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Some properties of analytic sets with proper projections

Abstract. We give an effective criterion when an analytic set with proper projection is algebraic. We take an ideal of polynomials vanishing on the set then we construct a polydisc convenient for reduction. If this polydisc is "large enough" we can apply the division theorem in the ring of formal power series convergent in this polydisc to prove that the set is algebraic.

1. Introduction

The main results of this paper are Theorem 3.2 and Theorem 3.4 which give an affective criterion of algebraicity of analytic sets with proper projection. For convenience of the readers we start with a short introduction into the well-know theories of Gröbner bases for polynomial ideals. Then we recall some facts about division in the ring of everywhere convergent power series and formal power series convergent in "large enough" polydisc. These facts will be used to prove the main theorems.

2. Notation and basic facts

We introduce some notions which will be used throughout this paper. The basic algebraic structures involved are the polynomial ring $\mathcal{R} := \mathbb{K}[X] = \mathbb{K}[X_1, \ldots, X_n]$, the ring $\mathbb{K}[[X]] = \mathbb{K}[[X_1, \ldots, X_n]]$ of formal power series and the rings

$$E_r := \{f \in \mathbb{K}[[X]] : \ f \text{ is absolutely convergent at the point } r\}$$

corresponding to $r = (r_1, \ldots, r_n) \in \mathbb{R}_+^n$. Note that if $f \in E_r$ then f is absolutely uniformly convergent in the closure of the polydisc P_r, where

$$P_r := \{(x_1, \ldots, x_n) \in \mathbb{K}^n : \ |x_1| < r_1, \ldots, |x_n| < r_n\}.$$

Since we are interested in convergence, we restrict ourself to the fields of complex $(\mathbb{K} = \mathbb{C})$ or real $(\mathbb{K} = \mathbb{R})$ numbers.

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Let $X^\alpha := X_1^{\alpha_1} \cdot \ldots \cdot X_n^{\alpha_n}$. For $f = \sum_{\alpha \in \mathbb{N}^n} c_\alpha X^\alpha \in \mathbb{K}[[X]]$ the *support* of f is defined as

$$\operatorname{supp} f = \{\alpha : \ c_\alpha \neq 0\}.$$

For a set $F \subset \mathbb{K}[[X]]$ we put $\operatorname{supp} F = \bigcup_{f \in F} \operatorname{supp} f$.

Let $f = \sum_{\alpha \in \mathbb{N}^n} c_\alpha X^\alpha \in E_r$, where $r = (r_1, \ldots, r_n) \in \mathbb{R}_+^n$. The space E_r with the norm

$$\|f\|_r := \sum_{\alpha \in \mathbb{N}^n} |c_\alpha| r^\alpha$$

is a Banach space (for more details about this space, see [3], p. 23–27). For a given nonempty subset $\mathcal{D} \subseteq \mathbb{N}^n$ we define

$$E_r(\mathcal{D}) := \{f \in E_r : \ \operatorname{supp} f \subseteq \mathcal{D}\}.$$

Let $\prec$ be a fixed admissible term ordering in $\mathbb{N}^n$. Then, by definition, $X^\alpha \prec X^\beta$ if $\alpha \prec \beta$. If $f = \sum_{\alpha \in \mathbb{N}^n} c_\alpha X^\alpha \in \mathcal{R}$, $f \neq 0$ then the *exponent, leading term, leading coefficient, leading monomial* and *tail* of f are defined as

$$\begin{aligned}
\exp_\prec f &:= \max_\prec \{\alpha : \ \alpha \in \operatorname{supp} f\},\\
\mathrm{LT}_\prec f &:= X^{\exp_\prec f},\\
\mathrm{LC}_\prec f &:= c_{\exp_\prec f},\\
\mathrm{LM}_\prec f &:= \mathrm{LC}_\prec f \, \mathrm{LT}_\prec f,\\
\mathrm{tail}_\prec f &:= f - \mathrm{LM}_\prec f,
\end{aligned}$$

respectively.

For $F \subset \mathcal{R}$ we define

$$\begin{aligned}
\Delta_F &:= \begin{cases} \bigcup_{f \in F} (\exp_\prec f + \mathbb{N}^n) & \text{if } F \nsubseteq \{0\},\\ \varnothing & \text{if } F \subseteq \{0\},\end{cases}\\
\mathcal{D}_F &:= \mathbb{N}^n \setminus \Delta_F.
\end{aligned}$$

Let $I \subset \mathcal{R}$ be a nonzero ideal and let $\prec$ be an admissible term ordering. A finite subset $G \subset I$ is called a *Gröbner basis of I with respect to $\prec$* if $\Delta_G = \Delta_I$. For convenience to the readers we recall the division theorem in $\mathcal{R}$.

