}}+x^{2n+1}+y^{2n+1}\Big)=3x^{4n+2}\\
&\quad=3y^{4n+2}.
\end{aligned}$$

(2) *If* $3\mid n$, *then we have*

$$\begin{aligned}
&(x^2+xy+y^2)\mid(x^{2(2n+1)}+(xy)^{2n+1}+y^{2(2n+1)}),\\
&(x^2+xy+y^2)^2\mid\Big(x^{2(2n+1)}+(xy)^{2n+1}+y^{2(2n+1)}-(2n+1)\\
&\qquad\times(x^2+xy+y^2)(xy(x+y))^{\frac{4}{3}n}\Big),\\
&(x^2+xy+y^2)^4\mid\Big(x^{2(2n+1)}+(xy)^{2n+1}+y^{2(2n+1)}\\
&\qquad-(2n+1)(x^2+xy+y^2)(xy(x+y))^{\frac{4}{3}n}\\
&\qquad+\frac{1}{3}n(2n+1)(x^{2n+1}+y^{2n+1})(x^2+xy+y^2)^2(xy(x+y))^{\frac{2}{3}n}\Big).
\end{aligned}$$

(3) *If* $n\equiv 2\,(\mathrm{mod}\,3)$, *then we have*

$$\begin{aligned}
&(x^2+xy+y^2)\mid(x^{2(2n+1)}+(xy)^{2n+1}+y^{2(2n+1)}),\\
&(x^2+xy+y^2)^2\mid\Big(x^{2(2n+1)}+(xy)^{2n+1}+y^{2(2n+1)}+(2n+1)\\
&\qquad\times(x^{2n+1}+y^{2n+1})(x^2+xy+y^2)(xy(x+y))^{\frac{2n-1}{3}}\Big),\\
&(x^2+xy+y^2)^4\mid\Big(x^{2(2n+1)}+(xy)^{2n+1}+y^{2(2n+1)}\\
&\qquad+(2n+1)(x^{2n+1}+y^{2n+1})(x^2+xy+y^2)(xy(x+y))^{\frac{2n-1}{3}}\\
&\qquad+\frac{1}{3}(n+1)(2n+1)(x^2+xy+y^2)^2(xy(x+y))^{\frac{4n-2}{3}}\Big).
\end{aligned}$$

Proof of Theorem 5 runs in the similar way as the proof of Theorem 3 and will be omitted here.

Corollary 6
The following relation holds true

$$(x^2+xy+y^2)\mid(x^{2n}+(xy)^n+y^{2n})\iff 3\nmid n.$$

We note that

$$(x^2-xy+y^2)\mid(x^{2n}-(xy)^n+y^{2n})\iff n\ \textit{is odd and}\ 3\nmid n.$$

3. Limits of quotients of polynomials

The following three sequences of limits hold

(i)

$$\lim_{\substack{x,y\in\mathbb{C}\setminus\{0\},\ \theta:=y/x,\\ 1+\theta+\theta^2\neq 0,\\ \frac{(\theta+\theta^2)^2}{(1+\theta+\theta^2)^3}\to g}} \frac{q_n(x,y)}{(x^2+x\,y+y^2)^n} = \sum_{k=0}^{\lfloor n/3\rfloor} \frac{2n}{n-k}\binom{n-k}{2k} g^k,$$

$$\lim_{-\shortparallel-}\Big(\frac{q_n(x,y)}{(x^2+x\,y+y^2)^n}-2\Big) = \sum_{k=1}^{\lfloor n/3\rfloor} \frac{2n}{n-k}\binom{n-k}{2k} g^k,$$

$$\lim_{-\shortparallel-}\Big(\frac{q_n(x,y)}{(x^2+x\,y+y^2)^n}-2-n(n-2)g\Big) = \sum_{k=2}^{\lfloor n/3\rfloor} \frac{2n}{n-k}\binom{n-k}{2k} g^k,$$

etc.

Sketch of the proof. From (4) we get

$$\frac{q_n(x,y)}{(x^2+xy+y^2)^n} = \sum_{k=0}^{\lfloor n/3\rfloor} \frac{2n}{n-k}\binom{n-k}{2k}\left(\frac{\big(\frac{y}{x}(1+\frac{y}{x})\big)^2}{\big(1+\frac{y}{x}+(\frac{y}{x})^2\big)^3}\right)^k,$$

which easily implies the final relations.

(ii)

$$\lim_{-\shortparallel-}\frac{M_n(x,y)}{(x^2+xy+y^2)^n} = \sum_{k=0}^{\lfloor n/3\rfloor}(-1)^k\frac{n}{n-2k}\binom{n-2k}{k} g^k,$$

$$\lim_{-\shortparallel-}\Big(\frac{M_n(x,y)}{(x^2+xy+y^2)^n}-1\Big) = \sum_{k=1}^{\lfloor n/3\rfloor}(-1)^k\frac{n}{n-2k}\binom{n-2k}{k} g^k,$$

$$\lim_{-\shortparallel-}\Big(\frac{M_n(x,y)}{(x^2+xy+y^2)^n}-1+n\,g\Big) = \sum_{k=2}^{\lfloor n/3\rfloor}(-1)^k\frac{n}{n-2k}\binom{n-2k}{k} g^k,$$

etc.

Sketch of the proof. From (2) we obtain

$$\frac{M_n(x,y)}{(x^2+xy+y^2)^n} = \sum_{k=0}^{\lfloor n/3\rfloor}(-1)^k\frac{n}{n-2k}\binom{n-2k}{k}\left(\frac{\big(\frac{y}{x}(1+\frac{y}{x})\big)^2}{\big(1+\frac{y}{x}+(\frac{y}{x})^2\big)^3}\right)^k,$$

which implies the above relations.

(iii)

$$\lim_{\substack{x,y\in\mathbb{C}\setminus\{0\},\ x+y\neq 0,\\ \theta:=y/x,\ 1+\theta+\theta^2\neq 0,\\ \frac{(\theta+\theta^2)^2}{(1+\theta+\theta^2)^3}\to g}} \frac{p_n(x,y)}{xy(x+y)(x^2+xy+y^2)^{n-1}} = \sum_{k=0}^{\lfloor (n-1)/3\rfloor} \frac{2n+1}{n-k}\binom{n-k}{2k+1} g^k,$$

$$\lim_{-\shortparallel-}\Big(\frac{p_n(x,y)}{xy(x+y)(x^2+xy+y^2)^{n-1}}-2n-1\Big) = \sum_{k=1}^{\lfloor (n-1)/3\rfloor} \frac{2n+1}{n-k}\binom{n-k}{2k+1} g^k,$$

$$\lim_{-\|-}\left(\frac{p_n(x,y)}{xy(x+y)(x^2+xy+y^2)^{n-1}}-2n-1-\frac{1}{6}(2n+1)(n-2)(n-3)g\right)$$
$$=\sum_{k=2}^{\lfloor (n-1)/3\rfloor}\frac{2n+1}{n-k}\binom{n-k}{2k+1}g^k,$$

etc.

Sketch of the proof. Immediately from (3) we obtain

$$\frac{p_n(x,y)}{xy(x+y)(x^2+xy+y^2)^{n-1}}$$
$$=\sum_{k=0}^{\lfloor (n-1)/3\rfloor}(-1)^k\frac{2n+1}{n-k}\binom{n-k}{2k+1}\left(\frac{\left(\frac{y}{x}\left(1+\frac{y}{x}\right)\right)^2}{\left(1+\frac{y}{x}+\left(\frac{y}{x}\right)^2\right)^3}\right)^k,$$

which implies our limits.

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