Annales Academiae Paedagogicae Cracoviensis
Folia 23 Studia Mathematica IV (2004)

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Functorial prolongations of some functional bundles

To Andrzej Zajtz, on the occasion of his 70th birthday

Abstract. We discuss two kinds of functorial prolongations of the functional bundle of all smooth maps between the fibers over the same base point of two fibered manifolds over the same base. We study the prolongation of vector fields in both cases and we prove that the bracket is preserved. Our proof is based on several new results concerning the finite dimensional Weil bundles.

Introduction

Let E_1 and E_2 be two classical fiber bundles over the same base M. The differential geometric investigation of the functional bundle $\mathcal{F}(E_1, E_2) \longrightarrow M$ of all smooth maps from a fiber of E_1 into the fiber of E_2 over the same base point was iniciated by the paper by A. Jadczyk and M. Modugno on the Schrödinger connection, [6], [7]. The simpliest cases of the tangent bundle $T\mathcal{F}(E_1, E_2) \longrightarrow TM$ and of the r-th jet prolongation $J^r\mathcal{F}(E_1, E_2) \longrightarrow M$ are discussed in [1]. In the present paper we first clarify that the essential assumption for these constructions is that T is a product preserving bundle functor on the classical category $\mathcal{M}f$ of all smooth manifolds and all smooth maps and J^r is a fiber product preserving bundle functor on the category $\mathcal{FM}_m$ of all fibered manifolds with m-dimensional bases and of all fibered manifold morphisms covering local diffeomorphisms. Every product preserving bundle functor F on $\mathcal{M}f$ is a Weil functor $F = T^A$, where A is a Weil algebra, [12]. The general construction of $T^A\mathcal{F}(E_1, E_2) \longrightarrow T^AM$ was presented by the third author in [9], [10], see also Section 2 of the present paper. We underline that this construction is based on the covariant approach to Weil bundles and their natural transformations, [8], [12]. On the other hand, in [13] it was deduced that every fiber product preserving bundle functor G on $\mathcal{FM}_m$ is of

AMS (2000) Subject Classification: 58A20.

The second and third authors were supported by the Ministry of Education of the Czech Republic under the project MSM 143100009.

the form $G = (A, H, t)$, where A is a Weil algebra, H is a group homomorphism $H: G^r_m \longrightarrow \operatorname{Aut} A$ of the r-th jet group G^r_m in dimension m into the group of all algebra automorphisms of A and $t: \mathbb{D}^r_m \longrightarrow A$ is an equivariant algebra homomorphism, where $\mathbb{D}^r_m = J^r_0(\mathbb{R}^m, \mathbb{R})$ is the Weil algebra corresponding to the functor of (m, r)-velocities. In Section 6 of the present paper we construct $G\mathcal{F}(E_1, E_2) \longrightarrow M$ in a way that generalizes the case of $J^r\mathcal{F}(E_1, E_2) \longrightarrow M$.

Our main geometric problem is the prolongation of vector fields on $\mathcal{F}(E_1, E_2)$ with respect to F and G. Since we cannot use the flow in the functional case, we start from the fact that the classical flow prolongation with respect to T^A of a vector field $M \longrightarrow TM$ coincides with the composition of its T^A-prolongation $T^AM \longrightarrow T^ATM$ with the exchange map $\kappa^A_M: T^ATM \longrightarrow TT^AM$. We apply this idea to a vector field X on $\mathcal{F}(E_1, E_2)$ and we say the composition $\mathcal{T}^AX = \kappa^A_{\mathcal{F}(E_1,E_2)} \circ T^AX$ to be the field prolongation of X. The bracket of vector fields on $\mathcal{F}(E_1, E_2)$ is defined in terms of the strong difference, [1], [12]. Proposition 3.2 in Section 3 reads that $\mathcal{T}^A$ preserves the bracket of vector fields even in the functional case. To deduce it, we develop, in Sections 4 and 5, a purely algebraic proof of the fact that $\mathcal{T}^A$ preserves bracket in the manifold case. For this purpose we need certain new lemmas concerning the classical Weil bundles, which are collected in Sections 4 and 5. In particular, we present a complete description of the strong difference in terms of Weil algebras. In Section 7 we study the prolongation of vector fields to $G\mathcal{F}(E_1, E_2)$ and we prove that the bracket is preserved even in this case. Finally we remark that an interesting kind of exchange morphism, which was introduced recently for the manifold case in [11], can be extended to the functional bundles as well.

In Section 1 we present a simplified version of the theory of smooth spaces in the sense of A. Frölicher, [4], which we call F-smooth spaces, and of F-smooth bundles. Special attention is paid to the functorial character of the construction of $\mathcal{F}(E_1, E_2)$ and to the concept of finite order morphism.

If we deal with finite dimensional manifolds and maps between them, we always assume they are of class C^∞, i.e. smooth in the classical sense. Unless otherwise specified, we use the terminology and notation from the monograph [12].

