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Anna Bahyrycz

On systems of equations with unknown multifunctions related to the plurality function

Abstract. Let T be a nonempty set. Inspired by a problem posed by Z. Moszner in [10] we investigate for which additional assumptions put on multifunctions $Z(t)\colon T \to 2^{\mathbb{R}(m)}$, which fulfil condition

$$\bigcup_{t\in T} Z(t) = \mathbb{R}(m)$$

and the system of conditions

$$Z(t_1)^{k_1} \cap Z(t_2)^{k_2} + Z(t_1)^{l_1} \cap Z(t_2)^{l_2} \subset Z(t_1)^{k_1 l_1} \cap Z(t_2)^{k_2 l_2}$$

for all $t_1, t_2 \in T$ and for all $k_1, k_2, l_1, l_2 \in \{0,1\}$ such that $k_1 l_1 + k_2 l_2 \neq 0$, where $\mathbb{R}(m) := [0,+\infty)^m \setminus \{0_m\}$, $Z(t)^1 := Z(t)$, $Z(t)^0 := \mathbb{R}(m) \setminus Z(t)$, the multifunctions are also satisfying system of equations obtained by replacing the inclusion in the above conditions by the equality. Next we study if this system of equations are equivalent to some system of conditional equations.

F.S. Roberts, generalizing the mathematical description of choices introduced by himself in [11, 12] considers functions $f\colon \mathbb{R}(m) := [0,+\infty)^m \setminus \{0_m\} \to \mathbb{R}(m)$, which satisfy, among others, the following conditional functional equation:

$$\forall_{x,y\in\mathbb{R}(m)} \quad f(x)\cdot f(y) \neq 0_m \implies f(x+y) = f(x)\cdot f(y), \tag{1}$$

where $0_m := (0,\dots,0) \in \mathbb{R}^m$ and $x+y := (x_1+y_1,\dots,x_m+y_m)$, $x\cdot y := (x_1\cdot y_1,\dots,x_m\cdot y_m)$ for $x=(x_1,\dots,x_m)\in\mathbb{R}(m)$, $y=(y_1,\dots,y_m)\in\mathbb{R}(m)$.

Mathematical theory of this approach was developed by Z.S. Rosenbaum [13], Z. Moszner [3, 7, 8, 9, 10], G.L. Forti and L. Paganoni [5, 6] and A. Bahyrycz [1, 2, 3, 4].

It may be shown (see [7]) that the description of all the solutions $f=(f_1,\dots, f_m)$ of equation (1) takes the form:

$$f_\nu(x) = \begin{cases} \exp a_\nu(x) & \text{for } x \in Z_\nu, \\ 0 & \text{for } x \in \mathbb{R}(m)\setminus Z_\nu, \end{cases}$$

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where $a_\nu\colon \mathbb{R}^m \to \mathbb{R}$ are additive functions for $\nu = 1, \ldots, m$, whereas the sets Z_ν satisfy the conditions

$$Z_1 \cup \ldots \cup Z_m = \mathbb{R}(m), \tag{2}$$

$$ij \neq 0_m \implies Z_1^{i_1} \cap \ldots \cap Z_m^{i_m} + Z_1^{j_1} \cap \ldots \cap Z_m^{j_m} \subset Z_1^{i_1 j_1} \cap \ldots \cap Z_m^{i_m j_m}, \tag{3}$$

where $i = (i_1, \ldots, i_m)$, $j = (j_1, \ldots, j_m) \in 0(m) := \{0,1\}^m \setminus \{0_m\}$, $E_1 + E_2 := \{x + y :\ x \in E_1 \text{ and } y \in E_2\}$ for $E_1, E_2 \subset \mathbb{R}^m$, $E^1 := E$, $E^0 := \mathbb{R}(m) \setminus E$ for $E \subset \mathbb{R}(m)$.

Let us notice that the parameters determining the solutions of equation (1) consist of sets $Z_1, \ldots, Z_m$ satisfying conditions (2) and (3), as well as additive functions $a_\nu\colon \mathbb{R}^m \to \mathbb{R}$.

