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Classes of multivalent analytic functions with Montel's normalization

Abstract. In this paper we define classes of functions with Montel's normalization. We investigate the coefficients estimates, distortion properties, the radii of starlikeness and convexity, subordination theorems, partial sums and integral means inequalities for the defined classes of functions. Some remarks depicting consequences of the main results are also mentioned.

1. Introduction and basic notations

Let $\mathcal{A}$ denote the class of functions which are *analytic* in $\mathcal{U} = \mathcal{U}(1)$, where

$$\mathcal{U}(r) = \{z \in \mathbb{C} : \ |z| < r\}$$

is an open disc and let $\mathcal{A}(p, k)$ $(p, k \in \mathbb{N} = \{1, 2, 3, \ldots\},\ p < k)$ denote the class of functions $f \in \mathcal{A}$ of the form

$$f(z) = a_p z^p + \sum_{n=k}^{\infty} a_n z^n \qquad (z \in \mathcal{U};\ a_p > 0). \tag{1}$$

For a multivalent function $f \in \mathcal{A}(p, k)$ the normalization

$$\left.\frac{f(z)}{z^{p-1}}\right|_{z=0} = 0 \qquad \text{and} \qquad \left.\frac{f'(z)}{z^{p-1}}\right|_{z=0} = p$$

is classical. One can obtain interesting results by applying Montel's normalization (*cf.* [11]) of the form

$$\left.\frac{f(z)}{z^{p-1}}\right|_{z=0} = 0 \qquad \text{and} \qquad \left.\frac{f'(z)}{z^{p-1}}\right|_{z=\rho} = p, \tag{2}$$

where $\rho = |\rho| e^{i\eta}$ is a fixed point of the unit disk $\mathcal{U}$.

We denote by $\mathcal{A}_\rho(p, k)$ the class of functions $f \in \mathcal{A}(p, k)$ with Montel's normalization (2) and call it the class of functions with two fixed points.

Also, by $\mathcal{T}^\eta(p, k)$ $(\eta \in \mathbb{R})$ we denote the class of functions $f \in \mathcal{A}(p, k)$ for

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which all of non-vanishing coefficients a_n satisfy the condition

$$\arg(a_n) = \pi + (p-n)\eta \qquad (n = k, k+1, \ldots). \tag{3}$$

For $\eta = 0$ we obtain the class $\mathcal{T}^0(p,k)$ of functions with negative coefficients. Moreover, we define

$$\mathcal{T}(p,k) := \bigcup_{\eta\in\mathbb{R}} \mathcal{T}^\eta(p,k).$$

The classes $\mathcal{T}(p,k)$ and $\mathcal{T}^\eta(p,k)$ are called the classes of functions with varying argument of coefficients. The class $\mathcal{T}(1,2)$ was introduced by Silverman [16] (see also [22]).

Let $\alpha \in \langle 0,p)$, $r \in (0,1\rangle$. A function $f \in \mathcal{A}(p,k)$ is said to be *convex of order* α in $\mathcal{U}(r)$ if

$$\operatorname{Re}\left(1 + \frac{zf''(z)}{f'(z)}\right) > \alpha \qquad (z \in \mathcal{U}(r)).$$

A function $f \in \mathcal{A}(p,k)$ is said to be *starlike of order* α in $\mathcal{U}(r)$ if

$$\operatorname{Re}\left(\frac{zf'(z)}{f(z)}\right) > \alpha \qquad (z \in \mathcal{U}(r)). \tag{4}$$

We denote by $\mathcal{S}_p^c(\alpha)$ the class of all functions $f \in \mathcal{A}(p,p+1)$, which are convex of order α in $\mathcal{U}$ and by $\mathcal{S}_p^*(\alpha)$ we denote the class of all functions $f \in \mathcal{A}(p,p+1)$ which are starlike of order α in $\mathcal{U}$.

It is easy to show that for a function f from the class $\mathcal{T}(p,k)$ the condition (4) is equivalent to the following condition

$$\left|\frac{zf'(z)}{f(z)} - p\right| < p - \alpha \qquad (z \in \mathcal{U}(r)). \tag{5}$$

Let $\mathcal{B}$ be a subclass of the class $\mathcal{A}(p,k)$. We define *the radius of starlikeness of order* α and *the radius of convexity of order* α for the class $\mathcal{B}$ by

$$R_\alpha^*(\mathcal{B}) = \inf_{f\in\mathcal{B}} \left\{\sup\{r \in (0,1] : \ f \text{ is starlike of order } \alpha \text{ in } \mathcal{U}(r)\}\right\},$$
$$R_\alpha^c(\mathcal{B}) = \inf_{f\in\mathcal{B}} \left\{\sup\{r \in (0,1] : \ f \text{ is convex of order } \alpha \text{ in } \mathcal{U}(r)\}\right\},$$

respectively.

We say that a function $f \in \mathcal{A}$ is a *subordinate* to a function $F \in \mathcal{A}$ and write $f(z) \prec F(z)$ (or simply $f \prec F$), if and only if there exists a function $\omega \in \mathcal{A}$ ($\omega(0) = 0$, $|\omega(z)| < 1$, $z \in \mathcal{U}$), such that

$$f(z) = F(\omega(z)) \qquad (z \in \mathcal{U}).$$

In particular, if F is univalent in $\mathcal{U}$, we have the following equivalence

$$f(z) \prec F(z) \iff [f(0) = F(0) \wedge f(\mathcal{U}) \subset F(\mathcal{U})].$$

For functions $f, g \in \mathcal{A}$ of the form

$$f(z) = \sum_{n=0}^{\infty} a_n z^n \qquad \text{and} \qquad g(z) = \sum_{n=0}^{\infty} b_n z^n,$$

by $f * g$ we denote *the Hadamard product* (or *convolution*) of f and g, defined by

$$(f * g)(z) = \sum_{n=0}^{\infty} a_n b_n z^n \qquad (z \in \mathcal{U}).$$

Let A, B, δ be real parameters, $\delta \geq 0$, $0 \leq B \leq 1$, $-1 \leq A < B$, and let $\varphi, \phi \in \mathcal{A}(p, k)$.