1. *F*-smooth bundles

We shall use the following simplified version, [2], of the theory of smooth spaces by A. Frölicher, [4].

Definition 1.1
An F-smooth space is a set S along with a set C_S of maps $c: \mathbb{R} \longrightarrow S$, which are called F-smooth curves, satisfying the following two conditions:

(i) each constant curve $\mathbb{R} \longrightarrow S$ belongs to C_S,

(ii) if $c \in C_S$ and $\gamma \in C^\infty(\mathbb{R}, \mathbb{R})$, then $c \circ \gamma \in C_S$.

If $(S', C_{S'})$ is another F-smooth space, a map $f\colon S \longrightarrow S'$ is said to be F-smooth, if $f \circ c$ is an F-smooth curve on S' for every F-smooth curve c on S.

So we obtain the category $\mathcal{S}$ of F-smooth spaces. Every subset $\bar{S} \subset S$ is also an F-smooth space, if we define $C_{\bar{S}} \subset C_S$ to be the subset of the curves with values in $\bar{S}$. In particular every smooth manifold M turns out to be an F-smooth space by assuming as F-smooth curves just the smooth curves. Moreover, a map between smooth manifolds is F-smooth, if and only if it is smooth.

We find it useful to define the concept of F-smooth bundle in a more general form than in [2].

Definition 1.2
An F-smooth bundle is a triple of an F-smooth space S, a smooth manifold M and a surjective F-smooth map $p\colon S \longrightarrow M$. If $p'\colon S' \longrightarrow M'$ is another F-smooth bundle, then a morphism of S into S' is a pair of an F-smooth map $f\colon S \longrightarrow S'$ and a smooth map $\underline{f}\colon M \longrightarrow M'$ satisfying $\underline{f} \circ p = p' \circ f$.

Thus we obtain the category $\mathcal{SB}$ of F-smooth bundles. Every subset $\bar{S} \subset S$ satisfying $p(\bar{S}) = M$ is also an F-smooth bundle.

An important class of F-smooth bundles are the bundles of smooth maps between the fibers over the same base point of two classical fibered manifolds $p_1\colon E_1 \longrightarrow M$ and $p_2\colon E_2 \longrightarrow M$. We write

$$\mathcal{F}(E_1, E_2) = \bigcup_{x \in M} C^\infty(E_{1x}, E_{2x})$$

and denote by $p\colon \mathcal{F}(E_1, E_2) \longrightarrow M$ the canonical projection. A curve $c\colon \mathbb{R} \longrightarrow \mathcal{F}(E_1, E_2)$ is called F-smooth, if $\underline{c} := p \circ c\colon \mathbb{R} \longrightarrow M$ is a smooth map and the induced map

$$\tilde{c}\colon \underline{c}^* E_1 \longrightarrow E_2, \qquad \tilde{c}(t, y) = c(t)(y), \qquad p_1(y) = \underline{c}(t)$$

is also smooth, [1].

Write $\mathcal{FM}^I \subset \mathcal{FM}$ for the subcategory of locally trivial fibered manifolds whose morphisms are diffeomorphisms on the fibers. Let $\mathcal{FM}^I \times_{\mathcal{B}} \mathcal{FM}$ denote the category whose objects are pairs (E_1, E_2) with $E_1 \longrightarrow M$ in $\mathcal{FM}^I$ and $E_2 \longrightarrow M$ in $\mathcal{FM}$ and morphisms are pairs (f_1, f_2) with $f_1\colon E_1 \longrightarrow E_3$ in $\mathcal{FM}^I$ and $f_2\colon E_2 \longrightarrow E_4$ in $\mathcal{FM}$ over the same base map $\underline{f}\colon M \longrightarrow N$, where N is the common base of E_3 and E_4. If we define $\mathcal{F}(f_1, f_2)\colon \mathcal{F}(E_1, E_2) \longrightarrow \mathcal{F}(E_3, E_4)$ by

$$\mathcal{F}(f_1, f_2)(h) = f_2(x) \circ h \circ f_1^{-1}(\underline{f}(x)), \qquad h \in C^\infty(E_{1x}, E_{2x}), \tag{1.1}$$

then $\mathcal{F}$ is a functor on $\mathcal{FM}^I \times_{\mathcal{B}} \mathcal{FM}$ with values in the category $\mathcal{SB}$.

Definition 1.3
Every F-smooth subbundle $S \subset \mathcal{F}(E_1, E_2)$ will be called a functional F-smooth bundle.

If $S' \subset \mathcal{F}(E_3, E_4)$ is another functional F-smooth bundle and (f_1, f_2) has the property $\mathcal{F}(f_1, f_2)(S) \subset S'$, then the restricted and corestricted map will be interpreted as an $\mathcal{SB}$-morphism $S \longrightarrow S'$.

Consider a smooth map $q\colon E_3 \longrightarrow E_1$.