Theorem 2.1 (see [2], p. 79, Proposition 1)
Let $G = \{g_1, \ldots, g_s\}$ be a Gröbner basis for an ideal $I \in \mathcal{R}$ with respect to an admissible term ordering $\prec$ and let $f \in \mathcal{R}$. Then there is a unique $f_{\mathrm{red}} \in \mathcal{R}$ with the following properties:

(i) *no term of f_{red} is divisible by any of* $\mathrm{LT}_\prec g_1, \ldots, \mathrm{LT}_\prec g_s$,

(ii) *there is $g \in I$ such that $f = g + f_{\mathrm{red}}$.*

We will call f_{red} a *reduction of* f with respect to I.

This theorem can be generalized to ideal generated by polynomials in the ring E_r, for some $r \in \mathbb{R}_+^n$ which depend on a fixed ideal and an admissible term ordering. We can reduce $f = \sum_{\alpha\in\mathbb{N}^n} c_\alpha X^\alpha \in \mathbb{K}[[X]]$ as follow. We reduce each X^α with respect to polynomial ideal I and admissible term ordering $\prec$, we receive X^α_{red}. Then we create a formal power series $\sum_{\alpha\in\mathbb{N}^n} c_\alpha X^\alpha_{\mathrm{red}}$. And questions are when this series is convergent and if we can control the norm of it.

The first generalization appeared in [1] and it involves reduction of everywhere convergent power series (see [1], Theorem 3.7). The ideas from this paper can be used to prove analogous facts for the ring E_r.

At first we define a polydisc convenient for reduction with respect to a given ideal $I \subset \mathcal{R}$ and a given admissible term ordering $\prec$ as follow (see [6], Definition 4.1)

Definition 2.2
We say that a polydisc P_r, $r \in \mathbb{R}_+^n$, is *convenient for reduction with respect to* I *and* $\prec$, if

$$\|\mathrm{LM}_\prec g\|_r > \|\mathrm{tail}_\prec g\|_r \qquad \text{for } g \in G,$$

where G is the reduced Gröbner basis of I with respect to $\prec$.

For any ideal I and ordering $\prec$ the set $\mathcal{M}_{I,\prec}$ of all $r \in \mathbb{R}_+^n$, where P_r is a polydisc convenient for reduction with respect to I and $\prec$, is nonempty (see [6], Lemma 2.6) and open (see [6], Remark 3.3).

Theorem 2.3 ([6], Theorem 3.4)
Let $\prec$ be an admissible term ordering. Let $G \subset \mathcal{R}$ be a Gröbner basis of an ideal I and let P_r be a polydisc convenient for reduction with respect to I and $\prec$. Then

(i) *if $f = \sum_{\alpha\in\mathbb{N}^n} c_\alpha X^\alpha \in E_r$ then the series $\sum_{\alpha\in\mathbb{N}^n} c_\alpha X^\alpha_{\mathrm{red}}$ is convergent to an $f_{\mathrm{red}} \in E_r(\mathcal{D}_I)$,*

(ii) *the mapping* $\mathrm{red}\colon E_r \ni f \mapsto f_{\mathrm{red}} \in E_r(\mathcal{D}_I)$ *gives a continuous projection of E_r onto $E_r(\mathcal{D}_I)$,*

(iii) *$\|f_{\mathrm{red}}\|_r \le \|f\|_r$ for $f \in E_r$,*

(iv) *$f_{\mathrm{red}} = 0$ if and only if $f \in IE_r$ for an $f \in E_r$,*

(v) *$E_r = IE_r \oplus E_r(\mathcal{D}_I)$ (direct sum).*

3. When an analytic set with proper projection is algebraic

At first we recall some facts. Let $\Omega \subset \mathbb{C}^n$ be a domain. A subset X of Ω is said to be a *Nash subset of* Ω if for every $x_0 \in \Omega$ there exist a neighbourhood $U \subset \Omega$ of x_0 and Nash functions $f_1, \dots, f_r$ on U such that

$$X \cap U = \{x \in U : \ f_1(x) = f_2(x) = \ldots = f_r(x) = 0\}.$$

A subset X of $\mathbb{C}^n$ is said to be *algebraic in* Ω if $X \cap \Omega = \overline{X} \cap \Omega$, where $\overline{X}$ is the Zariski closure of X.