As a generalization, Z. Moszner in [10] considers multifunctions $Z(t)\colon T \to 2^G$, where T is an arbitrary nonempty set, $(G, +)$ is an arbitrary grupoid. He replaces (2) and (3) by

$$\bigcup_{t \in T} Z(t) = G, \tag{4}$$

$$(\exists_{t \in T}\ \ i(t)j(t) \neq 0) \implies \bigcap_{t \in T} Z(t)^{i(t)} + \bigcap_{t \in T} Z(t)^{j(t)} \subset \bigcap_{t \in T} Z(t)^{i(t)j(t)}, \tag{5}$$

where $Z(t)^1 := Z(t)$, $Z(t)^0 := G \setminus Z(t)$, and $i(t), j(t)\colon T \to \{0,1\}$ are arbitrary functions not identically equal to zero. Then he proves (see Theorem 9 in [10]) that the multifunction $Z(t)\colon T \to 2^G$ fulfilling condition (4), satisfies condition (5) if and only if $Z(t)$ satisfies condition

$$Z(t_1)^{k_1} \cap Z(t_2)^{k_2} + Z(t_1)^{l_1} \cap Z(t_2)^{l_2} \subset Z(t_1)^{k_1 l_1} \cap Z(t_2)^{k_2 l_2} \tag{6}$$

for all $t_1, t_2 \in T$ and for all $k_1, k_2, l_1, l_2 \in \{0,1\}$ such that $k_1 l_1 + k_2 l_2 \neq 0$.

Ispirated by a problem posed by Z. Moszner in [10], we will investigate for which additional assumptions put on multifunctions $Z(t)\colon T \to 2^{\mathbb{R}(m)}$, which fulfil condition

$$\bigcup_{t \in T} Z(t) = \mathbb{R}(m) \tag{7}$$

and system of conditions (6), the multifunctions are also satisfying system of equations obtained by replacing the inclusion in the above conditions by the equality.

We start with the following definition.

Definition 1

For every subset $L \subset \{1, \ldots, m\}$ we denote by B_L the set

$$B_L := \{(x_1, \ldots, x_m) \in \mathbb{R}(m) :\ \forall_l\ [l \in L \Rightarrow x_l = 0]\},$$

and we denote by $\mathbb{B}$ the family

$$\mathbb{B} := \{B_L :\ L \subset \{1, \ldots, m\}\}.$$

Let us notice that if the multifunction $Z(t)\colon T \to 2^{\mathbb{R}(m)}$ satisfies condition (7), then for every $a \in \mathbb{R}(m)$ and every $t \in T$ there exists a unique non-zero function $i(t)\colon T \to \{0,1\}$ such that $a \in \bigcap_{t \in T} Z(t)^{i(t)}$ (further on it will be denoted by $i_a(t)$).

Theorem 1
Let T be an arbitrary set with at least 2 elements and let the multifunction $Z(t)\colon T \to 2^{\mathbb{R}(m)}$ fulfils condition (7). The multifunction $Z(t)$ satisfies equation

$$Z(t_1)^{k_1} \cap Z(t_2)^{k_2} + Z(t_1)^{l_1} \cap Z(t_2)^{l_2} = Z(t_1)^{k_1 l_1} \cap Z(t_2)^{k_2 l_2} \tag{8}$$

for all $t_1, t_2 \in T$ and for all $k_1, k_2, l_1, l_2 \in \{0,1\}$ such that $k_1 l_1 + k_2 l_2 \neq 0$ if and only if $Z(t)$ satisfies the system of conditions (6) for all $t_1, t_2 \in T$ and for all $k_1, k_2, l_1, l_2 \in \{0,1\}$ such that $k_1 l_1 + k_2 l_2 \neq 0$ and the condition

$$\begin{aligned} \forall_{B\in\mathbb{B}}\ \forall_{\emptyset\neq T_1\subsetneq T}\ \exists_{k(t)\colon T_2=T\setminus T_1\to\{0,1\},\,k\not\equiv 0}\ & \bigcap_{t\in T_1} Z(t)^1 \cap \bigcap_{t\in T_2} Z(t)^0 \cap B \neq \emptyset \\ & \Longrightarrow \bigcap_{t\in T_1} Z(t)^1 \cap \bigcap_{t\in T_2} Z(t)^{k(t)} \cap B \neq \emptyset. \end{aligned} \tag{9}$$