By $\mathcal{W}(p, k; \phi, \varphi; A, B; \delta)$ we denote the class of functions $f \in \mathcal{A}(p, k)$ such that

$$(\varphi * f)(z) \neq 0 \qquad (z \in \mathcal{U} \setminus \{0\}) \tag{6}$$

and

$$\frac{(\phi * f)(z)}{(\varphi * f)(z)} - \delta \left| \frac{(\phi * f)(z)}{(\varphi * f)(z)} - 1 \right| \prec \frac{1 + Az}{1 + Bz}. \tag{7}$$

If $0 < B < 1$, then the condition (7) is equivalent to the following inequality

$$\left| \frac{(\phi * f)(z)}{(\varphi * f)(z)} - \delta \left| \frac{(\phi * f)(z)}{(\varphi * f)(z)} - 1 \right| - \frac{1 - AB}{1 - B^2} \right| < \frac{B - A}{1 - B^2} \qquad (z \in \mathcal{U}), \tag{8}$$

and if $B = 1$, then it is equivalent to the following

$$\delta \left| \frac{(\phi * f)(z)}{(\varphi * f)(z)} - 1 \right| - \operatorname{Re} \left\{ \frac{(\phi * f)(z)}{(\varphi * f)(z)} - 1 \right\} < \frac{1 - A}{2} \qquad (z \in \mathcal{U}). \tag{9}$$

Now, we define the classes of functions with varying argument of coefficients related to the class $\mathcal{W}(p, k; \phi, \varphi; A, B; \delta)$. Let us denote

$$\begin{aligned}
\mathcal{W}_\rho(p, k; \phi, \varphi; A, B; \delta) &:= \mathcal{A}_\rho(p, k) \cap \mathcal{W}(p, k; \phi, \varphi; A, B; \delta), \\
\mathcal{TW}^\eta(p, k; \phi, \varphi; A, B; \delta) &:= \mathcal{T}^\eta(p, k) \cap \mathcal{W}(p, k; \phi, \varphi; A, B; \delta), \\
\mathcal{TW}^\eta_\rho(p, k; \phi, \varphi; A, B; \delta) &:= \mathcal{A}_\rho(p, k) \cap \mathcal{TW}^\eta(p, k; \phi, \varphi; A, B; \delta), \\
\mathcal{TW}_\rho(p, k; \phi, \varphi; A, B; \delta) &:= \mathcal{T}(p, k) \cap \mathcal{W}_\rho(p, k; \phi, \varphi; A, B; \delta).
\end{aligned}$$

In this article we assume that φ, ϕ are functions of the form

$$\varphi(z) = z^p + \sum_{n=k}^{\infty} \alpha_n z^n, \quad \phi(z) = z^p + \sum_{n=k}^{\infty} \beta_n z^n \qquad (z \in \mathcal{U}),$$

where the sequences $\{\alpha_n\}$, $\{\beta_n\}$ are real, and

$$0 \leq \alpha_n < \beta_n \qquad (n = k, k+1, \ldots).$$

Moreover, we define

$$d_n := (\delta + 1)(1 + B)\beta_n - (\delta B + A + \delta + 1)\alpha_n \qquad (n = k, k+1, \ldots). \tag{10}$$

The family $\mathcal{W}(p,k;\phi,\varphi;A,B;\delta)$ unifies various new and also well-known classes of analytic functions. We list a few of them in the last section.

The objective of the present paper is to study the coefficients estimates, distortion properties, the radii of starlikeness and convexity, subordination theorems, partial sums and integral means inequalities for the classes of functions with varying argument of coefficients. Some remarks depicting consequences of the main results are also mentioned.

2. Coefficients estimates

First we mention a sufficient condition for the function to belong to the class $\mathcal{W}(p,k;\phi,\varphi;A,B;\delta)$.

Theorem 2.1
Let $0 \leq B \leq 1$ *and* $-1 \leq A < B$*. If* $f \in \mathcal{A}_\rho(p,k)$ *and*

$$\sum_{n=k}^{\infty} d_n |a_n| \leq (B-A)a_p, \tag{11}$$

then $f \in \mathcal{W}(p,k;\phi,\varphi;A,B;\delta)$.

Proof. If $0 \leq B < 1$, then we have

$$\begin{aligned} &\left| \frac{(\phi * f)(z)}{(\varphi * f)(z)} - \delta\left|\frac{(\phi * f)(z)}{(\varphi * f)(z)} - 1\right| - \frac{1-AB}{1-B^2} \right| \\ &\qquad \leq (\delta+1)\left|\frac{(\phi * f)(z)}{(\varphi * f)(z)} - 1\right| + \frac{B(B-A)}{1-B^2} \\ &\qquad \leq (\delta+1)\frac{\sum_{n=k}^{\infty}(\beta_n - \alpha_n)|a_n||z|^{n-p}}{a_p - \sum_{n=k}^{\infty}\alpha_n|a_n||z|^{n-p}} + \frac{B(B-A)}{1-B^2}. \end{aligned}$$

Thus, by (11) we obtain (8) and consequently $f \in \mathcal{W}(p,k;\phi,\varphi;A,B;\delta)$. Let now $B=1$. Then simply calculations give

$$\begin{aligned} \delta\left|\frac{(\phi * f)(z)}{(\varphi * f)(z)} - 1\right| - \mathrm{Re}\left\{\frac{(\phi * f)(z)}{(\varphi * f)(z)} - 1\right\} &\leq (\delta+1)\left|\frac{(\phi * f)(z)}{(\varphi * f)(z)} - 1\right| \\ &\leq (\delta+1)\frac{\sum_{n=k}^{\infty}(\beta_n - \alpha_n)|a_n||z|^{n-p}}{a_p - \sum_{n=k}^{\infty}\alpha_n|a_n||z|^{n-p}}, \end{aligned}$$

and, by (11) we obtain (9). Hence $f \in \mathcal{W}(p,k;\phi,\varphi;A,B;\delta)$ and the proof is complete.

Our next theorem shows that the condition (11) is necessary for functions of the form (1), with property (3) to belong to the class $\mathcal{TW}^\eta(p,k;\phi,\varphi;A,B;\delta)$.