Theorem 3.1 (see [7], Theorem 2.12)
Every irreducible Nash subset of $\mathbb{C}^n$ is an algebraic irreducible subset of $\mathbb{C}^n$.

Let consider a ring of polynomials of two group of variables

$$\mathbb{K}[X_1, \ldots, X_n, Y_1, \ldots, Y_m].$$

We define *elimination term ordering for* Y as an admissible term ordering in $\mathbb{N}^n \times \mathbb{N}^m$ such that

$$X^\alpha \prec_Y Y^\beta \qquad \text{for } \alpha \in \mathbb{N}^n,\ \beta \in \mathbb{N}^m \setminus \{0\}.$$

Now we consider an analytic set of codimension 1. Then the following theorem holds true.

Theorem 3.2
Let $\Omega \subset \mathbb{C}^n$ be a domain. Let $N \subset \Omega \times \mathbb{C}$ be a purely n-dimensional analytic set such that the restriction of natural projection

$$\pi\,|_N \colon N \ni (x, y) \mapsto x \in \Omega$$

is a proper mapping and the ideal $I = I(N)$ of polynomials vanishing on N is a proper ideal of the ring $\mathcal{R}$. Let $P_r = P_\rho \times P_t$, where $r = (\rho, t) \in \mathbb{R}^n_+ \times \mathbb{R}_+$, be a polydisc convenient for reduction with respect to the ideal I and $\prec_Y$, an elimination ordering for Y. If $P_\rho \subset \Omega$ and the set N is algebraic in P_r then the set N is algebraic in $\Omega \times \mathbb{C}$.

Proof. Since N is a purely n-dimensional analytic subset of $\Omega \times \mathbb{C}$ and the restriction of natural projection $\pi|_N$ is proper, there exists the unique system $\sigma_1, \ldots, \sigma_s$ of holomorphic functions on Ω such that

$$N = \{(x, y) \in \Omega \times \mathbb{C} :\ y^s + \sigma_1(x) y^{s-1} + \ldots + \sigma_s(x) = 0\}.$$

Since the Zariski closure of N is an algebraic set of codimension 1, the reduced Gröbner basis G of I with respect to $\prec_Y$ consists of only one polynomial g of the form

$$g(X, Y) = a_k(X) Y^k + a_{k-1}(X) Y^{k-1} + \ldots + a_0(X)$$

with $k \geq s$ ($\pi|_N$ is an s-sheeted branched covering) and $a_k \neq 0$, and so $\mathrm{LM}_{\prec_Y} g = X^\alpha Y^k$ with an $\alpha \in \mathbb{N}^n$.

Let $f(X, Y) := Y^s + \sigma_1(X) Y^{s-1} + \ldots + \sigma_s(X)$. Since $f(X, Y)$ vanishes on N and N in algebraic in P_r, $f(X, Y) \in I\mathcal{O}(P_r)$, where $\mathcal{O}(P_r)$ is the ring of holomorphic functions in P_r (see e.g. [5], Theorem 4.6). The set $\mathcal{M}_{I, \prec_Y}$ is open. Thus, we can find $\widetilde{r} \in \mathcal{M}_{I, \prec_Y}$ such that the closure of $P_{\widetilde{r}}$ is contained in P_r and all the assumptions of Theorem 3.2 are satisfied with respect to $P_{\widetilde{r}}$. Since $I\mathcal{O}(P_{\widetilde{r}}) \subset E_{\widetilde{r}}$, we have $f(X, Y) \in IE_r$ and so

$$(-Y^s)_{\mathrm{red}} = (\sigma_1(X) Y^{s-1} + \ldots + \sigma_s(X))_{\mathrm{red}},$$

where "red" is the reduction in $E_{\widetilde{r}}$. On the other hand, $(-Y^s)_{\mathrm{red}}$ is a polynomial. Since $s - 1 < k$,

$$\mathrm{supp}\,(\sigma_1(X) Y^{s-1} + \ldots + \sigma_s(X)) \subset \mathcal{D}_I.$$

Hence

$$(\sigma_1(X)Y^{s-1} + \ldots + \sigma_s(X))_{\mathrm{red}} = \sigma_1(X)Y^{s-1} + \ldots + \sigma_s(X).$$

Then $f(X, Y)$ is a polynomial, which completes the proof.