Proof. ($\Rightarrow$) Take arbitrary $B \in \mathbb{B}$ and $\emptyset \neq T_1 \subsetneq T$ such that

$$\bigcap_{t\in T_1} Z(t)^1 \cap \bigcap_{t\in T_2} Z(t)^0 \cap B \neq \emptyset.$$

Fix an $x \in \bigcap_{t\in T_1} Z(t)^1 \cap \bigcap_{t\in T_2} Z(t)^0 \cap B$ and $t_1 \in T_1$, $t_2 \in T_2$. By (8) we obtain that there exist $y \in Z(t_1)^1 \cap Z(t_2)^0$ and $z \in Z(t_1)^1 \cap Z(t_2)^1$ such that $y + z = x \in Z(t_1)^1 \cap Z(t_2)^0$. Since $x \in B$, we have $y \in B$ and $z \in B$. Obviously, $y \in \bigcap_{t\in T} Z(t)^{i_y(t)}$, $z \in \bigcap_{t\in T} Z(t)^{i_z(t)}$ and from (8) for every $t \in T$ we get

$$y + z \in Z(t_1)^1 \cap Z(t)^{i_y(t)} + Z(t_1)^1 \cap Z(t)^{i_z(t)} = Z(t_1)^1 \cap Z(t)^{i_y(t) i_z(t)},$$

so $i_y(t) i_z(t) = i_x(t)$ for $t \in T$, hence $i_y(t) = i_z(t) = 1$ for all $t \in T_1$.

We define a function $k(t)\colon T_2 \to \{0,1\}$ in the following way

$$k(t) := i_z(t) \qquad \text{for } t \in T_2.$$

We note that the function $k(t)$ is not identically equal to zero (because $k(t_2) = 1$) and

$$z \in \bigcap_{t\in T_1} Z(t)^1 \cap \bigcap_{t\in T_2} Z(t)^{k(t)} \cap B.$$

($\Leftarrow$) Take arbitrary $t_1, t_2 \in T$ and $k_1, k_2, l_1, l_2 \in \{0,1\}$ such that $k_1 l_1 + k_2 l_2 \neq 0$. We will show that

$$Z(t_1)^{k_1 l_1} \cap Z(t_2)^{k_2 l_2} \subset Z(t_1)^{k_1} \cap Z(t_2)^{k_2} + Z(t_1)^{l_1} \cap Z(t_2)^{l_2}. \tag{10}$$

We consider two cases:

a) $k_1 = l_1$ and $k_2 = l_2$,

b) $k_1 \neq l_1$ or $k_2 \neq l_2$.

Case a) By Theorem 10 in [10] the set $Z := Z(t_1)^{k_1} \cap Z(t_2)^{k_2}$ is a cone over $\mathbb{Q}_+$ (i.e. $x + y \in Z$ and $\frac{n}{k}x \in Z$ for all $x, y \in Z$ and $n, k \in \mathbb{N}$). Then condition (10) holds, because for every $x \in Z$ we have $x = \frac{1}{2}x + \frac{1}{2}x \in Z + Z$.

Case b) Then either $k_1 = l_1 = 1$ and $k_2 \neq l_2$ or $k_1 \neq l_1$ and $k_2 = l_2 = 1$. Without loss of generality we may assume that $k_1 = l_1 = 1$, $k_2 = 1$ and $l_2 = 0$.