Definition 1.4
An $\mathcal{SB}$-morphism $D\colon \mathcal{F}(E_1, E_2) \longrightarrow \mathcal{F}(E_3, E_4)$ is said to be of the order r, if for every $\varphi, \psi\colon E_{1x} \longrightarrow E_{2x}$ and $v \in E_3$, $p_1(q(v)) = x$,

$$j^r_{q(v)}\varphi = j^r_{q(v)}\psi \quad \text{implies} \quad D(\varphi)(v) = D(\psi)(v). \tag{1.2}$$

Consider the fibered manifold

$$\mathcal{F}J^r(E_1, E_2) = \bigcup_{x \in M} J^r(E_{1x}, E_{2x}) \longrightarrow E_1 . \tag{1.3}$$

By (1.2), D induces the so called associated map

$$\mathcal{D}\colon \mathcal{F}J^r(E_1, E_2) \times_{E_1} E_3 \longrightarrow E_4 .$$

In the same way as in [1] one proves that $\mathcal{D}$ is a smooth map.

We express the coordinate form of $\mathcal{D}$ in the case $q\colon E_3 \longrightarrow E_1$ is an $\mathcal{FM}$-morphism that is a surjective submersion on each fiber of E_3. Let x^i or u^a be some local coordinates on M or N and y^p or z^s or (y^p, v^b) or w^c be some additional fiber coordinates on E_1 or E_2 or E_3 or E_4, respectively. Then z^s_α are the induced coordinates on $\mathcal{F}J^r(E_1, E_2)$, where $0 \le |\alpha| \le r$ is a multiindex, the range of which is the fiber dimension of E_1, and the coordinate expression of $\mathcal{D}$ is

$$u^a = f^a(x^i), \qquad w^c = f^c(x^i, y^p, z^s_\alpha, v^b), \tag{1.4}$$

where f^a and f^c are smooth functions.

The concept of r-th order morphism can be modified to a functional F-smooth bundle $S \subset \mathcal{F}(E_1, E_2)$ analogously to [12], Section 18.

2. The tangent-like case

Let A be a Weil algebra of the width k. Under the covariant approach, [8], [12], the elements of a Weil bundle $T^A M$ are the A-velocities $j^A g$ of smooth maps $g\colon \mathbb{R}^k \longrightarrow M$. For a smooth map $f\colon M \longrightarrow N$, we define $T^A f\colon T^A M \longrightarrow T^A N$ by

$$T^A f(j^A g) = j^A (f \circ g). \tag{2.1}$$

If B is another Weil algebra of the width l, then every algebra homomorphism $\mu\colon A \longrightarrow B$ can be generated by a B-velocity $j^B h$ of a map $h\colon \mathbb{R}^l \longrightarrow \mathbb{R}^k$. The natural transformation $\mu_M\colon T^A M \longrightarrow T^B M$ induced by μ has the form of a reparametrization

$$\mu_M(j^A g) = j^B(g \circ h). \tag{2.2}$$

Consider $\mathcal{F}(E_1, E_2)$. We have $T^A p_i\colon T^A E_i \longrightarrow T^A M$ and we write

$$T^A_X E_i := (T^A p_i)^{-1}(X), \qquad X \in T^A M,\ i = 1, 2.$$

Let $g_1, g_2\colon \mathbb{R}^k \longrightarrow \mathcal{F}(E_1, E_2)$ be two F-smooth maps satisfying $j^A(p \circ g_1) = j^A(p \circ g_2) \in T^A M$. Then we construct the associated maps $T^A_0 g_i\colon T^A_X E_1 \longrightarrow T^A_X E_2$,

$$T^A_0 g_i(j^A f(u)) = j^A g_i(u)(f(u)), \qquad u \in \mathbb{R}^k\,,$$

where $f\colon \mathbb{R}^k \longrightarrow E_1$ satisfies $p \circ g_i = p_1 \circ f$, $i = 1, 2$. If $T^A_0 g_1 = T^A_0 g_2$, we say that g_1 and g_2 determine the same A-velocity $j^A g_1 = j^A g_2$. The set $T^A \mathcal{F}(E_1, E_2)$ of all such A-velocities is a subspace in $\mathcal{F}(T^A E_1, T^A E_2) \longrightarrow T^A M$, so a functional F-smooth bundle. In the product case $E_i = M \times Q_i$, $i = 1, 2$, the third author deduced in [9]

$$T^A(M \times Q_1, M \times Q_2) = T^A M \times C^\infty(Q_1, T^A Q_2). \tag{2.3}$$

In [9] it was also clarified that the idea of reparametrization (2.2) can be applied to $j^A g \in T^A \mathcal{F}(E_1, E_2)$ as well. So every algebra homomorphism $\mu = j^B h\colon A \longrightarrow B$ induces an F-smooth map

$$\mu_{\mathcal{F}(E_1,E_2)}\colon T^A \mathcal{F}(E_1, E_2) \longrightarrow T^B \mathcal{F}(E_1, E_2), \qquad j^A g \longmapsto j^B(g \circ h)\,. \tag{2.4}$$

Consider a functional F-smooth bundle $S \subset \mathcal{F}(E_1, E_2)$. Then $T^A S \subset T^A \mathcal{F}(E_1, E_2)$ means the subset of all $j^A g$, $g\colon \mathbb{R}^k \longrightarrow S$.

