Annales Academiae Paedagogicae Cracoviensis
Folia 23 Studia Mathematica IV (2004)

Bogdan Choczewski, Ilie Corovei, Anna Matkowska

On some functional equations related to Steffensen's inequality

Dedicated to Professor Andrzej Zajtz on his seventieth birthday

Abstract. We consider the problem, proposed by the second author (cf. [1]) of solving functional equations stemming from the Steffensen integral inequality (S), which is applicable in actuarial problems, cf. [4]. Imposing some regularity conditions we find solutions of two equations in two variables, one with two and another with three unknown functions.

1. Introduction

J.F. Steffensen proved in his paper [4] from 1918 entitled *On certain inequalities between mean values and their applications to actuarial problems* the following

Proposition
If $f\colon [a,b] \to \mathbb{R}$ *is a decreasing function and* $g\colon [a,b] \to [0,1]$ *is an integrable function, then*

$$\int_{b-c}^{b} f(t)\,dt \le \int_{a}^{b} f(t)g(t)\,dt \le \int_{a}^{a+c} f(t)\,dt, \qquad c := \int_{a}^{b} g(t)\,dt. \tag{S}$$

[Cf. also J. Dieudonné [2], p. 50, and, for this and for several related inequalities, D.S. Mitrinović [3], Section 2.16, pp. 105-116.]

The second author proposed in [1] to look for f and g such that the medial term in inequalities (S) is the arithmetic mean of the two others. Let x and y vary in $[a,b]$ and let us write the relevant functional equation with the unknown functions f and g:

AMS (2000) Subject Classification: 39B22, 26D15.

$$\int_x^{x+\gamma(x,y)} f(t)\,dt + \int_{y-\gamma(x,y)}^{y} f(t)\,dt = 2\int_x^y f(t)g(t)\,dt, \qquad \gamma(x,y) := \int_x^y g(t)\,dt, \tag{E}$$

where $(x,y) \in [a,b]^2$.

In this paper we deal with equation (E) for differentiable f and continuous g. We also consider a functional equation related to (E), with three, sufficiently regular, unknown functions: f, g and h, the latter replacing those limits of integration in (E) which contain $\gamma(x,y)$, cf. [1].

2. Equation with two unknown functions

Let us first note that if f in (E) is a constant function then the equation is satisfied by an arbitrary (integrable) function g. In the theorem that follows we determine functions f, corresponding to a wide variety of functions g, such that (f,g) be the solution to (E). It turns out that in most cases f is a constant function.

Theorem 1
Assume that $g\colon [a,b] \to [0,1]$ *is a continuous function and either:*

(i) $g(x) = K$ *for* $x \in [a,b]$, *and* $K \notin \{0, 1, \frac{1}{2}\}$

or

(ii) $0 < g(x) < 1$, $x \in (a,b)$ *and either* $g(a) = 0$, $g(b) = 1$ *or* $g(a) = 1$, $g(b) = 0$.

Then the function $f\colon [a,b] \to \mathbb{R}$, *differentiable in* $[a,b]$, *satisfies equation* (E) *if and only if it is of the form:*

in case (i)

$$f(x) = \alpha x + \beta, \qquad x \in [a,b],$$

in case (ii)

$$f(x) = A, \qquad x \in [a,b],$$

where α, β, A *are arbitrary real numbers.*

Proof. Assume (i). Since now $\gamma(x,y) = K(y-x)$, equation (E) becomes:

$$\int_x^{x+K(y-x)} f(t)\,dt + \int_{y-K(y-x)}^{y} f(t)\,dt = 2K\int_x^y f(t)\,dt, \qquad (x,y) \in [a,b]^2. \tag{1}$$

We take the derivatives, with respect to x of both sides of (1):

$$f(x+K(y-x))(1-K)-f(x)-f(y-K(y-x))K=-2Kf(x).$$

as f is continuous in $[a,b]$. We differentiate again, but with respect to y, and divide the formula obtained by $K(1-K)$:

$$f'(x+K(y-x))-f'(y-K(y-x))=0, \qquad (x,y)\in[a,b]^2.$$

The transformation

$$T\colon \mathbb{R}^2\to\mathbb{R}^2;\quad (x,y)\mapsto(X,Y):=((1-K)x+Ky,Kx+(1-K)y)$$

maps bijectively $[a,b]^2$ onto itself. Indeed, $K\neq\frac{1}{2}$ implies injectivity of T and $K\in(0,1)$ says that both X and Y are convex combinations of x and y. Therefore we have the relation

$$f'(t)=f'(s), \qquad (s,t)\in[a,b]^2.$$

Consequently, the function f' is constant and f is affine, on $[a,b]$, as claimed.