Remark 3.3

Under the assumptions of Theorem 3.2 every irreducible analytic component of N has nonempty intersection with the polydisc P_r. Indeed, let $\widetilde{N}$ be a sum of these components of the set N, which have nonempty intersection with P_r. Assume that $\widetilde{N} \neq N$. Since $\widetilde{N}$ is a purely n-dimensional analytic set such that the restriction of natural projection is proper, there exists an unique system u_i, $i = 1, \ldots, q$ of holomorphic functions on Ω such that

$$\widetilde{N} = \{(x, y) \in \Omega \times \mathbb{C} : \ y^q + u_1(x)y^{q-1} + \ldots + u_q(x) = 0\},$$

with $q < k$. Just like in the proof of Theorem 3.2 one can prove that $h(X, Y) := Y^q + u_1(X)Y^{q-1} + \ldots + u_q(X)$ is a polynomial. Then $g(X, Y)$ divides $h(X, Y)$, which contradicts condition $q < k$.

Now we consider an analytic set with higher codimension.

Theorem 3.4

Let $\Omega \subset \mathbb{C}^n$ be a domain. Let $N \subset \Omega \times \mathbb{C}^m$ be a purely n-dimensional analytic set such that the restriction of natural projection

$$\pi\,|_N : N \ni (x, y) \mapsto x \in \Omega$$

is a proper mapping and the ideal $I = I(N)$ of polynomials vanishing on N is a proper ideal of the ring $\mathcal{R}$. Let $P_r = P_\rho \times P_t$, where $r = (\rho, t) \in \mathbb{R}^n_+ \times \mathbb{R}^m_+$, be a polydisc convenient for reduction with respect to the ideals $I \cap \mathbb{C}[X, Y_j]$ for $j = 1, \ldots, m$ and $\prec_Y$, an elimination ordering for Y. If $P_\rho \subset \Omega$, the set N is algebraic in P_r, and each $N_j := \pi_j(N)$ is algebraic in $P_{(\rho, t_j)} \subset \mathbb{C}^{n+1}$, where

$$\pi_j : \Omega \times \mathbb{C}^m \ni (x, y) \mapsto (x, y_j) \in \Omega \times \mathbb{C}$$

for $j = 1, \ldots, m$ then the set N is algebraic in $\Omega \times \mathbb{C}^m$.

Proof. From Lemma 3.1 [6] it follows that there exists $r \in \mathbb{R}^{n+m}_+$ such that the polydisc P_r is convenient for reduction with respect to the ideals $I \cap \mathbb{C}[X, Y_j]$ for $j = 1, \ldots, m$ and the ordering $\prec_Y$.

Let for a fixed $j \in \{1, \ldots, m\}$

$$L_j : \mathbb{C}^m \ni (y_1, \ldots, y_m) \mapsto y_j \in \mathbb{C},$$
$$\Phi_j : \mathbb{C}^n \times \mathbb{C}^m \ni (x, y) \mapsto (x, y_j) \in \mathbb{C}^n \times \mathbb{C}.$$

Then $N_j = \Phi_j(N)$ is a purely n-dimensional analytic subset of $\Omega \times \mathbb{C}$ and the restriction of the natural projection

$$\widetilde{\pi_j}\,|_{N_j} : N_j \ni (x, y_j) \mapsto x \in \Omega$$

is proper. From Theorem 3.2 the set N_j is algebraic in $\Omega \times \mathbb{C}$.

Let $W := \bigcap_{j=1}^{m} \Phi_j^{-1}(N_j)$. Then the set N is a subset of W and the sets $\overline{N}$, $\overline{W}$ are purely n-dimensional sets, when the closure is in Zarisky topology. Let $\overline{W} = \overline{W_1} \cup \ldots \cup \overline{W_s}$ be an algebraic decomposition of W. Then, after change of order we claim that

$$\overline{N} = \overline{W_1} \cup \ldots \cup \overline{W_l}, \qquad l \leq s.$$

Since N is algebraic in P_r,

$$N \cap P_r = (\overline{W_1} \cup \ldots \cup \overline{W_l}) \cap P_r.$$

Since the sets N_j, $j = 1, \ldots, m$, are algebraic in $\Omega \times \mathbb{C}$, $N \cap P_r$ has nonempty intersection with any irreducible analytic components of $\overline{W_1} \cup \ldots \cup \overline{W_l}$ (Remark 3.3). Since N is analytic, we have

$$\overline{N} \cap (\Omega \times \mathbb{C}^m) = N,$$

which ends the proof.

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