Definition 2.1
An $\mathcal{SB}$-morphism $D\colon S \longrightarrow \mathcal{F}(E_3, E_4)$ is called A-differentiable, if the rule

$$T^A D(j^A g) = j^A(D \circ g)$$

defines an F-smooth map $T^A S \longrightarrow T^A \mathcal{F}(E_3, E_4)$. We say D is strongly differentiable, if it is A-differentiable for every Weil algebra A.

If D is strongly differentiable, then $T^A D$ is also strongly differentiable. Indeed, analogously to the finite dimensional case one verifies easily $T^B(T^A D) = T^{B\otimes A} D$. In particular, every finite order morphism is strongly differentiable, for its associated map is smooth. Further, each morphism $\mathcal{F}(f_1, f_2)$ is strongly differentiable and we have

$$T^A\mathcal{F}(f_1, f_2)(j^A g(u)) = j^A\big(f_2(p(g(u))) \circ g(u) \circ f_1^{-1}(\underline{f}(p(g(u))))\big)\,.$$

Thus, $T^A\mathcal{F}$ is a functor on the category $\mathcal{FM}^I \times_{\mathcal{B}} \mathcal{FM}$ with values in $\mathcal{SB}$.

Analogously to the finite dimensional case, [3], we define an A-field on $\mathcal{F}(E_1, E_2)$ as a strongly differentiable section $\mathcal{F}(E_1, E_2) \longrightarrow T^A\mathcal{F}(E_1, E_2)$. In the case $A = \mathbb{D}$ of the algebra of dual numbers, we obtain a vector field $X\colon \mathcal{F}(E_1, E_2) \longrightarrow T\mathcal{F}(E_1, E_2)$.

3. Prolongation of vector fields

In the manifold case, the exchange algebra homomorphism $\kappa^A\colon A \otimes \mathbb{D} \longrightarrow \mathbb{D}\otimes A$ defines a natural transformation $\kappa^A_M\colon T^A TM \longrightarrow TT^A M$. For a classical vector field $X\colon M \longrightarrow TM$, its flow prolongation $\mathcal{T}^A X\colon T^A M \longrightarrow TT^A M$ coincides with $\kappa^A_M \circ T^A X$, [12]. For a vector field $X\colon \mathcal{F}(E_1, E_2) \longrightarrow T\mathcal{F}(E_1, E_2)$, we also can construct $T^A X\colon T^A\mathcal{F}(E_1, E_2) \longrightarrow T^A T\mathcal{F}(E_1, E_2)$ and apply $\kappa^A_{\mathcal{F}(E_1,E_2)}\colon T^A T\mathcal{F}(E_1, E_2) \longrightarrow TT^A\mathcal{F}(E_1, E_2)$. In this way we obtain a vector field on $T^A\mathcal{F}(E_1, E_2)$.

Definition 3.1
The vector field $\mathcal{T}^A X := \kappa^A_{\mathcal{F}(E_1,E_2)} \circ T^A X$ will be called the field prolongation of X.

We recall that the bracket of two vector fields X, Y on $\mathcal{F}(E_1, E_2)$ was defined by using the strong difference, [1],

$$[X, Y] = (TY \circ X) \div (TX \circ Y). \tag{3.1}$$

(For classical vector fields $X, Y\colon M \longrightarrow TM$, (3.1) coincides with the classical bracket, [1].) We are going to deduce

Proposition 3.2
For every vector fields X, Y on $\mathcal{F}(E_1, E_2)$,

$$\mathcal{T}^A([X, Y]) = [\mathcal{T}^A X, \mathcal{T}^A Y]. \tag{3.2}$$

The proof will be based on the algebraic results of the next two sections.

4. The algebraic form of the strong difference

Write $p^T_M \colon TM \longrightarrow M$ for the bundle projection. We recall that two elements $X, Y \in TT_xM$ satisfying

$$p^T_{TM}X = Tp^T_MY, \qquad p^T_{TM}Y = Tp^T_MX \tag{4.1}$$

determine the strong difference

$$X \div Y \in T_xM, \tag{4.2}$$

[12]. Denote by SM the domain of definition of the strong difference, i.e., $SM \subset TTM \times_M TTM$ is the subset of all pairs (X, Y) satisfying (4.1), and by $\sigma_M \colon SM \longrightarrow TM$ the map (4.2). For every smooth map $f \colon M \longrightarrow N$, one verifies easily that (TTf, TTf) transforms SM into SN. So we obtain a map

$$Sf \colon SM \longrightarrow SN$$

and S is a bundle functor on $\mathcal{M}f$. Moreover, the strong difference map is a natural transformation