Theorem 2.2
Let $f \in \mathcal{T}^\eta(p,k)$*. Then* $f \in \mathcal{TW}^\eta(p,k;\phi,\varphi;A,B;\delta)$ *if and only if the condition* (11) *holds true.*

Proof. In the view of Theorem 2.1 we need only to show that each function $f \in \mathcal{TW}^{\eta}(p,k;\phi,\varphi;A,B;\delta)$ satisfies the coefficient inequality (11). Let $f \in \mathcal{TW}^{\eta}(p,k;\phi,\varphi;A,B;\delta)$. Putting $z = re^{i\eta}$ in the conditions (8) and (9) we obtain

$$(\delta+1)\frac{\sum_{n=2}^{\infty}(\beta_n-\alpha_n)|a_n|r^{n-p}}{a_p-\sum_{n=2}^{\infty}\alpha_n|a_n|r^{n-p}} < \frac{B-A}{1+B}.$$

Thus, by (6) we have

$$\sum_{n=2}^{\infty} d_n|a_n|r^{n-p} < (B-A)a_p,$$

which, upon letting $r \to 1^-$, readily yields the assertion (11).

From Theorem 2.2 we can deduce the following result.

Theorem 2.3
Let $f \in \mathcal{T}^{\eta}(p,k)$. Then $f \in \mathcal{TW}^{\eta}_{\rho}(p,k;\phi,\varphi;A,B;\delta)$ if and only if it satisfies (2) and

$$\sum_{n=k}^{\infty}\left(pd_n-(B-A)\frac{n}{p}|\rho|^{n-p}\right)|a_n| \le B-A. \tag{12}$$

Proof. For a function $f \in \mathcal{T}^{\eta}(p,k)$ with the normalization (2) we have

$$a_p = 1+\sum_{n=k}^{\infty}\frac{n}{p}|a_n||\rho|^{n-p}. \tag{13}$$

Then the conditions (11) and (12) are equivalent.

From Theorem 2.3 we obtain the following lemma.

Lemma 2.4
If there exists an integer $n_0 \in \mathbb{N}_k = \{k, k+1, \dots\}$ such that

$$pd_{n_0}-(B-A)n_0|\rho|^{n_0-p} \le 0,$$

then the function

$$f_{n_0}(z) = \left(1+a\frac{n_0}{p}\rho^{n_0-p}\right)z^p - ae^{i(p-n_0)\eta}z^{n_0} \qquad (z \in \mathcal{U})$$

belongs to the class $\mathcal{TW}^{\eta}_{\rho}(p,k;\phi,\varphi;A,B;\delta)$ for all positive real numbers a. Moreover, for all $n \in \mathbb{N}_k$ such that

$$pd_n-(B-A)n|\rho|^{n-p} > 0,$$

the functions

$$f_n(z) = \left(1+a\frac{n_0}{p}\rho^{n_0-p}+b\frac{n}{p}z^{n-p}\right)z^p - ae^{i(p-n_0)\eta}z^{n_0} - be^{i(p-n)\eta}z^n \qquad (z \in \mathcal{U}),$$

where

$$b=\frac{p(B-A)+((B-A)n_0|\rho|^{n_0-p}-pd_{n_0})a}{pd_n-(B-A)n|\rho|^{n-p}},$$

belong to the class $\mathcal{TW}^{\eta}_{\rho}(p,k;\phi,\varphi;A,B;\delta)$.

By Lemma 2.4 and Theorem 2.3, we have the following two corollaries.

Corollary 2.5
Let

$$pd_n-(B-A)n|\rho|^{n-p}\geq 0 \qquad (n=k,k+1,\ldots).$$

If $pd_n-(B-A)n|\rho|^{n-p}>0$, *then the* n*-th coefficient of the class* $\mathcal{TW}^{\eta}_{\rho}(p,k;\phi,\varphi;A,B;\delta)$ *satisfies the following inequality*

$$|a_n|\leq\frac{B-A}{pd_n-(B-A)n|\rho|^{n-p}}.$$

The result is sharp, the function $f_{n,\eta}$ *of the form*

$$f_{n,\eta}(z)=\frac{pd_nz^p-p(B-A)e^{i(p-n)\eta}z^n}{pd_n-(B-A)n|\rho|^{n-p}} \qquad (z\in\mathcal{U}) \tag{14}$$

is an extremal function.

Corollary 2.6
Let $\{d_n\}$ *be defined as in* (10). *If*

$$pd_n-(B-A)n|\rho|^{n-p}=0,$$

then the n*-th coefficient of the class* $\mathcal{TW}^{\eta}_{\rho}(p,k;\phi,\varphi;A,B;\delta)$ *is unbounded. Moreover, if there exists* $n_0\in\mathbb{N}_k=\{k,k+1,\ldots\}$ *such that*

$$pd_{n_0}-(B-A)n_0|\rho|^{n_0-p}<0,$$

then all of the coefficients of the class $\mathcal{TW}^{\eta}_{\rho}(p,k;\phi,\varphi;A,B;\delta)$ *are unbounded.*

Putting $\rho=0$ in Theorem 2.3 and Corollary 2.5 we have the corollaries listed below.

Corollary 2.7
Let $f\in\mathcal{T}^{\eta}(p,k)$. *Then* $f\in\mathcal{TW}^{\eta}_{0}(p,k;\phi,\varphi;A,B;\delta)$ *if and only if*

$$\sum_{n=k}^{\infty}d_n|a_n|\leq B-A.$$

Corollary 2.8
If $f\in\mathcal{TW}^{\eta}_{0}(p,k;\phi,\varphi;A,B;\delta)$, *then*

$$|a_n|\leq\frac{B-A}{d_n}.$$

The result is sharp. The functions $f_{n,\eta}$ of the form

$$f_{n,\eta}(z) = z^p - \frac{B-A}{d_n} e^{i(p-n)\eta} z^n \qquad (z \in \mathcal{U}) \tag{15}$$

are extremal.

3. Distortion theorems

From Theorem 2.2 we have the following lemma.