We take arbitrary $z = (z_1, \ldots, z_m) \in Z(t_1)^1 \cap Z(t_2)^0$. Obviously, $z \in \bigcap_{t\in T_1} Z(t)^1 \cap \bigcap_{t\in T_2} Z(t)^0$ for some $T_1 \subset T$ $(t_1 \in T_1)$ and $T_2 = T \setminus T_1$ $(t_2 \in T_2)$. We denote

$$S := \{j \in \{1, \ldots, m\} : \ z_j = 0\},$$

$$\mathcal{T} := \Big\{A : \ T_1 \subset A \subset T \text{ and } \bigcap_{t\in A} Z(t)^1 \cap \bigcap_{t\in T\setminus A} Z(t)^0 \cap B_S \neq \emptyset\Big\}.$$

The set $\mathcal{T}$ is not empty $(T_1 \in \mathcal{T})$ and partially ordered by the inclusion. Every chain has as the upper bound the union of its elements. By the Kuratowski–Zorn's Lemma there exists in $\mathcal{T}$ a maximal element $\mathbf{T}$. We assume that $\mathbf{T} \neq T$. Then, by applying condition (9), we obtain that there exists a function $k(t)\colon T \setminus \mathbf{T} \to \{0, 1\}$, not identically equal to zero, such that

$$\bigcap_{t\in\mathbf{T}} Z(t)^1 \cap \bigcap_{t\in T\setminus\mathbf{T}} Z(t)^{k(t)} \cap B_S \neq \emptyset,$$

because $\bigcap_{t\in\mathbf{T}} Z(t)^1 \cap \bigcap_{t\in T\setminus\mathbf{T}} Z(t)^0 \cap B_S \neq \emptyset$. It means that the set

$$\mathbf{T} \cup \{t \in T \setminus \mathbf{T} : \ k(t) = 1\} \in \mathcal{T} \qquad \text{and} \qquad \mathbf{T} \subsetneq \mathbf{T} \cup \{t \in T \setminus \mathbf{T} : \ k(t) = 1\},$$

which contradicts the fact that $\mathbf{T}$ is a maximal element of $\mathcal{T}$, hence

$$\bigcap_{t\in T} Z(t)^1 \cap B_S \neq \emptyset.$$

Let $x = (x_1, \ldots, x_m) \in \bigcap_{t\in T} Z(t)^1 \cap B_S$. For every $l \in \{1, \ldots, m\} \setminus S$ there exists $q_l \in \mathbb{Q}_+$ such that $z_l > q_l x_l$.

We denote

$$q := \min\{q_l : \ l \in \{1, \ldots, m\} \setminus S\}.$$

Then $z - qx \in B_S$, because $z_j - qx_j = 0$ for $j \in S$ and $z_l - qx_l > 0$ for $l \in \{1, \ldots, m\} \setminus S$. By condition (5) (which is equivalent to condition (6) if condition (7) is assumed) we have

$$qx + (z - qx) \in \bigcap_{t\in T} Z(t)^1 + \bigcap_{t\in T} Z(t)^{i_{z-qx}(t)} \subset \bigcap_{t\in T} Z(t)^{i_{z-qx}(t)}.$$

On the other hand

$$qx + (z - qx) = z \in \bigcap_{t\in T_1} Z(t)^1 \cap \bigcap_{t\in T_2} Z(t)^0.$$

Therefore $i_{z-qx}(t) = i_z(t)$ for all $t \in T$ and

$$z - qx \in \bigcap_{t\in T_1} Z(t)^1 \cap \bigcap_{t\in T_2} Z(t)^0,$$

thus

$$z = qx + (z - qx) \in Z(t_1)^1 \cap Z(t_2)^1 + Z(t_1)^1 \cap Z(t_2)^0.$$

Since $z \in Z(t_1)^1 \cap Z(t_2)^0$ was arbitrary

$$Z(t_1)^1 \cap Z(t_2)^0 \subset Z(t_1)^1 \cap Z(t_2)^1 + Z(t_1)^1 \cap Z(t_2)^0,$$

which completes the proof.

Remark 1
As an immediate consequence of Theorem 1, we obtain that if the multifunction $Z(t)\colon T \to 2^{\mathbb{R}(m)}$ which fulfils condition (7) and does not fulfil condition (6) or condition (9), does not satisfy system of equations obtained by replacing the inclusion in conditions (6) by the equality.

The following examples show that neither condition (6) nor condition (9) in Theorem 1 may be omitted.