$$\sigma_M \colon SM \longrightarrow TM. \tag{4.3}$$

The fact $S\mathbb{R}^m = \overset{5}{\times}\mathbb{R}^m$ implies that S preserves products. Write $\mathbb{S}$ for the corresponding Weil algebra. In general, the sum of two Weil algebras $A = \mathbb{R} \times N_A$ and $B = \mathbb{R} \times N_B$ is defined by

$$A + B = \mathbb{R} \times N_A \times N_B$$

with the induced multiplication that satisfies $ab = 0$ for all $a \in N_A$, $b \in N_B$. Clearly, we have

$$T^AM \times_M T^BM = T^{A+B}M.$$

Write $\mathbb{D} = \{a_0 + a_1e\}$, $e^2 = 0$. Then TT corresponds to $\mathbb{D} \otimes \mathbb{D}$, which is linearly generated by 1, e_1, e_2, e_1e_2. Let $\{1, E_1, E_2, E_1E_2\}$ be the linear generators of another copy of $\mathbb{D} \otimes \mathbb{D}$. So $\mathbb{S}$ is a subalgebra of $\mathbb{D} \otimes \mathbb{D} + \mathbb{D} \otimes \mathbb{D}$ and (4.1) implies directly that the elements of $\mathbb{S}$ are of the form

$$X = a_0 + a_1(e_1 + E_2) + a_2(e_2 + E_1) + a_3e_1e_2 + a_4E_1E_2,$$

$a_0, \ldots, a_4 \in \mathbb{R}$. By the definition of the strong difference, [12], the algebra homomorphism $\sigma \colon \mathbb{S} \longrightarrow \mathbb{D}$ corresponding to (4.2) is

$$\sigma(X) = a_0 + (a_3 - a_4)e. \tag{4.4}$$

Write $p^A_M \colon T^AM \longrightarrow M$ for the bundle projection. Since $SM \subset TTM \times_M TTM$ is defined by (4.1), $T^ASM \subset T^ATTM \times_{T^AM} T^ATM$ is the set of all pairs (X, Y) satisfying

$$T^A p^T_{TM} X = T^A T p^T_M Y, \qquad T^A p^T_{TM} Y = T^A T p^T_M X. \tag{4.5}$$

On the other hand, $ST^A M \subset TTT^A M \times_{T^A M} TTT^A M$ is characterized by

$$p^T_{TT^A M} X = T p^T_{T^A M} Y, \qquad p^T_{TT^A M} Y = T p^T_{T^A M} X. \tag{4.6}$$

We have $T^A\sigma_M \colon T^A SM \longrightarrow T^A TM$, $\kappa^A_{TM} \colon T^A TTM \longrightarrow TT^A TM$ and $T\kappa^A_M \colon TT^A TM \longrightarrow TTT^A M$. For technical reasons, we postpone the proof of the following assertion to Section 5.

Proposition 4.1

The map $T\kappa^A_M \circ \kappa^A_{TM} \colon T^A TTM \longrightarrow TTT^A M$ induces a diffeomorphism $K^A_M \colon T^A SM \longrightarrow ST^A M$ and the following diagram commutes

$$\begin{tikzcd}
T^A SM \arrow[r, "K^A_M"] \arrow[d, "T^A\sigma_M"'] & ST^A M \arrow[d, "\sigma_{T^A M}"] \\
T^A TM \arrow[r, "\kappa^A_M"] & TT^A M
\end{tikzcd} \tag{4.7}$$

Now we first show how (4.7) implies that the flow prolongation $\mathcal{T}^A$ of classical vector fields $X, Y \colon M \longrightarrow TM$ preserves the bracket. We have $(TY \circ X, TX \circ Y) \colon M \longrightarrow SM$ and

$$[X, Y] = \sigma_M \circ (TY \circ X, TX \circ Y). \tag{4.8}$$

Then $T^A(TY \circ X, TX \circ Y) \colon T^A M \longrightarrow T^A SM$. Adding K^A_M we obtain

$$\begin{aligned} T\kappa^A_M \circ \kappa^A_{TM} \circ T^A TY \circ T^A X &= T\kappa^A_M \circ TT^A Y \circ \kappa^A_M \circ T^A X \\ &= T\mathcal{T}^A Y \circ \mathcal{T}^A X \end{aligned}$$

and the same for $TX \circ Y$. So in (4.7) we clockwise obtain $[\mathcal{T}^A X, \mathcal{T}^A Y]$. Counterclockwise, we first get $T^A[X, Y]$ and then $\mathcal{T}^A[X, Y]$.