Lemma 3.1
Let $f \in \mathcal{TW}_\rho^\eta(p,k;\phi,\varphi;A,B;\delta)$. If

$$0 < d_k - (B-A)\frac{k}{p}|\rho|^{k-p} \leq d_n - (B-A)\frac{n}{p}|\rho|^{n-p}, \tag{16}$$

then

$$\sum_{n=k}^{\infty} |a_n| \leq \frac{B-A}{d_k - (B-A)\frac{k}{p}|\rho|^{k-p}}.$$

Moreover, if

$$0 < \frac{d_k - (B-A)\frac{k}{p}|\rho|^{k-p}}{k} \leq \frac{d_n - (B-A)\frac{n}{p}|\rho|^{n-p}}{n}, \tag{17}$$

then

$$\sum_{n=k}^{\infty} n|a_n| \leq \frac{k(B-A)}{d_k - (B-A)\frac{k}{p}|\rho|^{k-p}}.$$

Remark 3.2
The second part of Lemma 3.1 may be formulated in terms of σ-neighborhood N_σ defined by

$$N_\sigma = \left\{ f(z) = a_p z^p + \sum_{n=k}^{\infty} a_n z^n \in \mathcal{T}^\eta(p,k) : \sum_{n=k}^{\infty} n|a_n| \leq \sigma \right\}$$

as the following corollary.

Corollary 3.3
If the sequence $\{d_n\}$ defined by (10) satisfies (17), then

$$\mathcal{TW}_\rho^\eta(p,k;\phi,\varphi;A,B;\delta) \subset N_\sigma,$$

where

$$\delta = \frac{k(B-A)}{d_k - (B-A)\frac{k}{p}|\rho|^{k-p}}.$$

Theorem 3.4
Let $f \in \mathcal{TW}_\rho^\eta(p,k;\phi,\varphi;A,B;\delta)$ *and let* $|z| = r < 1$*. If the sequence* $\{d_n\}$ *satisfies* (16)*, then*

$$pa_p r^p - \frac{B-A}{d_k - (B-A)\frac{k}{p}|\rho|^{k-p}} r^k \le |f(z)| \le \frac{d_k r^p + (B-A)r^k}{d_k - (B-A)\frac{k}{p}|\rho|^{k-p}}. \tag{18}$$

Moreover, if (17) *holds, then*

$$\phi'(r) \le |f'(z)| \le \frac{d_k r^p + k(B-A)r^{k-1}}{d_k - (B-A)\frac{k}{p}|\rho|^{k-p}}, \tag{19}$$

where

$$\phi(r) := \begin{cases} r^p & (r \le \rho), \\ \dfrac{d_k r^p - (B-A)r^k}{d_k - (B-A)\frac{k}{p}|\rho|^{k-p}} & (r > \rho). \end{cases}$$

The result is sharp, with the extremal functions $f_{k,\eta}$ *of the form* (14) *and* $f(z) = z$ $(z \in \mathcal{U})$.

Proof. Suppose that $f \in \mathcal{TW}_\rho^\eta(p,k;\phi,\varphi;A,B;\delta)$. By Lemma 3.1 we have

$$\begin{aligned} |f'(z)| &= \left| pa_p z^{p-1} + \sum_{n=k}^{\infty} na_n z^{n-1} \right| \le r^{p-1}\left(pa_p + \sum_{n=k}^{\infty} n|a_n| r^{n-p} \right) \\ &\le r^{p-1}\left(p + \sum_{n=k}^{\infty} n|a_n||\rho|^{n-p} + \sum_{n=k}^{\infty} n|a_n| r^{n-p} \right) \\ &\le r^{p-1}\left(p + (|\rho|^{k-p} + r^{k-p}) \sum_{n=k}^{\infty} n|a_n| \right) \\ &\le \frac{d_k r^{p-1} + k(B-A)r^k}{d_k - (B-A)\frac{k}{p}|\rho|^{k-p}}, \end{aligned}$$

and

$$|f'(z)| \ge r^{p-1}\left(pa_p - \sum_{n=k}^{\infty} n|a_n| r^{n-p} \right) = r^{p-1}\left(p + \sum_{n=k}^{\infty} (|\rho|^{n-p} - r^{n-p}) n|a_n| \right). \tag{20}$$

If $r \le \rho$, then we obtain $|f'(z)| \ge r^{p-1}$. If $r > \rho$, then the sequence $\{(\rho^{n-p} - r^{n-p})\}$ is decreasing and negative. Thus, by (20), we obtain

$$|f'(z)| \ge r^{p-1}\left(p - (r^{k-p} - |\rho|^{k-p}) \sum_{n=2}^{\infty} a_n \right) \ge \frac{pd_k r^p - k(B-A)r^k}{d_k - (B-A)\frac{k}{p}|\rho|^{k-p}},$$

and we have the assertion (19). Making use of Lemma 3.1, in conjunction with (13), we readily obtain the assertion (18) of Theorem 3.4.

Corollary 3.5

Let $f \in \mathcal{TW}_0^\eta(p,k;\phi,\varphi;A,B;\delta)$. *If* $d_k \le d_n$ $(n = k, k+1, \dots)$, *then*

$$r^p - \frac{B-A}{d_k} r^k \le |f(z)| \le r^p + \frac{B-A}{d_k} r^k \qquad (|z| = r < 1).$$

Moreover, if $nd_k \le kd_n$, *then*

$$pr^{p-1} - \frac{k(B-A)}{d_k} r^{k-1} \le |f'(z)| \le pr^{p-1} + \frac{k(B-A)}{d_k} r^{k-1} \qquad (|z| = r < 1).$$

The result is sharp, with the extremal function $f_{k,\eta}$ *of the form* (15).

4. The radii of convexity and starlikeness

Theorem 4.1

The radius of starlikeness of order α *for the class* $\mathcal{TW}^\eta(p,k;\phi,\varphi;A,B;\delta)$ *is given by*

$$R_\alpha^*(\mathcal{TW}^\eta(p,k;\phi,\varphi;A,B;\delta)) = \inf_{n\ge k} \Big(\frac{(p-\alpha)d_n}{(n-\alpha)(B-A)}\Big)^{\frac{1}{n-p}}. \tag{21}$$

The functions

$$f_{n,\eta}(z) = a_p\Big(z^p - \frac{B-A}{d_n} e^{i(p-n)\eta} z^n\Big) \qquad (z \in \mathcal{U};\ n = k, k+1, \dots;\ a_p > 0) \tag{22}$$

are extremal.