Let T be an arbitrary set with at least 2 elements and take an arbitrary $t^* \in T$. Now, we define two multifunctions $Z_1(t), Z_2(t)\colon T \to 2^{\mathbb{R}(m)}$ by

$$Z_1(t) = \begin{cases} \emptyset & \text{for } t = t^*, \\ \mathbb{R}(m) & \text{for } t \in T \setminus \{t^*\} \end{cases}$$

and

$$Z_2(t) = \begin{cases} \{x = (x_1, \ldots, x_m) \in \mathbb{R}(m) :\ x_1 < 1\} & \text{for } t = t^*, \\ \mathbb{R}(m) & \text{for } t \in T \setminus \{t^*\}. \end{cases}$$

It can be easily checked that the multifunction $Z_1(t)$ satisfies conditions (6) and (7) but does not satisfy condition (9) (for $B = \mathbb{R}(m)$ and $T_1 = T \setminus \{t^*\}$).

It is obvious that the multifunction $Z_2(t)$ satisfies conditions (7). Now, we will show that the multifunction $Z_2(t)$ satisfies conditions (9). First we observe that

$$\bigcap_{t\in T_1} Z_2(t)^1 \cap \bigcap_{t\in T\setminus T_1} Z_2(t)^0 = \emptyset \qquad \text{for } T_1 \neq \emptyset \text{ such that } T \setminus \{t^*\} \neq T_1 \subsetneq T,$$

$$\bigcap_{t\in T\setminus\{t^*\}} Z_2(t)^1 \cap Z_2(t^*)^0 \neq \emptyset,$$

and $Z_2(t^*)^0 \cap B_S \neq \emptyset$ if and only if $1 \notin S$. We take an arbitrary $S \subset \{1, \ldots, m\} \setminus \{1\}$ and consider

$$(x_1, \ldots, x_m) \in \bigcap_{t\in T\setminus\{t^*\}} Z_2(t)^1 \cap Z_2(t^*)^0 \cap B_S.$$

Then

$$\Big(\frac{1}{2}, x_2, \ldots, x_m\Big) \in \bigcap_{t\in T} Z_2(t)^1 \cap B_S,$$

which means that condition (9) holds.

Condition (6) is not satisfied because for every $t \in T \setminus \{t^*\}$

$$\Big(\frac{1}{2}, 0, \ldots, 0\Big) + \Big(\frac{1}{2}, 0, \ldots, 0\Big) \in Z_2(t^*)^1 \cap Z_2(t)^1 + Z_2(t^*)^1 \cap Z_2(t)^1,$$

whereas

$$(1, 0, \ldots, 0) \in Z_2(t^*)^0 \cap Z_2(t)^1.$$

Remark 2

The condition (9) has a complicated form, so we give a simply condition, which if satisfied by the multifunction $Z(t)$ implicate that $Z(t)$ does not fulfil condition (9). Namely, we will show that if the multifunction $Z(t)\colon T \to 2^{\mathbb{R}(m)}$ satisfies conditition (7) and condition

$$\exists_{\emptyset \neq B^* \in \mathbb{B}} \ \exists_{\emptyset \neq T^* \subset T} \quad \bigcap_{t \in T^*} Z(t)^1 \cap B^* = \emptyset, \tag{11}$$

then $Z(t)$ does not fulfil condition (9).

Let $Z(t)\colon T \to 2^{\mathbb{R}(m)}$ be an arbitrary multifunction fulfilling conditions (7) and (11). We take an arbitrary $x \in B^*$, obviously, $x \in \bigcap_{t \in T} Z(t)^{i_x(t)}$ and there is at least one $t^* \in T$ such that $i_x(t^*) = 1$. We put

$$\mathcal{T} := \Big\{ A : \ t^* \in A \subset T \text{ and } \bigcap_{t \in A} Z(t)^1 \cap \bigcap_{t \in T \setminus A} Z(t)^0 \cap B^* \neq \emptyset \Big\}.$$

The set $\mathcal{T}$ is not empty, because $\{t^*\} \in \mathcal{T}$, and it is partially ordered by the inclusion. Every chain has as an upper bound the union of its elements. By the Kuratowski–Zorn's Lemma there exists in $\mathcal{T}$ a maximal element $\mathbf{T}$ and $\mathbf{T} \neq T$, because $\bigcap_{t \in T} Z(t)^1 \cap B^* \subset \bigcap_{t \in T^*} Z(t)^1 \cap B^* = \emptyset$.