Consider now the case of $\mathcal{F}(E_1, E_2)$. According to the general fact that the homomorphisms of Weil algebras extend to the functional case, (4.7) yields a commutative diagram

$$\begin{tikzcd}
T^A S\mathcal{F}(E_1, E_2) \arrow[r, "K^A_{\mathcal{F}(E_1,E_2)}"] \arrow[d, "\kappa^A_{\mathcal{F}(E_1,E_2)}"'] & ST^A \mathcal{F}(E_1, E_2) \arrow[d, "\sigma_{T^A \mathcal{F}(E_1,E_2)}"] \\
T^A T\mathcal{F}(E_1, E_2) \arrow[r, "\kappa^A_{\mathcal{F}(E_1,E_2)}"] & TT^A \mathcal{F}(E_1, E_2)
\end{tikzcd} \tag{4.9}$$

For two vector fields X, Y on $\mathcal{F}(E_1, E_2)$, we first construct

$$(TY \circ X, TX \circ Y) \colon \mathcal{F}(E_1, E_2) \longrightarrow S\mathcal{F}(E_1, E_2).$$

Then we deduce (3.2) in the same way as in the manifold case. This proves Proposition 3.2.

5. Some Weilian lemmas

The elements of $A = T^A\mathbb{R}$ are of the form $j^A g$, $g\colon \mathbb{R}^k \longrightarrow \mathbb{R}$. For a vector space V, the map $V \times A \longrightarrow T^A V$, $(v, j^A g) \longmapsto j^A(gv)$ is bilinear and defines an identification $T^A V = V \otimes A$. If W is another vector space and $f\colon V \longrightarrow W$ is a linear map, then $T^A f\colon T^A V \longrightarrow T^A W$ is of the form

$$T^A f = f \otimes \mathrm{id}_A\colon V \otimes A \longrightarrow W \otimes A, \tag{5.1}$$

[12]. Further, let $\mu\colon A \longrightarrow B$ be an algebra homomorphism. Then the induced natural transformation $\mu_V\colon T^A V \longrightarrow T^B V$ is of the form

$$\mu_V = \mathrm{id}_V \otimes \mu\colon V \otimes A \longrightarrow V \otimes B. \tag{5.2}$$

This follows from the fact that V is isomorphic to $\mathbb{R}^n$ and we have a product preserving functor.

In particular, if C is another Weil algebra, then (5.1) implies that the natural transformation $T^C\mu_M\colon T^C T^A M \longrightarrow T^C T^B M$ corresponds to the algebra homomorphism

$$\mathrm{id}_C \otimes \mu\colon C \otimes A \longrightarrow C \otimes B. \tag{5.3}$$

Further, the maps $\mu_{T^C M}\colon T^A T^C M \longrightarrow T^B T^C M$ form a natural transformation $T^A T^C \longrightarrow T^B T^C$ that corresponds to the algebra homomorphism

$$\mu \otimes \mathrm{id}_C\colon A \otimes C \longrightarrow B \otimes C. \tag{5.4}$$

The trivial bundle functor on $\mathcal{M}f$ transforming every manifold M into $\mathrm{id}_M\colon M \longrightarrow M$ and every smooth map f into (f, f) corresponds to the trivial Weil algebra $\mathbb{R}$. The natural transformation $p^A_M\colon T^A M \longrightarrow M$ is determined by the canonical "real part projection" $\rho_A\colon A = \mathbb{R} \times N_A \longrightarrow \mathbb{R}$. So $T^B p^A_M\colon T^B T^A M \longrightarrow T^B M$ corresponds to the canonical map

$$\mathrm{id}_B \otimes \rho_A\colon B \otimes A \longrightarrow B \otimes \mathbb{R} = B. \tag{5.5}$$

Write $\kappa^{A,B}\colon A \otimes B \longrightarrow B \otimes A$ for the exchange map. This defines the exchange natural transformation $\kappa^{A,B}_M\colon T^A T^B M \longrightarrow T^B T^A M$. By (5.4), $\kappa^{A,B}_{T^C M}\colon T^A T^B T^C M \longrightarrow T^B T^A T^C M$ corresponds to the exchange $A \otimes B \otimes C \longrightarrow B \otimes A \otimes C$. By (5.3), $T^B\kappa^{A,C}_M\colon T^B T^A T^C M \longrightarrow T^B T^C T^A M$ corresponds to the exchange $B \otimes A \otimes C \longrightarrow B \otimes C \otimes A$.

Lemma 5.1
The following diagram commutes

$$\begin{tikzcd}
T^A T^B T^C M \arrow[r, "T^B\kappa^{A,C}_M \circ \kappa^{A,B}_{T^C M}"] \arrow[d, "T^A p^B_{T^C M}"'] & T^B T^C T^A M \arrow[d, "p^B_{T^C T^A M}"] \\
T^A T^C M \arrow[r, "\kappa^{A,C}_M"] & T^C T^A M
\end{tikzcd} \tag{5.6}$$

Proof. At the algebra level, we have a commutative diagram

$$\begin{array}{ccccc} A\otimes B\otimes C & \longrightarrow & B\otimes A\otimes C & \longrightarrow & B\otimes C\otimes A \\ \downarrow & & & & \downarrow \\ A\otimes C & & \longrightarrow & & C\otimes A \end{array}$$