Proof. A function $f \in \mathcal{T}^\eta(p,k)$ of the form (1) is starlike of order α in the disk $\mathcal{U}(r)$ if and only if it satisfies the condition (5). Let $|z| = r < 1$. Since

$$\left|\frac{zf'(z)}{f(z)} - p\right| = \left|\frac{\sum_{n=k}^\infty (n-p)a_n z^n}{a_p z^p + \sum_{n=k}^\infty a_n z^n}\right| \le \frac{\sum_{n=k}^\infty (n-p)|a_n||z|^{n-p}}{a_p - \sum_{n=k}^\infty |a_n||z|^{n-p}},$$

the condition (5) is true if

$$\sum_{n=k}^\infty \frac{n-\alpha}{p-\alpha}|a_n| r^{n-p} \le a_p. \tag{23}$$

By Theorem 2.2, we have

$$\sum_{n=k}^\infty \frac{d_n}{B-A}|a_n| \le a_p. \tag{24}$$

Thus, the condition (23) is true if

$$\frac{n-\alpha}{p-\alpha} r^{n-p} \le \frac{d_n}{B-A} \qquad (n = k, k+1, \dots),$$

that is, if

$$r \le \Big(\frac{(p-\alpha)d_n}{(n-\alpha)(B-A)}\Big)^{\frac{1}{n-p}}.$$

It follows that each function $f \in \mathcal{TW}^{\eta}(p,k;\phi,\varphi;A,B;\delta)$ is starlike of order α in the disk $\mathcal{U}(r)$, where

$$r = \inf_{n\geq k}\Big(\frac{(p-\alpha)d_n}{(n-\alpha)(B-A)}\Big)^{\frac{1}{n-p}}.$$

The functions $f_{n,\eta}$ of the form (22) realize equality in (24), and the radius r cannot be larger. Thus we have (21).

The following result may be proved in much the same way as Theorem 4.1.

THEOREM 4.2
The radius of convexity of order α for the class $\mathcal{TW}^{\eta}(p,k;\phi,\varphi;A,B;\delta)$ is given by

$$R^c_{\alpha}(\mathcal{TW}^{\eta}(p,k;\phi,\varphi;A,B;\delta)) = \inf_{n\geq k}\Big(\frac{(p-\alpha)d_n}{n(n-\alpha)(B-A)}\Big)^{\frac{1}{n-p}}.$$

The functions $f_{n,\eta}$ of the form (22) are the extremal functions.

It is clear that for

$$a_p = \frac{d_n}{d_n-(B-A)\frac{k}{p}|\rho|^{n-p}} > 0$$

the extremal functions $f_{n,\eta}$ of the form (22) belong to the class $\mathcal{TW}^{\eta}_{\rho}(p,k;\phi,\varphi;A,B;\delta)$. Moreover, we have

$$\mathcal{TW}^{\eta}_{\rho}(p,k;\phi,\varphi;A,B;\delta) \subset \mathcal{TW}^{\eta}(p,k;\phi,\varphi;A,B;\delta).$$

Thus, by Theorems 4.1 and 4.2 we have the following results.

COROLLARY 4.3
Let the sequence $\{d_n-(B-A)\frac{n}{p}|\rho|^{n-p}\}$ be positive. The radius of starlikeness of order α for the class $\mathcal{TW}^{\eta}_{\rho}(p,k;\phi,\varphi;A,B;\delta)$ is given by

$$R^*_{\alpha}(\mathcal{TW}^{\eta}_{\rho}(p,k;\phi,\varphi;A,B;\delta)) = \inf_{n\geq k}\Big(\frac{(p-\alpha)d_n}{(n-\alpha)(B-A)}\Big)^{\frac{1}{n-p}}.$$

The functions $f_{n,\eta}$ of the form (22) are the extremal functions.

COROLLARY 4.4
Let the sequence $\{d_n-(B-A)\frac{n}{p}|\rho|^{n-p}\}$ be positive. The radius of convexity of order α for the class $\mathcal{TW}^{\eta}_{\rho}(p,k;\phi,\varphi;A,B;\delta)$ is given by

$$R^c_{\alpha}(\mathcal{TW}^{\eta}_{\rho}(p,k;\phi,\varphi;A,B;\delta)) = \inf_{n\geq k}\Big(\frac{(p-\alpha)d_n}{n(n-\alpha)(B-A)}\Big)^{\frac{1}{n-p}}.$$

5. Subordination results

Before stating and proving our subordination theorems for the classes $\mathcal{TW}^{\eta}(p, k; \phi, \varphi; A, B; \delta)$ and $\mathcal{TW}(p, k; \phi, \varphi; A, B; \delta)$ we need the following definition and lemma.

Definition 5.1
A sequence $\{b_n\}$ of complex numbers is said to be a subordinating factor sequence if for each function $f \in \mathcal{S}^c$ we have

$$\sum_{n=1}^{\infty} b_n a_n z^n \prec f(z) \qquad (a_1 = 1).$$

Lemma 5.2 ([23])
The sequence $\{b_n\}$ is a subordinating factor sequence if and only if

$$\mathrm{Re}\left\{1 + 2\sum_{n=1}^{\infty} b_n z^n\right\} > 0 \qquad (z \in \mathcal{U}). \tag{25}$$

Theorem 5.3
Suppose that the sequence $\{d_n\}$ satisfies the inequality (16). If $g \in \mathcal{S}^c$ and $f \in \mathcal{TW}^{\eta}(p, k; \phi, \varphi; A, B; \delta)$, then

$$\varepsilon \frac{f(z)}{z^{p-1}} * g(z) \prec g(z) \tag{26}$$

and

$$\mathrm{Re}\, \frac{f(z)}{z^{p-1}} > -\frac{1}{2\varepsilon} \qquad (z \in \mathcal{U}), \tag{27}$$

where

$$\varepsilon = \frac{d_k}{2a_p(B - A + d_k)}. \tag{28}$$

If p and $(k-p)$ are odd, and $\eta = 0$, then the constant factor ε cannot be replaced by a larger number.