We note that

$$\bigcap_{t \in \mathbf{T}} Z(t)^1 \cap \bigcap_{t \in T \setminus \mathbf{T}} Z(t)^0 \cap B^* \neq \emptyset.$$

Moreover for every function $k(t)\colon T \setminus \mathbf{T} \to \{0, 1\}$ such that $k \not\equiv 0$ we have

$$\bigcap_{t \in \mathbf{T}} Z(t)^1 \cap \bigcap_{t \in T \setminus \mathbf{T}} Z(t)^{k(t)} \cap B^* = \emptyset,$$

because $\mathbf{T}$ is the maximal element of the set $\mathcal{T}$. This means that condition (9) does not hold (for $B = B^*$ and $T_1 = \mathbf{T}$).

In particular condition (11) is satisfied, if either there exists $\emptyset \neq T^* \subset T$ such that $\bigcap_{t \in T^*} Z(t)^1 = \emptyset$ (because $\mathbb{R}(m) \in \mathbb{B}$) or if there exists $t^* \in T$ such that the set $Z(t^*)$ is a cone over $\mathbb{R}$, different from $\mathbb{R}(m)$ (because then there exists $l \in \{1, \ldots, m\}$ such that $B_{\{1,\ldots,m\} \setminus \{l\}} \cap Z(t^*)^1 = \emptyset$).

We note that according to Theorem 9 in [10] the condition (5) and (6) are equivalent under the assumption of condition (7). The question arises: if the system of conditional equations

$$(\exists_{t \in T} \ \ i(t)j(t) \neq 0) \implies \bigcap_{t \in T} Z(t)^{i(t)} + \bigcap_{t \in T} Z(t)^{j(t)} = \bigcap_{t \in T} Z(t)^{i(t)j(t)} \tag{12}$$

is equivalent to the system of equations (8) under the assumption of condition (7)?

We will prove the following

Theorem 2

Let T be an arbitrary set with at least 2 elements and let the multifunction $Z(t): T \to 2^{\mathbb{R}(m)}$ satisfy condition (7)*. If the multifunction $Z(t)$ fulfils the system of conditional equations* (12) *for the arbitrary functions $i(t), j(t): T \to \{0,1\}$ not identically equal to zero, then $Z(t)$ satisfies the system of equations* (8) *for all $t_1, t_2 \in T$ and all $k_1, k_2, l_1, l_2 \in \{0,1\}$ such that $k_1 l_1 + k_2 l_2 \neq 0$.*

Proof. As the conditions (5) and (6) are equivalent under the assumption of condition (7), it is enough to show that condition (10) holds for all $t_1, t_2 \in T$ and all $k_1, k_2, l_1, l_2 \in \{0,1\}$ such that $k_1 l_1 + k_2 l_2 \neq 0$. Take arbitrary $t_1, t_2 \in T$ and $k_1, k_2, l_1, l_2 \in \{0,1\}$ such that $k_1 l_1 + k_2 l_2 \neq 0$ and consider arbitrary $z \in Z(t_1)^{k_1 l_1} \cap Z(t_2)^{k_2 l_2}$. Clearly, $z \in \bigcap_{t \in T} Z(t)^{i_z(t)}$ and $i_z(t_1) = k_1 l_1$, $i_z(t_2) = k_2 l_2$. Putting $i(t) = j(t) = i_z(t)$ for $t \in T \setminus \{t_1, t_2\}$ and $i(t_1) = k_1, i(t_2) = k_2, j(t_1) = l_1, j(t_2) = l_2$, and applying condition (12) (because $k_1 k_2 = 1$ or $l_1 l_2 = 1$) we get

$$\bigcap_{t \in T \setminus \{t_1,t_2\}} Z(t)^{i_z(t) i_z(t)} \cap Z(t_1)^{k_1 l_1} \cap Z(t_2)^{k_2 l_2}$$
$$= \bigcap_{t \in T \setminus \{t_1,t_2\}} Z(t)^{i_z(t)} \cap Z(t_1)^{k_1} \cap Z(t_2)^{k_2} + \bigcap_{t \in T \setminus \{t_1,t_2\}} Z(t)^{i_z(t)} \cap Z(t_1)^{l_1} \cap Z(t_2)^{l_2}.$$