Now we are in position to prove Proposition 4.1. Comparing our general case with the situation in Section 4, we see $\kappa^{A,\mathbb{D}} = \kappa^A$ and $p^{\mathbb{D}}_M = p^T_M$. So if we put $B = \mathbb{D} = C$ into (5.6), we obtain

$$p^T_{TT^AM} \circ T\kappa^A_M \circ \kappa^A_{TM} = \kappa^A_M \circ T^A p^T_{TM}\,. \tag{5.7}$$

Every $X, Y \in T^A SM$ satisfy (4.5). The naturality of κ^A on $p^T_M \colon TM \longrightarrow M$ yields

$$\kappa^A_M \circ T^A T p^T_M = TT^A p^T_M \circ \kappa^A_{TM} \tag{5.8}$$

and the standard relation $p^T_{T^AM} \circ \kappa^A_M = T^A p^T_M$ implies

$$Tp^T_{T^AM} \circ T\kappa^A_M = TT^A p^T_M\,. \tag{5.9}$$

Hence we have

$$\begin{aligned}(p^T_{TT^AM} \circ T\kappa^A_M \circ \kappa^A_{TM})(X) &= \kappa^A_M(T^A p^T_{TM}(X)) = \kappa^A_M(T^A T p^T_M(Y)) \\ &= (TT^A p^T_M \circ \kappa^A_{TM})(Y) \\ &= (Tp^T_{T^AM} \circ T\kappa^A_M \circ \kappa^A_{TM})(Y).\end{aligned}$$

Thus, $(T\kappa^A_M \circ \kappa^A_{TM})(X)$ and $(T\kappa^A_M \circ \kappa^A_{TM})(Y)$ satisfy (4.6), so that K^A_M maps $T^A SM$ into $ST^A M$. In the case $M = \mathbb{R}^m$, we have $S\mathbb{R}^m = \overset{5}{\times}\mathbb{R}^m$ and $T^A\mathbb{R}^m = A^m$, so that $T^A S\mathbb{R}^m = \overset{5}{\times} A^m$ and $ST^A\mathbb{R}^m = \overset{5}{\times} A^m$. In this situation, $K^A_{\mathbb{R}^m}$ is the identity of $\overset{5}{\times} A^m$. Moreover, by (4.4) $\sigma_{\mathbb{R}^m}$ is determined by the difference of the fourth and fifth components. Taking into account that the vector addition in A is the T^A-prolongation of the addition of reals, we deduce that the diagram (4.7) commutes.

6. The jet-like case

Every fiber product preserving bundle functor G on $\mathcal{FM}_m$ is of the form $G = (A, H, t)$ where A is a Weil algebra, $H\colon G^r_m \longrightarrow \operatorname{Aut} A$ is a group homomorphism and $t\colon \mathbb{D}^r_m \longrightarrow A$ is an equivariant algebra homomorphism, [13]. For every manifold N, the natural transformations corresponding to $\operatorname{Aut} A$ determine an action H_N of G^r_m on $T^A N$. So we can construct the associated bundle

$P^rM[T^AN, H_N]$, where $P^rM \subset T^r_mM$ is the r-th order frame bundle of M. For a fibered manifold $\pi\colon E \longrightarrow M$, we define GE as a subset of $P^rM[T^AE, H_E]$ characterized by

$$GE = \{\{u, Z\},\ t_M u = T^A\pi(Z)\}, \qquad u \in P^rM,\ Z \in T^AE. \tag{6.1}$$

For an $\mathcal{FM}_m$-morphism $f\colon E \longrightarrow \bar{E}$ over a local diffeomorphism $\underline{f}\colon M \longrightarrow \bar{M}$, we have the induced principal bundle morphism $P^r\underline{f}\colon P^rM \longrightarrow P^r\bar{M}$ and an G^r_m-equivariant map $T^Af\colon T^AE \longrightarrow T^A\bar{E}$. So we can construct $P^r\underline{f}[T^Af]\colon P^rM[T^AE] \longrightarrow P^r\bar{M}[T^A\bar{E}]$ and we define

$$Gf = P^r\underline{f}[T^Af]|GE. \tag{6.2}$$

In the product case $E = \mathbb{R}^m \times Q$, we have $GE = \mathbb{R}^m \times T^AQ$, [13].

This construction extends directly to $\mathcal{F}(E_1, E_2)$. By (2.4), each element of $\operatorname{Aut} A$ determines an F-smooth isomorphism $T^A\mathcal{F}(E_1, E_2) \longrightarrow T^A\mathcal{F}(E_1, E_2)$. So we have an action $H_{\mathcal{F}(E_1,E_2)}$ of G^r_m on $T^A\mathcal{F}(E_1, E_2)$ and we can construct the F-smooth associated bundle

$$P^rM[T^A\mathcal{F}(E_1, E_2), H_{\mathcal{F}(E_1,E_2)}]. \tag{6.3}$$

Then we define $G\mathcal{F}(E_1, E_2)$ as the subset of (6.3) characterized by

$$\begin{aligned} G\mathcal{F}(E_1, E_2) = \{\{u, Z\},\ t_M u = T^Ap(Z)\},\\ u \in P^rM,\ Z \in T^A\mathcal{F}(E_1, E_2). \end{aligned} \tag{6.4}$$