Proof. Let $f \in \mathcal{TW}^{\eta}(p, k; \phi, \varphi; A, B; \delta)$ and suppose that a function g of the form

$$g(z) = \sum_{n=1}^{\infty} c_n z^n \qquad (c_1 = 1;\ z \in \mathcal{U})$$

belongs to the class $\mathcal{S}^c$. Then

$$\varepsilon \frac{f(z)}{z^{p-1}} * g(z) = \sum_{n=1}^{\infty} b_n c_n z^n \qquad (z \in \mathcal{U}),$$

where

$$b_n = \begin{cases} \varepsilon a_p & \text{if } n = 1, \\ 0 & \text{if } 2 \le n \le k-p, \\ \varepsilon a_{n+p-1} & \text{if } n > k-p. \end{cases}$$

Thus, by Definition 5.1 the subordination result (26) holds true if $\{b_n\}$ is the subordinating factor sequence. By (16) we have

$$\begin{aligned}\operatorname{Re}\left\{1+2\sum_{n=1}^{\infty}b_nz^n\right\} &= \operatorname{Re}\left\{1+2\varepsilon a_pz+\sum_{n=k}^{\infty}\frac{d_k}{B-A+d_k}a_nz^{n-p}\right\}\\ &\geq 1-2\varepsilon r-\frac{r}{(B-A+d_k)a_p}\sum_{n=k}^{\infty}d_n|a_n| \qquad (|z|=r<1).\end{aligned}$$

Thus, by Theorem 2.2 we obtain

$$\operatorname{Re}\left\{1+2\sum_{n=1}^{\infty}b_nz^n\right\}\geq 1-\frac{d_k}{B-A+d_k}r-\frac{B-A}{B-A+d_k}r>0.$$

This evidently proves the inequality (25) and hence (26). The inequality (27) follows from (26) by taking

$$g(z)=\frac{z}{1-z}=\sum_{n=1}^{\infty}z^n \qquad (z\in\mathcal{U}).$$

Next, we observe that the function $f_{k,\eta}$ of the form (22) belongs to the class $\mathcal{TW}^{\eta}(p,k;\phi,\varphi;A,B;\delta)$. If p and $(k-p)$ are odd, and $\eta=0$, then

$$\left.\frac{f_{k,\eta}(z)}{z^{p-1}}\right|_{z=-1}=-\frac{1}{2\varepsilon},$$

and the constant (28) cannot be replaced by any larger one.

Remark 5.4
If we use (13) *in Theorem 5.3, then we obtain the result related to the class* $\mathcal{W}^{\eta}_{\rho}(p,k;\phi,\varphi;A,B;\delta)$. *Moreover, putting* $\rho=0$ *we have the following corollary.*

Corollary 5.5
Let the sequence $\{d_n\}$ *satisfy the inequality* (16). *If* $g\in\mathcal{S}^c$ *and* $f\in\mathcal{TW}^{\eta}_0(p,k;\phi,\varphi;A,B;\delta)$*, then conditions* (26) *and* (27)*, with*

$$\varepsilon=\frac{d_k}{2(B-A+d_k)} \tag{29}$$

hold true. If p *and* $(k-p)$ *are odd, and* $\eta=0$*, then the constant factor* ε *in* (29) *cannot be replaced by a larger number.*

6. Integral means inequalities

Following Littlewood [9] we obtain integral means inequalities for the functions from the class $\mathcal{TW}^{\eta}(p,k;\phi,\varphi;A,B;\delta)$.

Lemma 6.1 ([9])
Let $f, g \in \mathcal{A}$. If $f \prec g$, then

$$\int_0^{2\pi} |f(re^{i\theta})|^\eta \, d\theta \le \int_0^{2\pi} |g(re^{i\theta})|^\eta \, d\theta \qquad (0 < r < 1,\ \eta > 0).$$

Theorem 6.2
Let the sequence $\{d_n\}$ satisfy the inequality (16). *If $f \in \mathcal{TW}^\eta(p, p+1; \phi, \varphi; A, B; \delta)$, then*

$$\int_0^{2\pi} |f(re^{i\theta})|^\lambda \, d\theta \le \int_0^{2\pi} |f_{p+1,\eta}(re^{i\theta})|^\lambda \, d\theta \qquad (0 < r < 1,\ \lambda > 0;\ z = re^{i\theta}), \tag{30}$$

where $f_{p+1,\eta}$ is defined by (22).

Proof. For a function $f \in \mathcal{TW}^\eta(p, p+1; \phi, \varphi; A, B; \delta)$ the inequality (30) is equivalent to the following inequality

$$\int_0^{2\pi} \left| a_p + \sum_{n=p+1}^{\infty} a_n z^{n-p} \right|^\lambda d\theta \le \int_0^{2\pi} \left| a_p - \frac{B-A}{d_{p+1}} e^{-i\eta} z \right|^\lambda d\theta \qquad (z = re^{i\theta}).$$

By Lemma 6.1, it suffices to show that

$$\sum_{n=p+1}^{\infty} a_n z^{n-p} \prec -\frac{B-A}{d_{p+1}} e^{-i\eta} z. \tag{31}$$

If we put

$$w(z) = -\sum_{n=p+1}^{\infty} \frac{d_{p+1} e^{i\eta}}{B-A} a_n \ z^{n-p} \qquad (z \in \mathcal{U}),$$

then by (16) and Theorem 2.2 we obtain

$$|w(z)| = \left| \sum_{n=p+1}^{\infty} \frac{d_{p+1}}{B-A} a_n z^{n-p} \right| \le |z| \sum_{n=p+1}^{\infty} \frac{d_n}{B-A} |a_n| \le |z| \qquad (z \in \mathcal{U}).$$

Since

$$\sum_{n=p+1}^{\infty} a_n z^{n-p} = -\frac{B-A}{d_{p+1}} e^{-i\eta} w(z) \qquad (z \in \mathcal{U}),$$

by definition of subordination we have (31), and this completes the proof.

Using (13) in Theorem 6.2 we have the following corollary.

Corollary 6.3
Assume that the sequence $\{d_n\}$ *satisfies the inequality* (16). *If* $f \in \mathcal{TW}_\rho^\eta(p, p+1; \phi, \varphi; A, B; \delta)$, *then*

$$\int_0^{2\pi} |f(re^{i\theta})|^\lambda \, d\theta \le \int_0^{2\pi} |f_{p+1,\eta}(re^{i\theta})|^\lambda \, d\theta \qquad (0 < r < 1,\ \lambda > 0;\ z = re^{i\theta}),$$

where $f_{p+1,\eta}(z)$ *is defined by* (5).

7. Partial sums

Following Silverman [15] and Silvia [17] in turn, we investigate partial sums f_m of the function f defined by

$$f_{k-1}(z) = a_p z^p \quad \text{and} \quad f_m(z) = a_p z^p + \sum_{n=k}^{m} a_n z^n \qquad (m = k, k+1, \ldots).$$

In this section we consider partial sums of functions in the class $\mathcal{TW}^\eta(p, k; \phi, \varphi; A, B; \delta)$ and obtain sharp lower bounds for the ratios of the real part of f to f_m and of f' to f'_m.