Thus there exist

$$x \in \bigcap_{t \in T \setminus \{t_1,t_2\}} Z(t)^{i_z(t)} \cap Z(t_1)^{k_1} \cap Z(t_2)^{k_2},$$
$$y \in \bigcap_{t \in T \setminus \{t_1,t_2\}} Z(t)^{i_z(t)} \cap Z(t_1)^{l_1} \cap Z(t_2)^{l_2}$$

such that $x + y = z$. Therefore

$$z = x + y \in Z(t_1)^{k_1} \cap Z(t_2)^{k_2} + Z(t_1)^{l_1} \cap Z(t_2)^{l_2},$$

which finishes the proof.

Remark 3

It is easily seen that the system of equations (12) and the system of equations (8) for a 2-element set T are identical.

Remark 4

The converse of Theorem 2 for the set T with at least 2 elements is not true.

Here is an example for $T = \{1, 2, 3\}$. Let H be a Hamel base of the space $\mathbb{R}^m$, such that

$$b_0 = (\sqrt{2}, 0, \ldots, 0) \in \mathbb{R}(m),$$
$$b_i = (0, \ldots, 0, \overset{(i)}{1}, 0, \ldots, 0) \in \mathbb{R}(m) \qquad \text{for } i = 1, \ldots, m$$

belong to H. Every $x \in \mathbb{R}^m$ has a representation, unique up to terms with coefficients zero

$$x = \sum_{l=0}^{k} q_l b_l,$$

where $q_l \in \mathbb{Q}$ and $b_l \in H$ for $l \in \{0, \ldots, k\}$.

We define the multifunction $Z(t)\colon \{1,2,3\} \to 2^{\mathbb{R}(m)}$ in the following way

$$Z(t) = \begin{cases} \{x \in \mathbb{R}(m) :\ q_o \geq 0\} & \text{for } t = 1, \\ \{x \in \mathbb{R}(m) :\ q_o = 0\} & \text{for } t = 2, \\ \{x \in \mathbb{R}(m) :\ q_o \leq 0\} & \text{for } t = 3. \end{cases}$$

We observe that $Z(1) \cup Z(2) \cup Z(3) = \mathbb{R}(m)$ and the equation (12) does not hold, because

$$\underset{\underset{\emptyset}{\|}}{Z(1)^1 \cap Z(2)^1 \cap Z(3)^0} + \underset{\underset{\emptyset}{\nparallel}}{Z(1)^1 \cap Z(2)^0 \cap Z(3)^0} \subsetneq \underset{\underset{\emptyset}{\nparallel}}{Z(1)^1 \cap Z(2)^0 \cap Z(3)^0}.$$

We will show that the multifuncion $Z(t)$ satisfies the system of conditions (6) for all $t_1, t_2 \in \{1,2,3\}$ and all $k_1, k_2, l_1, l_2 \in \{0,1\}$ such that $k_1 l_1 + k_2 l_2 \neq 0$ and condition (9) holds.

First we observe that the sets:

$$\begin{aligned} &Z(1)^1 \cap Z(2)^1 = Z(1)^1 \cap Z(3)^1 = Z(2)^1 \cap Z(3)^1 = Z(2)^1, \\ &Z(1)^1 \cap Z(2)^0 = Z(1)^1 \cap Z(3)^0 = \{x \in \mathbb{R}(m) :\ q_o > 0\}, \\ &Z(1)^0 \cap Z(3)^1 = Z(2)^0 \cap Z(3)^1 = \{x \in \mathbb{R}(m) :\ q_o < 0\}, \\ &Z(1)^0 \cap Z(2)^1 = Z(2)^1 \cap Z(3)^0 = \emptyset \end{aligned}$$

are cones over $\mathbb{Q}_+$, so for all $t_1, t_2 \in \{1,2,3\}$ and all $k_1, k_2 \in \{0,1\}$ such that $k_1 = 1$ or $k_2 = 1$ we get

$$Z(t_1)^{k_1} \cap Z(t_2)^{k_2} + Z(t_1)^{k_1} \cap Z(t_2)^{k_2} \subset Z(t_1)^{k_1} \cap Z(t_2)^{k_2}.$$