It is the matter of a straightforward calculation to verify that the functions given by $f(x)=\alpha x+\beta$, $g(x)=K$, $x\in[a,b]$, satisfy equation (1) for every real α and β.

Assume (ii). We differentiate equation (E) twice, first with respect to x, then with respect to y. Since $\gamma(x,y):=\int_x^y g(t)\,dt$ and g is continuous, $\frac{\partial\gamma}{\partial x}=-g(x)$, $\frac{\partial\gamma}{\partial y}=g(y)$. We get, consecutively,

$$f(x+\gamma(x,y))(1-g(x))-f(x)-f(y-\gamma(x,y))(-g(x))=-2f(x)g(x),$$

$$f'(x+\gamma(x,y))(1-g(x))g(y)-f'(y-\gamma(x,y))g(x)(1-g(y))=0. \qquad (\mathrm{E}')$$

Both equalities hold for $(x,y)\in[a,b]^2$.

Let now $g(a)=0$ and $g(b)=1$. We put $x=a$ in (E′):

$$f'(a+\gamma(a,y))g(y)=0, \qquad y\in(a,b].$$

Since the function $u\colon(a,b]\to\mathbb{R}$, $u(y)=a+\gamma(a,y)$, is strictly increasing in (a,b) ($u'(y)=g(y)>0$), it maps $(a,b]$ onto $(u(a),u(b)]=(a,a+c]$ (where $c=\int_a^b g(t)\,dt$, cf. (S)). Thus

$$f'(t)=0 \qquad (2)$$

for $t\in(a,a+c]$. Now we put $y=b$ in (E′):

$$f'(x+\gamma(x,b))(1-g(x))=0, \qquad x\in[a,b).$$

The function $v\colon[a,b)\to\mathbb{R}$, $v(x)=x+\gamma(x,b)$, is a strictly increasing bijection ($v'(x)=1-g(x)>0$) of $[a,b)$ onto $[a+c,b)$, whence (2) holds for $t\in(a+c,b]$. Finally, (2) is valid in (a,b), f is a constant function, $f(x)=A$ in (a,b). By the continuity, f is constant in $[a,b]$, as claimed.

Similarly, $g(a)=1$ implies (2) in $(a,b-c]$, whereas $g(b)=0$ yields (2) in $[b-c,b)$. It follows that $f'(t)$ vanishes in (a,b) also in the other case listed in (ii), whence f is a constant function on $[a,b]$.

To complete the proof let us remind that if f in (E) is a constant function then the equation is satisfied for every integrable function g.

REMARK 1
If in case (i) we have $K \in \{0, 1, \frac{1}{2}\}$, then equation (E) is an identity and f may be an arbitrary function.

REMARK 2
If the inequalities of (ii) hold but $g(a) = g(b) = 0$, or $g(a) = g(b) = 1$. then $f'(t) = 0$ in $(a, a+c] \cup [b-c, b)$. In the case where both intervals are disjoint, we get only the information that the restrictions of f to each of the intervals is a constant function.

REMARK 3
For $[a, b]$ replaced by $\mathbb{R}$ and $g(x) = K$, $x \in \mathbb{R}$, $|K| > 1$, the formula $f(x) = \alpha x + \beta$, $x \in \mathbb{R}$ also presents all differentiable in $\mathbb{R}$ solution of (E). Indeed, we may repeat the proof of (i) of Theorem 1, because the transformation T exploited there maps bijectively the plane onto itself.