Theorem 7.1
Assume that the sequence $\{d_n\}$ *is increasing and satisfies*

$$d_k \ge B - A.$$

If $f \in \mathcal{TW}^\eta(p, k; \phi, \varphi; A, B; \delta)$, *then*

$$\operatorname{Re}\left\{\frac{f(z)}{f_m(z)}\right\} \ge 1 - \frac{B-A}{d_{m+1}} \qquad (z \in \mathcal{U},\ m = k-1, k, \ldots) \tag{32}$$

and

$$\operatorname{Re}\left\{\frac{f_m(z)}{f(z)}\right\} \ge \frac{d_{m+1}}{B - A + d_{m+1}} \qquad (z \in \mathcal{U},\ m = k-1, k, \ldots). \tag{33}$$

The bounds are sharp, with extremal functions $f_{m+1,\eta}$ *defined by* (22).

Proof. Since

$$\frac{d_{n+1}}{B-A} > \frac{d_n}{B-A} > 1 \qquad (n = k, k+1, \ldots),$$

by Theorem 2.1 we have that

$$\sum_{n=k}^{m} |a_n| + \frac{d_{m+1}}{B-A} \sum_{n=m+1}^{\infty} |a_n| \le \sum_{n=k}^{\infty} \frac{d_n}{B-A} |a_n| \le a_p. \tag{34}$$

Let

$$\begin{aligned} g(z) &= \frac{d_{m+1}}{B-A}\Big\{\frac{f(z)}{f_m(z)} - \Big(1 - \frac{B-A}{d_{m+1}}\Big)\Big\} \\ &= 1 + \frac{\frac{d_{m+1}}{B-A}\sum_{n=m+1}^{\infty} a_n z^{n-p}}{a_p + \sum_{n=k}^{m} a_n z^{n-p}} \qquad (z \in \mathcal{U}). \end{aligned} \tag{35}$$

Then by (34), we find that

$$\left|\frac{g(z)-1}{g(z)+1}\right| \leq \frac{\frac{d_{m+1}}{B-A}\sum_{n=m+1}^{\infty}|a_n|}{2a_p - 2\sum_{n=2}^{n}|a_n| - \frac{d_{m+1}}{B-A}\sum_{n=m+1}^{\infty}|a_n|} \leq 1 \qquad (z \in \mathcal{U}).$$

Thus, we have

$$\operatorname{Re} g(z) \geq 0 \qquad (z \in U),$$

which by (35) readily yields the assertion (32) of Theorem 7.1. Similarly, if we take

$$h(z) = \Big(1 + \frac{d_{m+1}}{B-A}\Big)\Big\{\frac{f_m(z)}{f(z)} - \frac{d_{m+1}}{B-A+d_{m+1}}\Big\} \qquad (z \in \mathcal{U}),$$

then by (34), we can deduce that

$$\left|\frac{h(z)-1}{h(z)+1}\right| \leq \frac{(1+\frac{d_{m+1}}{B-A})\sum_{n=m+1}^{\infty}|a_n|}{2a_p - 2\sum_{n=k}^{m}|a_n| - (\frac{d_{m+1}}{B-A} - 1)\sum_{n=m+1}^{\infty}|a_n|} \leq 1 \qquad (z \in \mathcal{U}),$$

which leads us immediately to the assertion (33) of Theorem 7.1. In order to see that the function $f_{m+1,\eta}$ of the form (15) is extremal, we observe that

$$\begin{aligned} \frac{f_{m+1,\eta}(z)}{(f_{m+1,\eta})_m(z)} &= 1 - \frac{B-A}{d_{m+1}} \qquad (z = e^{i\eta}), \\ \frac{(f_{m+1,\eta})_m(z)}{f_{m+1,\eta}(z)} &= \frac{d_{m+1}}{B-A+d_{m+1}} \qquad (z = e^{i(\eta + \frac{\pi}{m-p+1})}). \end{aligned}$$

This completes the proof.

Theorem 7.2

Assume that the sequence $\{d_n\}$ is increasing and

$$d_k > (m+1)(B-A).$$

If $f \in \mathcal{TW}^{\eta}(p,k;\phi,\varphi;A,B;\delta)$, then

$$\begin{aligned} \operatorname{Re}\Big\{\frac{f(z)}{f_m(z)}\Big\} &\geq 1 - \frac{(m+1)(B-A)}{d_{m+1}} \qquad (z \in \mathcal{U},\ m = k-1, k, \ldots), \\ \operatorname{Re}\Big\{\frac{f_m(z)}{f(z)}\Big\} &\geq \frac{d_{m+1}}{(m+1)(B-A)+d_{m+1}} \qquad (z \in \mathcal{U},\ m = k-1, k, \ldots). \end{aligned}$$

The bounds are sharp, with extremal functions $f_{m+1,\eta}$ defined by (22).

Proof. If we define

$$g(z) = \frac{d_{m+1}}{B-A}\left\{\frac{f'(z)}{f'_m(z)} - \left(1 - \frac{(m+1)(B-A)}{d_{m+1}}\right)\right\} \qquad (z \in \mathcal{U}),$$

and

$$h(z) = \left(m+1+\frac{d_{m+1}}{B-A}\right)\left\{\frac{f'_m(z)}{f'(z)} - \frac{d_{m+1}}{(m+1)(B-A)+d_{m+1}}\right\} \qquad (z \in \mathcal{U}),$$

then the proof is analogous to that of Theorem 7.1, and we omit the details.

Remark 7.3
By using (13) *in Theorems 7.1 and 7.2 we obtain the results related to the class* $\mathcal{TW}^{\eta}_{\rho}(p,k;\phi,\varphi;A,B;\delta)$.

Remark 7.4
We observe that the results obtained for the class $\mathcal{TW}^{\eta}_{\rho}(p,k;\phi,\varphi;A,B;\delta)$ *are true for the class* $\mathcal{TW}_{\rho}(p,k;\phi,\varphi;A,B;\delta)$.