Of course,

$$\begin{aligned} &Z(1)^1 \cap Z(2)^1 + Z(1)^1 \cap Z(2)^0 \subset \{x \in \mathbb{R}(m) :\ q_o > 0\} = Z(1)^1 \cap Z(2)^0, \\ &Z(1)^1 \cap Z(2)^1 + Z(1)^0 \cap Z(2)^1 = \emptyset = Z(1)^0 \cap Z(2)^1, \\ &Z(1)^1 \cap Z(3)^1 + Z(1)^1 \cap Z(3)^0 \subset \{x \in \mathbb{R}(m) :\ q_o > 0\} = Z(1)^1 \cap Z(3)^0, \\ &Z(1)^1 \cap Z(3)^1 + Z(1)^0 \cap Z(3)^1 \subset \{x \in \mathbb{R}(m) :\ q_o < 0\} = Z(1)^0 \cap Z(3)^1, \\ &Z(2)^1 \cap Z(3)^1 + Z(2)^1 \cap Z(3)^0 = \emptyset = Z(2)^1 \cap Z(3)^0, \\ &Z(2)^1 \cap Z(3)^1 + Z(2)^0 \cap Z(3)^1 \subset \{x \in \mathbb{R}(m) :\ q_o < 0\} = Z(2)^0 \cap Z(3)^1, \end{aligned}$$

and the system of conditions (6) for every $t_1, t_2 \in \{1,2,3\}$ and every $k_1, k_2, l_1, l_2 \in \{0,1\}$ such that $k_1 l_1 + k_2 l_2 \neq 0$ is therefore satisfied.

Now, we take arbitrary $B_S \in \mathbb{B}$ and $\emptyset \neq T_1 \subsetneq \{1,2,3\}$ such that

$$\bigcap_{t\in T_1} Z(t)^1 \cap \bigcap_{t\in T_2} Z(t)^0 \cap B_S \neq \emptyset.$$

Then $S \subsetneq \{1,\dots,m\}$, because $B_S \neq \emptyset$.

We define

$$k(t) := 1 \qquad \text{for } t \in T_2.$$

We note that

$$x = \sum_{l\in\{1,\dots,m\}\setminus S} b_l \in Z(1)^1 \cap Z(2)^1 \cap Z(3)^1 \cap B_S,$$

therefore condition (9) holds.

Remark 5

In [10] (Theorem 13) it was shown that the multifunction $Z(t)\colon T \to 2^{\mathbb{R}(m)}$ identically equal to $\mathbb{R}(m)$ is the unique solution of the system of equations

$$\bigcap_{t\in T} Z(t)^{i(t)} + \bigcap_{t\in T} Z(t)^{j(t)} = \bigcap_{t\in T} Z(t)^{i(t)j(t)}$$

and equation (7). From this Theorem we obtain that $Z(t) = \mathbb{R}(m)$ is the unique solution of equation (7) and the system of conditional equations (12) for one-element set T. It is obvious, that the multifunction $Z(t) = \mathbb{R}(m)$ satisfies conditions (7) and (12) for every set T. For a 2-element set the system of conditions (7) and (12) have other solutions, for example the multifunction from Remark 4 for $T = \{1,3\}$. It is an open problem if there exist another multifunctions $Z(t)\colon T \to 2^{\mathbb{R}(m)}$, where $\overline{\overline{T}} > 2$, which fulfil (7) and (12)?

Remark 6

In our arguments in the paper we use Axiom of Choice. It is natural to ask to what extend this is really necessary.

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Institute of Mathematics
Pedagogical University
Podchorążych 2, 30-084 Kraków
Poland
E-mail: bah@ap.krakow.pl

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