3. Equation with three unknown functions

We pass to examining the equation related to (E) in which the limits of integration $x + \gamma(x, y)$, resp. $y - \gamma(x, y)$, are replaced by $h(xy + x + y)$, resp. $h(xy - x - y)$, where h is also an unknown function. Moreover, a "correcting term" has been added. The equation to be solved, with the unknown functions f, g, h, reads

$$\begin{aligned} &\int_x^{h(xy+x+y)} f(t)\,dt + \int_{h(xy-x-y)}^{y} f(t)\,dt + \int_{h(y^2+2y)}^{h(y^2-2y)} f(t)\,dt \\ &\qquad = 2\int_x^y f(t)g(t)\,dt, \end{aligned} \tag{H}$$

First of all, assuming the necessary regularity of the functions involved, on differentiating both sides of equation (H) with respect to x we obtain the equation

$$\begin{aligned} &(y+1)h'(xy+x+y)f(h(xy+x+y)) \\ &\qquad - (y-1)h'(xy-x-y)f(h(xy-x-y)) \\ &\quad = f(x)[1-2g(x)]. \end{aligned} \tag{H$'$}$$

The subsequent lemma establishes the equivalence of equations (H) and (H′).

LEMMA 1
The functions: $h, f, g\colon \mathbb{R} \to \mathbb{R}$; h differentiable on $\mathbb{R}$; f and g continuous on $\mathbb{R}$, satisfy equation (H) *on $\mathbb{R}^2$ if and only if they satisfy equation* (H′) *on $\mathbb{R}^2$.*

Proof. Clearly $(H) \Rightarrow (H')$. To get the converse implication let us rewrite equation (H′) as follows (cf. (4)):

$$\begin{aligned} &-(y+1)h'(sy+s+y)f(h(sy+s+y) \\ &\qquad + (y-1)h'(sy-s-y)f(h(sy-s-y)) + f(s) \\ &\quad = 2f(s)g(s). \end{aligned} \tag{3}$$

and integrate with respect to s their sides, LHS and RHS, over the interval $[x,y]$. After executing the substitutions: $t = h(sy+s+y)$ in the first integral of the LHS of (3) and $t = h(sy-s-y)$ in the second one we obtain

$$\int_{h(y^2+2y)}^{h(xy+x+y)} f(t)\,dt + \int_{h(y^2-2y)}^{h(xy-x-y)} f(t)\,dt + \int_x^y f(t)\,dt.$$

After adding and subtracting integrals with suitable limits of integration we find that this sum of integrals equals to the LHS of (H). The integral over $[x,y]$ of the RHS of (3) and RHS of (H) are the same expressions. Hence $(H') \Rightarrow (H)$.

In the sequel J will stand for an interval contained either in $(-\infty,-1)$ or in $(-1,1)$, or in $(1,+\infty)$. We are in position to prove the following

Theorem 2
If the function $h\colon \mathbb{R} \to J$ is three times differentiable on $\mathbb{R}$; the function $f\colon J \to \mathbb{R}$ is twice differentiable on J; and $g\colon J \to \mathbb{R}$ is continuous in J, then equation (H) *when postulated for $(x,y) \in J^2$, is equivalent to the system of the equalities (both valid for $x \in J$)*

$$\begin{cases} (x+1)h'(x)f(h(x)) = \alpha x + \beta; \\ (x^2-1)f(x)(1-2g(x)) = 2(\alpha x^2 - \beta), \end{cases} \tag{C}$$

where α and β are arbitrary real numbers, but $\alpha^2 + \beta^2 > 0$.

Proof. According to the Lemma it is enough to solve equation (H′). We denote, for short, by A and B the factors of the first product of functions occurring on the LHS of (H′):

$$A(x,y) := (y+1)h'(xy+x+y); \qquad B(x,y) := f(h(xy+x+y)).$$

Then the factors of the other product in LHS of (3) are equal

$$(y-1)h'(xy-x-y) = -\,A(-x,-y); \qquad f(h(xy-x-y)) = B(-x,-y).$$

Applying to A and B Maclaurin's formula (with the Peano reminder) in a neighbourhood of $y = 0$ yields:

$$\begin{aligned}A(x,y) &= h'(x) + y[h'(x) + (x+1)h''(x)] \\ &\quad + \frac{1}{2}y^2[2(x+1)h''(x) + (x+1)^2h'''(x)] \\ &\quad + o(y^2),\end{aligned}$$

$$\begin{aligned}B(x,y) &= f(h(x)) + y[(x+1)h'(x)f'(h(x))] \\ &\quad + \frac{1}{2}y^2[(x+1)^2h''(x)f'(h(x)) + (x+1)^2(h'(x))^2f''(h(x))] \\ &\quad + o(y^2),\end{aligned}$$

where the Landau symbol o refers to $y \to 0$.