8. Concluding remarks

We conclude this paper by observing that the family $\mathcal{W}(p,k;\phi,\varphi;A,B;\delta)$ unifies several new and also well-known classes of analytic functions. Let

$$\mathcal{W}^n_{\rho}(p,k;\varphi;A,B;\delta) := \mathcal{W}_{\rho}\left(p,k;\frac{z\varphi'(z)}{p},\sum_{l=0}^{n-1}\varphi(x^l z);A,B;\delta\right),$$

where $n \in \mathbb{N}$, $x^n = 1$, $x \neq 1$. The class $\mathcal{W}^n_{\rho}(p,k;\varphi;A,B;\delta)$ generalizes well-known classes, which were investigated in earlier works, see for example [1, 3, 7, 10, 12, 18, 20, 21]. In particular, the class $\mathcal{W}^n_{\rho}(p,k;\varphi;A,B;0)$ contains functions $f \in \mathcal{A}(p,k)$, which satisfy the condition

$$\frac{z(\varphi * f)'(z)}{\sum_{l=0}^{n-1}(\varphi * f)(x^l z)} \prec p\frac{1+Az}{1+Bz}.$$

It is related to the class of starlike functions with respect to n symmetric points. Moreover, putting $n = 1$, we obtain the class $\mathcal{W}^1_{\rho}(p,k;\varphi;A,B;0)$ defined by the following condition

$$\frac{z(\varphi * f)'(z)}{(\varphi * f)(z)} \prec p\frac{1+Az}{1+Bz}.$$

The class is related to the class of starlike functions. In particular, we have

$$\mathcal{S}^*_p(\alpha) := \mathcal{W}^1_{\rho}\left(p,p+1;\frac{z^p}{1-z};2\alpha-p,1;0\right).$$

Analogously, the class

$$\mathcal{W}^n_{\rho}(p,k;\varphi;2\gamma-p,1;\delta) \qquad (0 \leq \gamma < 1)$$

contains functions $f \in \mathcal{A}(p,k)$, which satisfy the condition

$$\mathrm{Re}\left\{\frac{z(\varphi * f)'(z)}{\sum_{l=0}^{n-1}(\varphi * f)(x^l z)} - \gamma\right\} > \delta\left|\frac{z(\varphi * f)'(z)}{\sum_{l=0}^{n-1}(\varphi * f)(x^l z)} - p\right| \qquad (z \in \mathcal{U}).$$

It is related to the class of δ-uniformly convex functions of order γ with respect to n symmetric points. Moreover, putting $n = 1$, we obtain the class $\mathcal{W}_n(p,k;\varphi;2\gamma-1,1;\delta)$ defined by the following condition:

$$\mathrm{Re}\left\{\frac{z(\varphi * f)'(z)}{(\varphi * f)(z)} - \gamma\right\} > \delta\left|\frac{z(\varphi * f)'(z)}{(\varphi * f)(z)} - p\right| \qquad (z \in \mathcal{U}).$$

The class is related to the class of δ-uniformly convex functions of order γ. The classes

$$UST(A,B;\delta) := \mathcal{W}_0\Big(1,2;\frac{z}{1-z};2\gamma-1,1;\delta\Big),$$
$$UCV(A,B;\delta) := \mathcal{W}_0\Big(1,2;\frac{z}{(1-z)^2};2\gamma-1,1;\delta\Big),$$

are the well-known classes of δ-starlike functions of order γ and δ-uniformly convex functions of order γ, respectively. In particular, the classes $UCV := UCV(1,0)$, $\delta - UCV := UCV(\delta,0)$ were introduced by Goodman [6], and Wiśniowska *et al.* ([19] and [8]), respectively.

We note that the class

$$\mathcal{H}_{\mathcal{T}}(\varphi;A,B;\delta) := \mathcal{T}^0(1,2) \cap \mathcal{W}_n(1,2;\varphi;2\gamma-1,1;\delta)$$

was introduced and studied by Raina and Bansal [13].

If we set

$$h(\alpha_1,z) := z_qF_s(\alpha_1,\dots,\alpha_q;\beta_1,\dots,\beta_s;z),$$

where ${}_qF_s$ is the generalized hypergeometric function, then we obtain the class

$$\mathcal{UH}(q,s,\lambda,A,B;\delta) := \mathcal{H}_{\mathcal{T}}(\lambda h(\alpha_1+1,z)+(1-\lambda)h(\alpha_1,z);A,B;\delta) \qquad (0 \le \lambda \le 1)$$

defined by Srivastava *et al.* [14].

Let λ be a convex parameter. A function $f \in \mathcal{A}(p,k)$ belongs to the class

$$\mathcal{V}_\lambda(\varphi;A,B) := \mathcal{W}\Big(\lambda\frac{\varphi(z)}{z} + (1-\lambda)\varphi'(z), z; A, B; 0\Big)$$

if it satisfies the following condition:

$$\lambda\frac{(\varphi * f)(z)}{z} + (1-\lambda)(\varphi * f)'(z) \prec \frac{1+Az}{1+Bz}.$$

Moreover, a function $f \in \mathcal{A}(p,k)$ belongs to the class

$$\mathcal{U}_\lambda(\varphi;A,B) := \mathcal{W}\Big(\lambda\frac{\varphi(z)}{z} + (1-\lambda)\varphi'(z); A, B; 0\Big)$$

if it satisfies the following condition:

$$\frac{z(\varphi * f)'(z) + (1-\lambda)z^2(\varphi * f)''(z)}{\lambda(\varphi * f)(z) + (1-\lambda)z(\varphi * f)'(z)} \prec \frac{1+Az}{1+Bz}.$$

The considered classes are defined by using the convolution $\varphi * f$ or equivalently by the linear operator

$$J_\varphi : \mathcal{A}(p,k) \to \mathcal{A}(p,k), \quad J_\varphi(f) = \varphi * f.$$

By changing the function φ, we can obtain a lot of important linear operators, and in consequence new and also well-known classes of functions. We list here some of these linear operators such as the Salagean operator, the Cho-Kim-Srivastava operator, the Dziok-Raina operator, the Hohlov operator, the Dziok-Srivastava operator, the Carlson-Shaffer operator, the Ruscheweyh derivative operator, the generalized Bernardi-Libera-Livingston operator, the fractional derivative operator, and so on (for the precise relationships see [4]).

If we apply the results presented in this paper to the classes discussed above, we can achieve further results. Some of these were obtained in earlier works, see for example [2, 4, 5, 13, 14].

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