Write $\mathcal{FM}^I_m = \mathcal{FM}^I \cap \mathcal{FM}_m$. For $(f_1, f_2) \in \mathcal{FM}^I_m \times_{\mathcal{B}} \mathcal{FM}_m$ with the common base map $\underline{f}$, we define

$$G\mathcal{F}(f_1, f_2) = P^r\underline{f}[T^A\mathcal{F}(f_1, f_2)]|G\mathcal{F}(E_1, E_2). \tag{6.5}$$

Hence $G\mathcal{F}$ is a functor on $\mathcal{FM}^I_m \times_{\mathcal{B}} \mathcal{FM}_m$ with values in $\mathcal{SB}$.

In the product case $E_1 = \mathbb{R}^m \times Q_1$, $E_2 = \mathbb{R}^m \times Q_2$, we have

$$G\mathcal{F}(E_1, E_2) = \mathbb{R}^m \times C^\infty(Q_1, T^AQ_2). \tag{6.6}$$

This shows that for $J^r = (\mathbb{D}^r_m, \mathrm{id}_{G^r_m}, \mathrm{id}_{\mathbb{D}^r_m})$ we obtain $J^r\mathcal{F}(E_1, E_2)$ constructed by means of the fiber r-jets in [1].

7. Vector fields in the jet-like case

In the manifold case, [11], if we have a principal bundle $P(M, C)$ with structure group C and a left C-space S, a right-invariant vector field φ on P and a left-invariant vector field ψ on S, the product vector field (φ, ψ) on

$P \times S$ is projectable to a vector field $\{\varphi, \psi\}$ on the associated bundle $P[S]$. In particular, if η is a projectable vector field on $E \longrightarrow M$ over a vector field ξ on M, then the flow prolongation $\mathcal{P}^r\xi$ is right-invariant on P^rM and $\mathcal{T}^A\eta$ is left-invariant on T^AE. In [11] we deduced that the flow prolongation $\mathcal{G}\eta$ of η coincides with the restriction of $\{\mathcal{P}^r\xi, \mathcal{T}^A\eta\}$ to $GE \subset P^rM[T^AE]$.

In the functional case, consider a vector field $X\colon \mathcal{F}(E_1, E_2) \longrightarrow T\mathcal{F}(E_1, E_2)$ over $\xi\colon M \longrightarrow TM$. Then (2.4) implies that the field prolongation $\mathcal{T}^AX$ is $H_{\mathcal{F}(E_1,E_2)}$-invariant. Hence we have the vector field $\{\mathcal{P}^r\xi, \mathcal{T}^AX\}$ on $P^rM[T^A\mathcal{F}(E_1, E_2)]$ and we define the field prolongation $\mathcal{G}X$ of X by

$$\mathcal{G}X = \{\mathcal{P}^r\xi, \mathcal{T}^AX\}|G\mathcal{F}(E_1, E_2). \tag{7.1}$$

This is a vector field $G\mathcal{F}(E_1, E_2) \longrightarrow TG\mathcal{F}(E_1, E_2)$ over ξ. For two vector fields X_i on $\mathcal{F}(E_1, E_2)$ over ξ_i , $i = 1, 2$, we have by the basic properties of the strong difference

$$[\mathcal{G}X_1, \mathcal{G}X_2] = \{[\mathcal{P}^r\xi_1, \mathcal{P}^r\xi_2], [\mathcal{T}^AX_1, \mathcal{T}^AX_2]\}.$$

Hence Proposition 3.2 yields

Proposition 7.1
We have

$$[\mathcal{G}X_1, \mathcal{G}X_2] = \mathcal{G}[X_1, X_2]\,.$$

At the end we remark that the third author, [11], constructed a map

$$\mu^G_E\colon J^rTM \times_{GTM} GTE \longrightarrow TGE$$

with the property that for every projectable vector field η on E over ξ on M

$$\mathcal{G}\eta = \mu^G_E \circ (j^r\xi \times_M G\eta)\,,$$

where $j^r\xi\colon M \longrightarrow J^rTM$ is the r-th jet prolongation of the section $\xi\colon M \longrightarrow TM$ and $G\eta : GE \longrightarrow GTE$ is the induced morphism. Analyzing this construction, one realizes that each step can be extended to our functional case. In other words, one can introduce in the same way an F-smooth morphism

$$\mu^G_{\mathcal{F}(E_1,E_2)}\colon J^rTM \times_{GTM} G\mathcal{F}(E_1, E_2) \longrightarrow TG\mathcal{F}(E_1, E_2)$$

with the property

$$\mathcal{G}X = \mu^G_{\mathcal{F}(E_1,E_2)} \circ (j^r\xi \times_M GX)$$

for every vector field X on $\mathcal{F}(E_1, E_2)$ with underlying vector field ξ on M.

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