Now we calculate the LHS(H′) $= A(x,y)B(x,y) + A(-x,-y)B(-x,-y)$, insert the formula obtained to (H′) and compare the free terms and the coefficients of y and y^2 of the resulting equation. This yields the following equalities:

$$h'(x)f(h(x)) + h'(-x)f(h(-x)) = f(x)[1 - 2g(x)], \tag{4}$$

$$F(x) = F(-x), \tag{5}$$

where

$$F(x) := [(x+1)h'(x)f(h(x))]' \tag{6}$$

and

$$(x+1)F'(x) = (x-1)F'(-x). \tag{7}$$

From (5) we have $F'(x) = -F'(-x)$. Eliminating $F'(-x)$ from this equation and from (7) we get $xF'(x) = 0$, $x \in J$, whence $F(x) = \alpha$ for $x \in J$. Integrating (6) we get the first equation of system (C). Inserting the resulting formula to (4) we arrive at the other equation of (C).

On the other hand, given some functions f, g, h satisfying (C) and regular as required in the theorem one checks by a direct calculation that they satisfy equation (H′) and, by Lemma, also equation (H). This completes the proof of the theorem.

Remark 4

The three unknown functions f, g, h, are linked by two conditions (C) only. Thus given arbitrarily one of the functions one may determine the others. For instance, if one of the functions h or f is the identity, we get the following triplets of solutions to (H):

$$\begin{cases} f(x) = \dfrac{\alpha x + \beta}{x+1}, \\ g(x) = \dfrac{(\beta - \alpha)x}{2(x-1)(\alpha x + \beta)}, \\ h(x) = x, \qquad\qquad x \in J, \alpha = 0 \text{ or } -\frac{\beta}{\alpha} \notin J; \end{cases}$$

$$\begin{cases} f(x) = x, \\ g(x) = \dfrac{1}{2} + \dfrac{\alpha x^2 - \beta}{x(1-x^2)}, \\ [h(x)]^2 = 2\alpha x + 2(\beta - \alpha)\log|x+1| + C, \qquad |x| > 1, \end{cases}$$

provided that the constants are so chosen that $h(x) \in J$. Among solutions of (H) when g is the identity there are those corresponding to $\alpha = \beta \neq 0$ in (C):

$$\begin{cases} f(x) = \dfrac{2\alpha}{1-2x}, \\ g(x) = x, \\ h(x) = C\ e^{-x} + \dfrac{1}{2}, \qquad |x| > 1, \end{cases}$$

since in this case h is a solution of the equation $2h'(x) + 2h(x) = 1$.

Acknowledgement

The authors want to express their thanks to the referees, whose remarks helped substantially to improve the paper.

References

[1] I. Corovei, *Functional-integral equations stemming from Steffensen's inequality* [5. Problems and Remark (presented by B. Choczewski)], in: Report of Meeting, 9th International Conference on Functional Equations and Inequalities, Złockie, September 7-13, 2003; Ann. Acad. Paedag. Cracoviensis, Studia Math., to appear.

[2] J. Dieudonné, *Calcul infinitésimal*, Hermann, Paris, 1968.

[3] D.S. Mitrinović, *Analitičke nejednakosti*, Gradevinska Knjiga, Beograd, 1970.

[4] J.F. Steffensen, *On certain inequalities between mean values and their applications to actuarial problems*, Skandinavisk Aktuarietidskrift, 1918, 82-97.

AGH University of Science and Technology
Faculty of Applied Mathematics
Al. Mickiewicza 30, Kraków, Poland
E-mail: smchocze@cyf-kr.edu.pl

Technical University
Institute of Mathematics
Cluj-Napoca, Romania
E-mail: Ilie.Corovei@math.utcluj.ro

AGH University of Science and Technology
Faculty of Managing
Al. Mickiewicza 30, Kraków, Poland